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RIGHT INVARIANT RIGHT HOLOIDS

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Right invariant right holoids H (r.i.r. holoids) are totally (and positively) ordered semigroups such that $a > b$ holds if and only if $a = bc$ for some $c \neq e$, e the identity of H. These holoids occur as semigroups of the prinicipal right ideals of right invariant right chain rings. We investigate in which way r.i.r. holoids of finite rank are built up from r.i.r. holoids of rank one which are known to be subsemigroups of the non-negative real numbers under addition. This is best described by conditions on $f(C, a)$ where C is a prime segment which is shifted over elements $a \in H$. The functional properties of $f(C, a)$ are studied, especially in the finite-rank-case. These results are then applied to the extension problem. Here, conditions are given under which the extension splits, however even under these assumptions an additional problem occurs. An element s is called a denominator for b if a solution x exists in H with $xs = b$. It is crucial to know the denominators sets and solution sets. Under certain condition it is possible to embed H into a r.i.r. holoid H' with larger sets of denominators.

1 Introduction

With every commutative valuation domain is associated a totally ordered group which is isomorphic to the group on non-zero principal (fractional) ideals of this ring. A right chain domain is an integral domain (not necessarily commutative) whose right ideals are linearly ordered by set inclusion. The non-zero principal right ideals of such a ring R will form a semigroup $H(R)$

with respect to multiplication if and only if R is right invariant, i.e. all right ideals are two-sided [3]. Such a semigroup $H(R)$ satisfies the following axioms:

(A) H is left cancellative: $hh_1 = hh_2$ implies $h_1 = h_2$ for $h_1, h_2, h \in H$.

(B) H is a right chain semigroup: For any elements $h_1, h_2 \in H$ there exists $h \in H$ with $h_1h = h_2$ or $h_2h = h_1$.

(C) If there exist elements $h', h'' \in H$ with $h_1h' = h_2$ and $h_2h'' = h_1$ for $h_1, h_2 \in H$, then $h_1 = h_2$.

We remark that (B) implies the existence of an identity.

Definition 1.1 A semigroup (with identity e) satisfying axioms (A), (B) and (C) is called a right holoid (r. holoid for short).

SCHEIN [8] calls such semigroups left holoids whereas SKORNYAKOV [9] uses the term valuation semigroups, however, his semigroups are not only left cancellative, but also right cancellative.

As these semigroups are originally derived from ring structures the condition of right invariance of a ring R leads to the following definition.

Definition 1.2 A semigroup H is called right invariant (r.i.) if $Ha \subseteq aH$ for all $a \in H$.

It can easily be verified that a r.i.r. holoid becomes a totally ordered semigroup by setting $a \leq b$ iff $b \in aH$. Finally H is positively ordered (see e.g. [7]), the only unit of H is e itself, $H \backslash \{e\}$ is the unique maximal right ideal which is the maximal two-sided ideal and which is completely prime.

We leave it to the reader to check the equivalence of the two descriptions:

Lemma 1.3 A left cancellative semigroup H with identity e is a r.i.r. holoid if and only if H is a right naturally totally ordered semigroup.
Semigroups of this type have been considered by SATYANARAYANA [7, Chap 3.] Earlier results about related structures can be found in CLIFFORD [4], CONRAD [5] and KLEIN-BARMEN [6].

Basic for the understanding of r.i.r. holoids is the fact that archimedian r.i.r. holoids are isomorphic to subsemigroups of $(\mathbb{R}^+, +)$ the non-negative real numbers under addition. This is essentially HÖLDER's result, and describes the rank one case if we define the rank of a r.i.r. holoid H as equal to the number of convex subsemigroups $\neq \{e\}$ of H, where e is the identity of H.

The structure of rank two r.i.r. holoids is known as well as the structure of right noetherian r.i.r. holoids, see [1].

If $H_{n-1} \subset H_n$ are neighbours in the chain of convex subsemigroups of a r.i.r. holoid H then $C_n = H_n \backslash H_{n-1}$ is called a prime segment. We investigate in Section 3 in which way prime segments are shifted over elements. These results can be applied to show that for r.i.r. holoids of finite rank $H_{n-1}a \subseteq aH_{n-1}$ for any $a \in H_n$ (Section 4). The factor semigroup H_n/H_{n-1} can be formed which is a semigroup of $(\mathbb{R}^+, +)$. The following problem is considered in Section 5: In which way can H_n be described in terms of H_{n-1} and H_n/H_{n-1} (*extension problem*)? Some conditions are given that assure that a semigroup R of representatives for H_n/H_{n-1} exists in H_n, see Theorems 5.1 and 5.2. The existence of a semigroup R of representatives is (unlike the group case) not sufficient to describe H_n in terms of H_{n-1} and $H_n/H_{n-1} \cong R$ even if one knows in which way the elements of H_{n-1} shift over the elements of R.

