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Left Valuation Rings, Left Cones, and a Question of Frege's

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Abstract. A left cone P in a group G is a submonoid of G so that the lattice of left ideals of P is totally ordered and that G is generated by P. A subring R of a skew field F is a left valuation ring of F if and only if $R^* = R\backslash\{0\}$ is a left cone in F^*. The ideal theory of these left cones is investigated, left cones in a wreath product of groups and in free groups are constructed which are not right cones. Such examples, which give an answer to a question of Frege's, were encountered in the mid seventies during the construction of left valuation rings. The final section deals with the question whether the existence of left cones in G implies the existence of certain cones in G.

1 Introduction

Let G be a group. A non-empty subset P of G is called a *left cone* of G if (i) $PP \subseteq P$, (ii) $PP^{-1} = P \cup P^{-1}$, and (iii) $P^{-1}P = G$. This is equivalent with the condition that P is a submonoid of G that generates G and for any two elements a, b in P either $Pa \supseteq Pb$ or $Pb \supset Pa$. A non-empty subset I of P is a *left ideal* of P if $PI \subseteq I$ and it follows that the lattice of left ideals of a left cone P is totally ordered. The intersection $P \cap P^{-1} = U(P)$ is the subgroup of units of P.

G. Frege [6] at around 1900 is concerned with, among other things, the construction of real numbers. He essentially asks whether a left cone P of a group G with $U(P) = \{e\}$ must also be a right cone. A negative answer is given in [1] where this problem is discussed in greater detail. An example of a left cone P with trivial unit group which is not a right cone was obtained earlier, see [2], during

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the construction of left valuation rings. See Section 3 where we also give further examples of such cones in free groups. The ideal theory of left cones is discussed in Section 2.

A *left valuation* ring of a skew field F is a subring R of F so that $R\backslash\{0\}$ is a left cone in the multiplicative group $F^* = (F\backslash\{0\}, \cdot)$ of F. A subset $I \supset \{0\}$ of R is a left ideal of the ring R if and only if $I\backslash\{0\}$ is a left ideal of the left cone $R\backslash\{0\}$: If a and b are elements in I with $b = ca$ for c in R, then $a - b = (1-c)a$. Such rings have interesting ring theoretical properties, see [8], [3]. W. Klingenberg in [9] considers projective planes over local rings R. The condition that R is a right valuation ring is equivalent with the geometrical condition that there is a line through any two distinct points in the plane. If, in addition, the lattice of left ideals is totally ordered, then R is the coordinate ring of a desarguesian projective plane: any two lines will intersect in at least one point.

If a group G contains a left cone P with $U(P) = \{e\}$, it contains a cone Q with $U(Q) = \{e\}$, i.e. G is then right ordered, see [10]. In the last section some related results are considered.

2 Ideal theory

Let P be a *left cone* in a group G. A *left ideal* of P is a non-empty subset I of P with $PI \subseteq I$. Right ideals and (two-sided) ideals are defined similarly. An ideal $I \neq P$ of P is a *prime ideal* if $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ for ideals A, B of P, and the ideal $I \neq P$ is *completely prime* if $ab \in I$ implies $a \in I$ or $b \in I$ for elements a, b in P.

Definition 2.1 A *prime segment* $Q_1 \supset Q_2$ of the left cone P consists of two completely prime ideals $Q_1 \neq Q_2$ of P so that no further completely prime ideal exists between Q_1 and Q_2. If Q_1 is a minimal completely prime ideal of P, then $Q_1 \supset Q_2$ with $Q_2 = \emptyset$ is also a prime segment.

If $P \neq G$ is a left cone of G with $U = U(P)$ as its subgroup of units, then $J(P) = P\backslash U$ is its unique maximal ideal which is also completely prime. The next result shows how further completely prime ideals of P can be obtained; see [4] for the next two results.

Proposition 2.2 *Let P be a left cone of a group G and $I \neq P$ an ideal of P. Then $Q = \bigcap_{n\in\mathbb{N}} I^n$ is either the empty set or a completely prime ideal of P. In particular, idempotent ideals $I = I^2 \neq P$ of P are completely prime.*

Prime segments $Q_1 \supset Q_2$ of the left cone P fall into one of three categories. Proposition 2.2 plays an essential role in the proof of this result.