An additional difficulty arises because of the possible presence of elements $s \neq e$ in H_{n-1} such that $b = xs$, $b \in R$, has a solution $x \in H$. Such an element s is called a *denominator* for b and results on denominator sets are obtained in Section 6. Related with this question are embedding problems. A construction theorem and some examples are considered in the final section.

2 Basic definitions and preliminary results

In the following H will always denote a r.i.r. holoid. We recall the definiton of a prime ideal resp. a completely prime ideal from [7, p. 2]. We list a few properties for later reference which partly can be found in [7, Theor. 3.47].

Lemma 2.1 Let H be a r.i.r. holoid.

(i) Each prime ideal $P \subseteq H$ is completely prime.

(ii) A convex subset $P \neq \emptyset$ of H is a prime ideal if and only if $a \in H \backslash P$ implies $a^2 \in H \backslash P$.

(iii) For any $t \in H$ the intersection $\bigcap t^n H = P$ is a prime ideal provided $P \neq \emptyset$.

Using prime ideals we define the rank of a holoid.

Definition 2.2 Let H be a r.i.r. holoid. H is said to be of rank n if there exist exactly $n+1$ prime ideals including H with $H = P_0 \supset P_1 \supset \cdots \supset P_n$.

Sometimes it is more convenient to work with prime ideals rather than with convex subsemigroups. Obviously there is a natural correspondence between convex semigroups and (completely) prime ideals. In the following we use the abbreviations $S_{i-1} = H\backslash P_i$ and $C_{i-1} = P_{i-1}\backslash P_i$ for $i = 1, \cdots, n$ and $C_n = P_n$.

Since only the left cancellation law is assumed for r.i.r. holoids the following concept plays a central role.

Definition 2.3 Let H be a r.i.r. holoid, $a \in H$. Then $E(a) = \{x \in H \mid xa = a\}$ is called the set of a - absorbed elements and $A(H) = \{a \in H \mid E(a) \neq \{e\}\}$ is the absorber radical of H.

Without proof we list a few properties of the "absorbing" process:

Lemma 2.4 Let H be a r.i.r. holoid, $a \in H\backslash\{e\}$. The following holds:

(i) $E(a)$ is a convex subsemigroup of H not containing a.

(ii) $E(a) \subset E(ab)$, $E(ba)$.

(iii) $A(H)$ is a (completely) prime ideal of H.

(iv) Let $s \in H\backslash A(H)$ and $as = bs$, then $a = b$.

The set $M = H\backslash A(H)$ forms a right Ore set in H and the semigroup $HM^{-1} = \{as^{-1} \mid a \in H, s \in M\}$ exists. The subset $H = \{as^{-1} \mid a \in J, s \in M, a \geq s\}$ is a subsemigroup of HM^{-1} which satisfies the statements in the following theorem. This result is not used in the remainder of the section, so the details of the proof are omitted.

Theorem 2.5 Let H be a r.i.r. holoid. Then there exists a r.i.r. holoid $\widehat{H}$ of the same rank as H in which H is embedded such that $\widehat{H}\backslash A(\widehat{H})$ is the positive cone of an ordered group and $A(H) = A(\widehat{H}) \cap H$.

3 Prime segments

Definition 3.1 Let H be a r.i.r. holoid and $P_1 \subset P$ neighbours in the chain of (completely) prime ideals of H. Then we say that $C = P\backslash P_1$ is a prime segment of H. If H has a minimal prime ideal P then P itself is called a prime segment, we allow $P_1 = \emptyset$.

Lemma 3.2 Let H be a r.i.r. holoid. Then every element $t \in H$ is associated with a prime segment $C(t)$.

PROOF: If $t = e$, then $C(e) = \{e\}$. Otherwise let $P = \bigcap P_i$, P_i prime then with $t \in P_i$ and $P_1 = \bigcap t^n H$. It is clear that $P \supset P_1$ and that for a prime ideal Q with $Q \supset P_1$ there exists n with $t^n \in Q$, and hence $t \in Q$ as Q is prime, thus $P \subseteq Q$. Set $C(t) = P \backslash P_1$. ■

We order the prime segments by writing $C < C'$ if and only if $C \neq C'$ and $sH \supset tH$ for $s \in C$, $t \in C'$.

For every $x, a \in H$ there exists a uniquely determined element $x' \in H$ with $xa = ax'$. It is evident that these *commutation rules* contain valuable informations on H, hence it would be desirable to understand the mappings φ_a from H to H that send x to x'. However, we describe in this section what happens to the prime segment $C(x)$ instead of the element x under φ_a. We are already familiar with one extreme case where a segment C is contained in $E(a)$ and $Ca = a$ follows.

Lemma 3.3 Let H be a r.i.r. holoid, $C = P \backslash P_1$ a prime segment and $a \in H$. Then there exists a prime segment $C' = P' \backslash P_1'$ with $Ca \subseteq aC'$.

PROOF: If $sa = a$ for some $s \in C$, then $C \subseteq E(a)$ and $Ca = a$. The statement follows with $C' = \{e\}$. Otherwise $sa = as_1$ for all $s \in C$ and $s_1 \neq e$. If there exist prime ideals P_1', Q and P' with $P_1' \subset Q \subset P'$ and $s_1 a = at_1$, $t_1 \in P' \backslash Q$ and $s_2 a = at_2$, $t_2 \in Q \backslash P_1'$ for $s_1, s_2 \in C$, we can assume that $s_1^n H \subseteq s_2 H$, $s_1^n = s_2 r$ say. We obtain $s_1^n a = at_1^n$ and $s_1^n a = s_2 ra = at_2 r'$ for some $r' \in H$. Hence, $t_1^n = t_2 r'$, a contradiction, since $\bigcap t_1^n H \supseteq Q$. ■

Corollary 3.4 Let H be a r.i.r. holoid, C a prime segment of H, $a \in H$. If for some $s \in C$ we have $sa \in aC$ then $Ca \subseteq aC$.

Lemma 3.2 allows us to define a function $f(a, -)$ for every $a \in H$ from the set of prime segments to itself by setting $f(a, C) = C'$ if and only if $Ca \subseteq aC'$.

Let $\{C_\lambda \mid \lambda \in \wedge\}$ be the set of prime segments such that $C_\alpha < C_\beta$ if and only if $\alpha < \beta$. Then we occasionally write $f(a, \lambda) = \gamma$ instead of $f(a, C_\lambda) = C_\gamma$. This function is almost injective; exept that, of course, prime segments in $E(a)$ are all mapped to $\{e\}$:

Lemma 3.5 Let H be a r.i.r. holoid, $C_1 \neq C_2$ two distinct prime segments in H. Then $f(a, C_1) = f(a, C_2)$ for some $a \in H$ implies $f(a, C_1) = \{e\}$.

PROOF: We can assume that every element in C_2 is larger than every element in C_1; let $s \in C_1$, $t \in C_2$ with $sa = as_1$, $ta = at_1$ with $s_1 \neq e \neq t_1$ in the same prime segment of H. This implies that there exists an integer n with $s_1^{n+1} H \subset$

$t_1H \subseteq s_1^nH$. It follows from the choice of s and t that $tH \subset s^{n+1}H$, so there exists $r \in H$ with $s^{n+1}r = t$. We obtain $s^{n+1}ra = s^{n+1}ar' = as_1^{n+1}r' = at_1r''r'$ where $ra = ar'$ and $s_1^{n+1} = t_1r''$ with $r', r'' \in H$, $r'' \neq e$. On the other hand: $s^{n+1}ra = ta = at_1$, leading to a contradiction. ∎

The following is an obvious Corollary to Lemma 3.4:

Corollary 3.6 If $C_1 < C_2$ for prime segments C_i in the r.i.r. holoid H, then $f(a, C_1) \leq f(a, C_2)$ and $f(a, C_1) < f(a, C_2)$ if $f(a, C_2) \neq \{e\}$.

Theorem 3.7 Let H be a r.i.r. holoid with minimum condition for prime ideals in H and $a \in H$. Then $f(a, C) \leq C$ for all prime segments C of H.