Theorem 2.3 *Let P be a left cone of a group G and let $Q_1 \supset Q_2$ a be a prime segment of P. Then exactly one of the following alternatives occurs:*

(a): *The prime segment $Q_1 \supset Q_2$ is left invariant; i.e. $aQ_1 \subseteq Q_1 a$ for all $a \in Q_1\backslash Q_2$.*

(b): *The prime segment $Q_1 \supset Q_2$ is simple; i.e. there are no further ideals of P between Q_1 and Q_2.*

(c): *The prime segment $Q_1 \supset Q_2$ is exceptional; i.e. there exists a prime ideal Q of P with $Q_1 \supset Q \supset Q_2$. Then there are no further ideals between Q_1 and Q and $\bigcap_{n\in\mathbb{N}} Q^n$ is equal to Q_2.*

This theorem provides useful information about the structure of the lattice of ideals of a left cone P.

More precise results can be obtained if P is a rank one left cone which is the case where $J(P)$ is the only completely prime ideal of P. The only segment $J(P) \supset \emptyset$ of P is then left invariant if and only if P is archimedean in the following sense: For $a, b \in J(P)$ there exists a natural number n with $Pb \supseteq Pa^n$. In that case, P is left invariant, i.e. $aP \subseteq Pa$ for all a in P. Then $U = U(P)$ satisfies $aU \subseteq Ua$ for all a in P and the quotient structure P/U consisting of the elements Ua, $a \in P$ can be formed; the operation in P/U is given by $Ua \cdot Ub = Uab$. The condition $aU \subseteq Ua$ for all a in P is not sufficient to guarantee that the semigroup P/U is embeddable into a group. This however follows from the assumption that P is a rank one left invariant cone: We show that left cancellation holds in P/U. Hence, assume that $UaUb = Uab = UaUc = Uac$ for a, b, c in P and, say, $c = db$ for d in P. Then $Ua = Uad$. We want to prove that this implies $d \in U$ and hence $Uc = Ub$.

If $d \in J(P)$ and $ad = d'a$ by the left invariance of P for some d' in P, then $Uad = Ud'a = Ua$ implies d' in U. Hence, $Pa = Pd'a = Pd'^n a = Pad^n \subseteq Pd^n$ for any n, which contradicts the assumption that P is archimedean. With Hölder's Theorem [7] we obtain the following result:

Theorem 2.4 *Let P be a left cone in G so that $J(P)$ is the only completely prime ideal of P and that $J(P) \supset \emptyset$ is a left invariant prime segment. Then P is left invariant, $P/U(P) = \overline{P}$ exists and is order isomorphic to a subsemigroup of $(\mathbb{R}, +)$.*

There are no similarly complete results for rank one left cones P for which the prime segment $J(P) \supset \emptyset$ is simple or exceptional. For examples and further results see [4], see also Example 4.2.

3 Left cones that are not cones

The authors in [1] rephrase a problem that Frege left unresolved in [6], Band 2, p. 172, as follows: Does there exist a group G with a non-empty subset H so that the following three conditions

α): $p, q \in H$ implies $pq \in H$;
β): $1 \notin H$;
γ): $p, q \in H$ implies $p = q$ or $pq^{-1} \in H$ or $qp^{-1} \in H$;
are satisfied, but the following fourth condition δ) is not satisfied?
δ): $p, q \in H$ implies $p = q$ or $q^{-1}p \in H$ or $p^{-1}q \in H$.

They construct such a subset H in the free group of rank 2 by considering regularly acting automorphisms of a non-linear upper semilinear set, which in turn is a set of finite subsets of the free group of rank 2.

We establish first a relationship between subsets H of a group G with α), β) and γ) and left cones P of G.