PROOF: Let us assume that the statement is wrong, so $f(a, C) = C_1 > C$ for some prime segment $C = C_0$. It follows from Lemma 3.5 that $f(a, C_i) < f(a, C_{i+1})$ where $f(a, C_i) = C_{i+1}$ is defined inductively. This is impossible if the minimum condition holds for prime ideals. ∎

Theorem 3.8 Let H be a r.i.r. holoid with maximum condition for prime ideals. Let a be an element in H and C be a prime segment then $f(a, C) \leq C$ implies either $f(a, C) = C$ or $f(a^n, C) = \{e\}$ for a suitable power of a.

PROOF: We can assume $C_1 = f(a, C) < C$. Obviously $f(a, C_1) = f(a^2, C)$. It follows from our assumption and Corollary 3.6 that $f(a^n, C) = \{e\}$ for a sufficiently large n. ∎

We end this section with an example that shows that the finiteness condition in Theorem 3.7 is necessary.

Example 3.9 Let $F = \mathbb{R}(t)$ be the function field in one indeterminate over the reals and let $\alpha = (a_0t^n + \cdots + a_{n-1}t + a_n)(b_0t^m + \cdots + b_m)^{-1}$ be an element in F with $a_0b_0 \neq 0$. We write $\alpha > 0$ if and only if $a_0b_0 > 0$. Consider the set H of all pairs (a, b) with $a \geq 1$, $b \in F$ and $b \geq 0$ if $a = 1$. H is a r.i.r. holoid if we define the operation $(a, b)(a', b') = (aa', ba' + b')$, the order of H will be the lexicographical order. H is the set of non-negative elements of an ordered group. Define sets $P_i = \{x \in H \mid x > (1, \alpha t^{i-1}) \text{ for all } \alpha \in \mathbb{R}\}$, $i \in \mathbb{Z}$. The set $H \backslash P_i$ is obviously convex and we want to show that it is a semigroup. Let $h_1 \leq (1, \alpha_1 t^{i-1}), h_2 \leq (1, \alpha_2 t^{i-1})$ for some $\alpha_1, \alpha_2 \in R$. Then $h_1h_2 \leq (1, \alpha_1 t^{i-1})(1, \alpha_2 t^{i-1}) = (1, (\alpha_1 + \alpha_2)t^{i-1})$ and $h_1h_2 \in H \backslash P_i$ follows. The P_i's are prime ideals with $P_i \supset P_{i+1}$. One can show that $P_i \backslash P_{i+1} = C_i$ is the prime segment containing $(1, t^i)$. Choose $a = (t, 0) \in H$. We have $(1, t^i)(t, 0) = (t, 0)(1, t^{i+1})$ and $f(a, C_i) = C_{i+1} > C_i$.

4 Right invariant right holoids of finite rank

The results of [3] apply in particular to those r.i.r. holoids which have finite rank.

Lemma 4.1 Let H be a r.i.r. holoid of rank n, $a \in H$ and $i \in \{0, \cdots, n\}$. Then we have: $f(ar, i) \leq f(a, i)$ for all $r \in H$. Moreover, $f(a, i) = i$ or $f(a^k, i) = 0$ for some $k \leq i$.

The proof follows immediately from Theorems 3.7 and 3.8. We study in the next result the way in which $f(b, i)$ changes as b ranges over a prime segment C_k.

Theorem 4.2 Let H be a r.i.r. holoid of rank n and C_k a prime segment of H. Then $f(b, i) = i$ for all $b \in C_k$ or $f(b, i) < i$ and hence $f(b^i, i) = 0$ for $b \in C_k$.

PROOF: Using Lemma 4.1, it is enough to show that $f(b, i) = i$ for all $b \in C_k$ whenever $f(a, i) = i$ for some $a \in C_k$. Let a be such an element and $b \in C_k$. By Lemma 2.1(iii) there exists a natural number m with $a^m H \subset bH \subseteq a^{m-1}H$. Thus $a^m = br$ for some $r \in H$ and $i = f(a^m, i) = f(br, i) \leq f(b, i) \leq i$. ∎

Since $f(b, i) \leq i$ for all $b \in H$ we can construct factor holoids from a given r.i.r. holoid of finite rank. More generally we have the following result.

Theorem 4.3 Let H be a r.i.r. holoid, $P = H \backslash S$ a prime ideal in H with $Sa \subseteq aS$ for all $a \in P$. Then there exists a factor semigroup H/S of H which is again a r.i.r. holoid, and a surjective homomorphism $\varphi : H \to H/S$ with $\varphi(a) = \varphi(b)$ if and only if $as = b$ or $bs = a$ for some $s \in S$. The maximal prime ideal in H/S is the image of P under φ.

PROOF: We consider the set of all equivalence classes $[a] = \{b \in H \mid as = b$ or $a = bs$ for some $s \in S\}$. This set with the operation $[a][b] = [ab]$ is a r.i.r. holoid H/S and $\varphi(a) = [a]$ is a surjective semigroup homomorphism. The details can be checked easily. ∎

Corollary 4.4 Let H be a r.i.r. holoid of finite rank, $P = H \backslash S$. Then we have rank(H/S) =rank$(H)-$rank(S).

The assumptions of Theorem 4.3 are satisfied since $f(b, i) \leq i$ holds if H has finite rank. If $\{e\} = S_0 \subset S_1 \subset \ldots \subset S_k = S \subset S_{k+1} \subset \ldots \subset S_n = H$ are exactly the convex subsemigroups of H, then $\varphi(S) = \varphi(e) \subset \varphi(S_{k+1}) \subset \cdots \subset \varphi(S_n) = \varphi(H)$ are exactly the convex subsemigroups of H/S.

Definition 4.5 Let H be a r.i.r. holoid, P a prime ideal in H. We say that P is discrete if $P^2 \neq P$. We say that H is discrete if all proper prime ideals in H are discrete.

Theorem 4.6 Let H be a r.i.r. holoid of finite rank, $P = P_k$ a non-discrete prime ideal. Then for $i < k$ either $f(p, i) = 0$ for all $p \in P$ and $S_i p = p$ or $f(b, i) = i$ for all $b \in P_k \backslash P_{k+1} = C_k$.

PROOF: We pick $b \in C_k$ and $i < k$. Since P is not discrete we have $P = P^k$. This means that we can find $b_j \in C_k$ with $b = b_1 \cdots b_k$. If $b_{i_0}H \supseteq b_j H$ for a certain i_0 and $j = 1, \cdots, k$ then $b_j = b_{j_0} r_j$ for some $r_j \in H$ and $b = b_{i_0}^k r$ for a certain $r \in H$. By Theorem 4.2 we have either $f(b, i) = f(b_{i_0}, i) = i$ or $f(b, i) < i$ and $f(b_{i_0}, i) < i$. But this implies $f(b, i) = f(b_{i_0}^k r, i) \leq f(b_{i_0}^k, i) = 0$ and $C_i b = b$ follows. This implies $S_i p = p$ for all $p \in P$ and proves the theorem. ∎

5 The extension problem

Let H be a r.i.r. holoid of finite rank n and the notation as in Section 4. It follows from Corollary 4.4 that the factor semigroup H/S_{n-1} exists and is a r.i.r. holoid of rank 1. By the classical result of HÖLDER this semigroup is isomorphic to a subsemigroup of $(\mathbb{R}^+, +)$, the non-negative real number under addition. If $P = P_n \neq P_n^2$ is discrete it follows that an element $[p]$ with p in $P \backslash P^2$ will be a least positive element in H/S_{n-1}, since otherwise there is an element $p_1 \in P$ with $e \neq [p_1] < [p]$ and $[p_1][p_2] = [p]$ with $p_2 \in P$. Then $p = p_1 p_2 s$ or $ps = p_1 p_2$ for some $s \in S_{n-1}$ and $p \in P^2$ in the first case. In the second case we have either $p_1 rs = p_1 p_2$, so $rs = p_2$, $r \in P$ and $p \in P^2$ or $ps = prp_2$, so $s = rp_2 \in P$ - a contradiction in every case. However, a r.i.r. holoid which is a subsemigroup of $(\mathbb{R}^+, +)$ with a least positive element is isomorphic to the semigroup of non-negative integers under addition. The semigroup H/S_{n-1} is equal to $\{[p]^n = [p^n] \mid n = 0, 1, 2, \cdots\}$ and $\{p^n \mid n = 0, 1, 2, \cdots,\}$ is a semigroup of representatives of H/S_{n-1} in H. We have proved the following result.

Theorem 5.1 Let H be a r.i.r. holoid of finite rank n, $P = P_n$ the minimal prime ideal of H, $P \neq P^2$ and $S = H \backslash P$. Then there exists in P an element p such that for every element $h \in H$ there is a unique non-negative integer n and a unique element $s \in S$ with either $h = p^n s$ or $hs = p^n$.