Proposition 3.1 *Let $G \neq \{1\}$ be a group. Then:*

a): *Let P be a left cone of G with $U(P) = \{1\}$. Then $H = J(P)$ generates G and satisfies α), β), γ); condition δ) is satisfied by H if and only if P is a right cone.*

b): *If H is a subset of G that satisfies α), β), γ), and generates G, then $P = H \cup \{1\}$ is a left cone of G with $U(P) = \{1\}$.*

c): *If P is a left cone of G and Q is a right and left cone of $U(P)$ with $U(Q) = \{1\}$, then $H = J(P) \cup J(Q)$ generates G and satisfies $\alpha)$, $\beta)$, and $\gamma)$. If P is not a right cone, H does not satisfy $\delta)$.*

Proof The proofs of a) and b) follow from the definitions. The statement c) follows from a) if $U(P) = \{1\}$. Otherwise, $J(Q)$ generates $U(P)$ and $H = J(P) \cup J(Q)$ generates G. To show that $\alpha)$ is satisfied by H it is sufficient to consider $p \in J(P)$ and $q \in J(Q) \subseteq U(P)$. Then pq, $qp \in J(P)$. To prove that condition $\gamma)$ is true for H let $p \neq q$ be elements in H. If both p and q are in $J(Q)$, then either pq^{-1} or qp^{-1} is in $J(Q)$. If p is in $J(P)$ and q is in $J(Q)$, then pq^{-1} is in $J(P) \subseteq H$. Finally, if both p and q are in $J(P)$, then either $p = cq$ or $q = cp$ for c in $J(P)$ or $p = uq$ for $u \in U(P) = Q$. In this last case it follows that u or u^{-1} is in $J(Q)$ which proves $\gamma)$ for H. If P is not a right cone, there exist $p, p' \in P$ with $p^{-1}p', p'^{-1}p \notin P$. It follows that $p \neq p'$ in $J(P) \subseteq H$ and H does not satisfy $\delta)$. □

We will now give several examples of subsets H in groups G that satisfy $\alpha)$, $\beta)$, $\gamma)$ and not $\delta)$. The first of these examples for the wreath product of two infinite cyclic groups was obtained during the construction of left valuation rings in [2].

Example 3.2 Let $B = \sum_{i \in \mathbb{Z}} \langle t_i \rangle$ be the direct sum of infinite cyclic groups $\langle t_i \rangle$. Then B can be ordered lexicographically, contains the subgroup $B_0 = \sum_{i=0}^{\infty} \langle t_i \rangle$, and admits the automorphism σ with $\sigma(t_i^{r_i})_{i \in \mathbb{Z}} = (t_{i+1}^{r_i})_{i \in \mathbb{Z}}$. The group G is the semidirect product $G = B \rtimes \langle y \rangle$ of (B, σ) with an infinite cyclic group $\langle y \rangle$; hence, the elements of G have the form by^n for $b \in B$, $n \in \mathbb{Z}$, and $yb = \sigma(b)y$.

The subset $H = \{by^n \in G | n > 0 \text{ and } b \in B_0, \text{ or } n = 0 \text{ and } e < b \in B_0\}$ satisfies the conditions $\alpha)$, $\beta)$, and $\gamma)$, where we observe that $p = p'q$ for $p, p', q \in P$ if and only if $p = b_1 y^{n_1}$, $q = b_2 y^{n_2}$ and either $n_1 > n_2$ or $n_1 = n_2$ and $b_1 > b_2$ in B_0.

In addition, $pq = (b_1 y^{n_1})(b_2 y^{n_2}) = b_1 \sigma^{n_1}(b_2) \sigma^{n_2}(b_1)^{-1} y^{n_2} b_1 y^{n_1} = q'p$ for any $p, q \in P$ and $q' = b_1 \sigma^{n_1}(b_2) \sigma^{n_2}(b_1)^{-1} y^{n_2} \in P$.

On the other hand, $p = y$ and $q = t_0 y$ (with $t_0 = (t_i^{r_i})_{i \in \mathbb{Z}}$, $r_0 = 1$, $r_i = 0$ otherwise) are elements in P, but $y \neq t_0 y p'$ and $t_0 y \neq y p'$ for all $p' \in H$, i.e. $\delta)$ does not holds for H.

Given a group G generated by a subset H that satisfies the conditions $\alpha)$, $\beta)$, and $\gamma)$. Then a binary relation $<$ can be defined on G by the rule that $a < b$ for a, b in G if and only if $b = pa$ for some $p \in H$. It follows immediately from the definition that $a < b$ implies $ag < bg$ for any g in G. The relation $<$ is called *limp*, i.e. left invariant under multiplication by positive elements, if $a < b$ for a, b in G and q in H implies $qa < qb$. Finally, the relation $<$ is called *dense* if H does not have a smallest element.

The relation defined in this way in Example 3.2 is not dense, since t_0 is the smallest element, but it is limp: $a < b$ for a, b in G implies $b = pa$ for p in H, and $qb = qpa = p'qa$ for q in H and a suitable q' in H.