Again, let H be a r.i.r. holoid of finite rank n with minimal prime ideal $P = P_n$ and $S = S_{n-1} = H \backslash P$. We now consider the case $P = P^2$ and – using Theorem 4.6 – have the following two possibilities: Either $Sp = p$ for all $p \in P$ or $n \geq 2$ and there exists an integer k with $1 \leq k \leq n$ with $S_{k-1} p = p$, but $f(p,i) = i$ for $k \le i < n$ and all $p \in P$. The next theorem deals with the first of these possibilities guarenteeing a semigroup of representatives for H/S.

Theorem 5.2 Let H be r.i.r. holoid of finite rank n, $P = P_n$ the minimal prime ideal of H and $E(p) = S = H\setminus P$ for every $p \in P$. Then there exists in H a semigroup R of representatives for H/S which is isomorphic to a subsemigroup of $(\mathbb{R}^+, +)$.

PROOF: The r.i.r. holoid H/S has rank one and is therefore embeddable into $(\mathbb{R}^+, +)$; in particular, H/S is commutative. If $(H/S)\setminus[e]$ contains a least element, P is discrete and this situation was already analyzed in Theorem 5.1. Otherwise we choose $p \in P$ arbitrarily. Let $[a] \ge [q]$ in H/S and there exists an element $r \in H$ with $[a] = [q][r] = [r][q]$. If $r_1 q$ is equivalent to $r_2 q$ with respect to the equivalence relation defined by S on H, i.e. if $[r_1 q] = [r_1][q] = [r_2 q] = [r_2][q]$ in H/S , then $[r_1] = [r_2]$ and so $r_1 \in r_2 S$ or $r_2 \in r_1 S$. This implies $r_1 q = r_2 q$ in every case. Therefore we choose rq as representative of $[a]$. The product $(rq)(sq) = (rqs)q$ is again an element of this form. Now let $\{[q_i]\}, [q_i] \ne [e]$, be a sequence of elements in H/S which is mapped to a sequence with limit equal to zero if H/S is embedded into $(\mathbb{R}^+, +)$. We assume further that $q_o = q$ and $[q_{i+1}] < [q_i]$. We derive another sequence $\{q'_i\}$ of elements in H from the sequence of the $[q_i]$ in the following way:

There exists an r_1 in H with $q_0 = q_1 r_1 = r_1 q'_1$ for some q'_1 in H. Obviously $[r_1][q_1] = [q_1][r_1] = [r_1][q'_1]$ so $[q_1] = [q'_1]$. We assume that q'_{i-1} has been defined with $[q'_{i-1}] = [q_{i-1}]$ and $q'_{i-2} = r_{i-1} q'_{i-1}$ we choose for q'_i the element in H with $q'_{i-1} = q_i r_i = r_i q'_i$ where r_i is in H. As before, $[q_i] = [q'_i]$.

Let $[a] > [e]$ be an element in H/S. Then there exists an i with $[a] > [q_i] = [q'_i]$, so $a = q'_i v$, $[a] = [q'_i][v] = [v][q'_i] = [vq'_i]$ for some $v \in H$. As before $[vq'_i] = [v'q'_i]$ implies $vg'_i = v'q'_i$. We choose vq'_i as the representative for $[a]$ if $[a] > [q'_i]$. If $[b] > [q'_i]$ has the representative wq'_i, then $vq'_i wq'_i$ is the representative of $[ab] = [a][b]$.

We write $R_i = \{vq'_i \mid [a] = [vq'_i] > [q'_i]\}$ for the set of representatives of elements $[a]$ with $[a] > [q'_i] = [q_i]$. Then $R_{i+1} \supset R_i$ follows, since for a representative $v_i q'_i$ of $[a]$ in R_i and $q'_i = r_{i+1} q'_{i+1}$ the element $v_i q'_i = v_i r_{i+1} q'_{i+1}$ is also the representative of $[a]$ in R_{i+1}. The union $R = \bigcup R_i \cup \{e\}$ is a semigroup of representatives for H/S in H. ∎

Let H be a r.i.r. holoid of finite rank n with minimal prime ideal $P = P_n$ and $S = S_{n-1} = H\setminus P$. Assume that a semigroup R of representatives for H/S exists in H. This is not enough to describe H in terms of S and R even if the mappings ϕ_p from S to S were known that send s to s' if $sp = ps'$ for $p \in R$. In addition one would have to know for which $s \in S$ the equations $p = xs$ have solutions in H and the actual solution sets.

A solution $h \in H$ of the equation $p = xs$ determines the element $s \in S$ and the element p in R uniquely. However, different elements h, $h' \in H$ may both satisfy $p = hs$, $p = h's$. If, say $ht = h'$, $t \in S$, then $ts = s$ and $t \in E(S)$ follows. Conversely, every element h' with $h' = ht$ or $h't = h$, $t \in E(S)$, satisfies $p = h's$ if $p = hs$. We will discuss these problems in the following section.

6 Denominators and quotient sets

We begin this section with a definition.

Definition 6.1 Let H be a r.i.r. holoid, P a prime ideal, $S = H \backslash P$ and $p \in P$. We write $N(p) = \{s \in S \mid \text{there exists } x \in H \text{ with } xs = p\}$ and say $N(p)$ is the set of denominators s of p and $L(p, s) = \{x \in H \mid xs = p\}$ is a quotient set.

The next result shows that in some instances the denominator sets do not depend on the chosen element p and that there is a one-to-one correspondence between quotient sets with the same s.

Theorem 6.2 Let H be a r.i.r. holoid of finite rank n, $P = P_n$ the minimal prime ideal of H and $S = H \backslash P$. Assume that there exists in H a semigroup R of representatives for H/S and let $p_1 \neq e \neq p_2$ be two elements in R. Then $N(p_1) = N(p_2)$ and there is a one-to-one mapping from $L(p_1, s)$ onto $L(p_2, s)$ for every $s \in N(p_1)$.

PROOF: We can assume that $p_1 < p_2$ in H and R. Then there exists an element p_2' in R with $p_1 p_2' = p_2$. However, R is commutative and $p_1 p_2' = p_2' p_1 = p_2$ follows. If we assume $s \in N(p_1)$ and $p_1 = xs$ for some $s \in H$ then obviously $p_2 = p_2' p_1 = (p_2' x)s$ and $s \in N(p_2)$. If $s' \in N(p_2)$ then $p_2 = ys'$ and $y = p_2' x$ for some $x \in H$ implies follows, since the other alternative $yr = p_2'$ implies $yrp_1 = p_2' p_1 = p_2 = ys'$, $rp_1 = s'$ in S, a contradiction, here r is a suitable element in H. Hence, $p_2 = p_2' p_1 = ys' = p_2' xs'$ and $p_1 = xs'$, $s' \in N(p_1)$. Thus $s \in N(p_1)$. ∎

The above arguments also show that with every $x \in L(p_1, s)$ the element $p_2' x$ is in $L(p_2, s)$. Distinct elements $x_1, x_2 \in L(p_1, s)$ correspond to distinct elements $p_2' x_1, p_2' x_2 \in L(p_2, s)$ and every element $y \in L(p_2, s)$ is of the form $y = p_2' x$ for some $x \in L(p_1, s)$. The denominator set $N(p)$, $p \in R$ is not uniquely determined by H and P. It varies with R, the semigroup of representatives of H/S, which is by no means unique as can easily be seen in the discrete case. Let H be a r.i.r. holoid of finite rank n, $P = P_n \neq P_n^2$ the minimal prime ideal, $S = H\backslash P$ and $p \in P\backslash P^2$. Then $R = \{p^n \mid n = 0, 1, 2, \cdots\}$ is a semigroup of representatives for H/S. In order to keep the denominator sets for the elements p^n small we try to choose p carefully, so that p "generates" the shifting process in respect to the function $f(p, -)$. By Lemma **4.1** one gets the idea to choose p as small as possible in the ordering of H. We introduce the following notation.

Definition 6.3 Let $P \neq P^2$ be a discrete prime ideal in a r.i.r. holoid H. We say the element $p \in P\backslash P^2$ is a typical shift locus for P if $f(p, C) = f(q, C)$ for every $q \in P$ with $pH \subseteq qH$ and any prime segment $C \subset H\backslash P$.

We observe that a similar definition for P non-discrete is not necessary because of Theorem 4.6.

Proposition 6.4 Let H be a r.i.r. holoid of finite rank n, $P \neq P^2$ its minimal ideal. Then there exists a typical shift locus for P.