To obtain a group G with a subset H that satisfies $\alpha)$, $\beta)$, $\gamma)$ but not $\delta)$ so that the corresponding relation $<$ is limp and dense, we modify the above example.

Example 3.3 The group $B = \sum_{i \in \mathbb{Q}} \langle t_i \rangle$ with $\langle t_i \rangle$ infinite cyclic for all i can again be ordered lexicographically. B contains the subgroup $B_0 = \sum_{0 < i \in \mathbb{Q}} \langle t_i \rangle$ and admits the automorphism σ with $\sigma(t_i^{r_i}) = (t_{i+1}^{r_i})$. Then G and H are defined as in Example 3.2.

In order to obtain subsets H in free groups F that satisfy conditions α), β), γ), but not δ), we combine the construction of left cones P in F given by P.M. Cohn in [5], p. 534/5, with Proposition 3.1.c).

Example 3.4 Let $X = \{x_\alpha | \alpha < \tau\}$ be a set of free generators of the free group F, where τ is an ordinal. Let U be the subgroup of F generated by the elements of the form

$$U_{\beta\alpha_1\alpha_2\ldots\alpha_r} = x_{\alpha_r}\ldots x_{\alpha_2}x_{\alpha_1}x_\beta x_{\alpha_1}^{-1}\ldots x_{\alpha_r}^{-1} \quad \text{with} \quad \beta < \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r.$$

Then P, the submonoid of F generated by U and X, is a left cone that is not a right cone if $|X| \geq 2$. Every element p in P can be written uniquely in the form $p = ux_{\alpha_r}\ldots x_{\alpha_2}x_{\alpha_1}$ with $u \in U = U(P)$, and $\alpha_r \leq \cdots \leq \alpha_2 \leq \alpha_1$. For u in U and x_α there exists u' in U with $x_\alpha u = u'x_\alpha$, and $x_\alpha x_\beta = (x_\alpha x_\beta x_\alpha^{-1}))x_\alpha = u_1 x_\alpha$ for $\beta < \alpha$ and $u_1 = x_\alpha x_\beta x_\alpha^{-1} \in U$. There does not exist an element h in P with $x_1x_2 = x_2h$ or $x_2 = x_1x_2h$. Since F can be linearly ordered, so can the subgroup U of F. If $\Pi = \{u \in U = U(P) | u > e\}$, then $H = J(P) \cup \Pi$ is a subset of F that satisfies α, β, and γ, but not δ).

4 Cones from left cones

A group G which has a right and left cone P with $U(P) = \{e\}$, i.e. a right ordered group, is in general not linearly ordered, it does not have a cone P' with $U(P') = \{e\}$ and $gP'g^{-1} = P'$ for all g in G. However, the following arguments, see [10], show that a group G with left cone P and $U(P) = \{e\}$ is right orderable: If $x_1, \ldots, x_n$ are elements $\neq e$ in G, then $x_i = q^{-1}p_i$ for some elements q, p_i in P. Hence, there exists an index i_0 with $Pp_{i_0} \subseteq Pp_i$ for $i = 1, \ldots, n$, and it follows that for $h = p_{i_0}$ the elements h and hx_i^{-1} are in P for all i. Then, either $Ph \subseteq Phx_i^{-1}$ and $hx_ih^{-1} \in P$, or $Phx_i^{-1} \subseteq Ph$ and $hx_i^{-1}h^{-1} \in P$. It follows that there exist $\varepsilon_i \in \{1, -1\}$ so that $hx_i^{\varepsilon_i}h^{-1} \in J(P)$ and that the sub-semigroup of G generated by the $hx_i^{\varepsilon_i}h^{-1}$ does not contain e. Hence, the sub-semigroup generated by the $x_i^{\varepsilon_i}$ does not contain e, and, by Conrad's criterion, G contains a right and left cone with $\{e\}$ as group of units $-G$ is right orderable.

We obtain:

Proposition 4.1 *If P is a left cone in the group G so that $U = U(P)$ is a normal subgroup of G, then G contains a right and left cone Q with $U(Q) = U$.*

To prove this we apply the above results to the factor group G/U.

However, there exist cones P in groups G so that $U(P)$ is not normal in G.

Example 4.2 Let $G = \langle x \rangle \ltimes_\sigma \{t^r | r \in \mathbb{R}\}$ be the semidirect product of the infinite cyclic group generated by x and the group $\{t^r | r \in \mathbb{R}\}$ isomorphic to $(\mathbb{R}, +)$ with $t^r x = xt^{2r}$, i.e. $\sigma(t^r) = t^{2r}$. Then $P = \{x^n t^r | n \in \mathbb{Z},\ r \geq 0\}$ is a cone of G with $U(P) = \langle x \rangle$. However, $txt^{-1} = xt \notin U(P)$ shows that U is not normal in G.

Let F be a skew field and R a left valuation ring of F. Then there is the following easy but somewhat surprising result:

Proposition 4.3 *Let R be a left valuation ring of the skew field F. Then R is an invariant valuation ring of F if and only if $U(R)$ is a normal subgroup of F^*.*

Proof If R is an invariant valuation ring of F, then $dRd^{-1} = R$ and $dU(R)d^{-1} = U(R)$ follows for $d \in F^*$; hence, $U(R)$ is normal in F^*. Conversely, assume that $U(R)$ is normal in F^*. An element r in R is either in $U(R)$ or in $J(R)$. In the last case, $1 + r \in U(R)$, hence, $d(1 + r)d^{-1} = 1 + drd^{-1} \in U(R)$, and $drd^{-1} \in R$. It follows that $dRd^{-1} \subseteq R$ for all d in F^* and $dRd^{-1} = R$. □

This last result is not true for left cones P in groups G or even cones P in G : The cone P, with $U(P) = \{e\}$ normal in G, of a right orderable group that is not linearly orderable, can serve as an example.

We consider one more case where the existence of a left cone P in a group G implies the existence of a cone $\widetilde{P}$ in G; however $U(\widetilde{P})$ is the normal closure of $U(P)$ in this case. Let P be a left cone in the group G so that every left ideal of P is an ideal; we say that P is left invariant. In this case $aP \subseteq Pa$ for all a in P and $P \subseteq a^{-1}Pa \subseteq a^{-1}b^{-1}Pba$ for a, b in P.

Lemma 4.4 (i) *Assume that P is a left invariant left cone of a group G. Then $\widetilde{P} = \cup a^{-1}Pa$, $a \in P$, is an invariant cone of G with $\widetilde{U} = U(\widetilde{P})$ the normal closure of $U(P)$.* (ii) *Assume that R is a left invariant left valuation ring of a skew field F. Then $\cup d^{-1}Rd = \widetilde{R}$, $d \in R^*$, is an invariant valuation ring of F with $\Gamma = F^*/\widetilde{U}$ as valuation group, where $\widetilde{U} = \cup d^{-1}U(R)d$, $d \in R^*$.*

Proof It is enough to prove (i). It follows from the observation before the lemma that the left cones $a^{-1}Pa$, $a \in P$, form a chain and $\widetilde{P}$ is therefore a left cone of G. If $\widetilde{p}$ is any element in $\widetilde{P}$, then $\widetilde{p} \in a^{-1}Pa$ for some a in P and $d^{-1}\widetilde{p}d$, $d\widetilde{p}d^{-1} \in \widetilde{P}$ follows for d in P. Since P generates G we have $g^{-1}\widetilde{P}g = \widetilde{P}$ for all g in G and all right ideals of P are ideals; $\widetilde{P}$ is a cone. Any unit of $\widetilde{P}$ is a unit in some $a^{-1}Pa$, hence contained in some $a^{-1}Ua$ for a in P. □

It appears to be a difficult problem to construct for a group G with left cone P a skew field F and a left valuation ring R of F so that $G \subseteq F^*$, $R \cap G = P$, and for every element $0 \neq r$ in R exist $u \in U(R)$ and $p \in P$ with $r = up$. Every principal left ideal $\neq (0)$ of R has then the form Rp for some $p \in P$ and $Rp_1 = Rp_2$, for $p_1, p_2 \in P$, only if $Pp_1 = Pp_2$. It follows that then there is a correspondence between the left ideal $\neq (0)$ of R and the left ideals of P. We say that R and P are associated if in this correspondence ideals, completely prime ideals and prime ideals correspond to ideals, completely prime ideals, and prime ideals respectively.

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