PROOF: If $P = pH$ we choose the generator p as typical shift locus for P and any $p \in P\backslash P^2$ can be chosen in case $E(p) = S = H\backslash P$ for all $p \in P$. In all other cases consider $T = P\backslash P^2$ and choose q in T with i_1 minimal such that $f(q, i_1) = 0$ and $f(q, i_1 + 1) > 0$. If q' is in P with $f(q', i_1') = 0$ and $i_1' > i_1$, then $q'H \subset qH$, since otherwise $q'r = q$ which implies $f(q, i_1') = 0$. In the set $T_1 = \{q \in T | f(q, i_1) = 0, f(q, i_1 + 1) > 0\}$ choose q with $f(q, i_1 + 1) = j_2$ maximal. If q' is an element in P with $f(q', i_1 + 1) < j_2$ then, as in the above argument, $q'H \subset qH$ since otherwise $q'r = q$ and $f(q, i_1 + 1) = f(q'r, i_1 + 1) \leq f(q', i_1 + 1) < j_2$, a contradiction.

We set $T_2 = \{q \in T_1 \mid f(q, i_1 + 1) = j_2\}$ and repeat the argument with $i_1 + 2$ in place of $i_1 + 1$ defining a set T_3 etc. The procedure ends after finitely many steps with sets $T \supset T_1 \supset T_2 \supset \cdots \supset T_k$, $k \leq n - 1$, and any element $p \in \bigcap T_i$ can be chosen as a typical shift locus. ∎

In the next lemma we list some conditions that must be satisfied by a denominator set $N(p)$.

Lemma 6.5 Let H be a r.i.r. holoid of finite rank n, P a prime ideal in H, $S = H\backslash P$ and $p \in P$.

(i) If $sp = ps'$ for $s \in S$, then $s' \in N(p)$.

(ii) If $s_1, s_2 \in N(p)$ then $Hs_1 \subseteq Hs_2$ or $Hs_2 \subseteq Hs_1$, i.e. $N(p)$ is right and left naturally ordered.

PROOF: (i) We have $sq = p$ for some $q \in P$. Therefore $sp = ps' = sqs'$ and $p = qs', s' \in N(p)$ follows using the fact that H has finite rank to conclude $s' \in S$.

(ii) Let $p = x_1 s_1 = x_2 s_2$ for $x_i \in H$, $s_i \in S$ then $x_1 r - x_2$ for $r \in H$ and $s_1 = rs_2$ follows. ∎

This result shows that on the one hand a denominator set $N(p)$ must contain certain elements, on the other hand it usually can not contain too many elements, since otherwise condition (ii) would be violated. To illustrate the last comment assume that a r.i.r. holoid of finite rank contains an element p in the prime ideal P and elements $e < s_1 < s_2$ in $S = H \backslash P$ such that $s_1 s_2 = s_2$ and $s_1 \notin N(s_2)$. Then s_1 and s_2 cannot be simultaneously in $N(p)$.

We observe also that in general $N(p)$ is larger than the set $\{s' \mid \text{there exists } s \in S = H\backslash P, sp = ps'\}$ for $p \in P$. To see this let H be a r.i.r. holoid of finite rank n with minimal prime ideal $P \neq P^2$. Assume that $f(p,k) \neq 0$ for $p \in P\backslash P^2$ and some $k < n$. It then follows that $f(p^n, k) = 0$ but still $N(p) = N(p^n)$ by Theorem 6.2.

It is not surprising that additional restrictions on $N(p)$ exist if p is a typical shift locus for a minimal prime ideal P in a r.i.r. holoid of finite rank.

Lemma 6.6 Let H be a r.i.r. holoid of finite rank n, $P \neq P^2$ a discrete prime ideal of H and p a typical shift locus for P. Then $f(s, f(p,k)) = f(p,k)$ for all k with $C_k \subseteq S = H\backslash P$ and $s \in N(p)$.

PROOF: Let $f(p,k) = k'$. We have $p = p's$ for some $p' \in P$. Hence $f(p,k) = f(p',k) = k'$ since p is a typical shift locus for P and $f(s,k') = k'$ follows immediately. ∎

We conclude this section with a result about prime principal right ideals. This result can be used to describe right noetherian r.i.r. holoids.

Theorem 6.7 Let H be a r.i.r. holoid, $P = pH$ a prime principal right ideal. Then $qp = p$ for all $q \in H\backslash P$.

PROOF: We have $qp = pq'$ and $p = qp' \in P$ for elements $q', p' \in H$. It follows that $p' \in P$, $p' = pr$ say, and $qp = qp'q'$, $p = p'q' = prq'$. This is possible for $r = q' = e$ only. ∎

7 A construction and an example

The following result describes a situation where a r.i.r. holoid can be extended by adding denominators for suitable elements and quotient sets.

Theorem 7.1 Let H be a r.i.r. holoid of finite rank and $P = pH$ a prime principal right ideal. Then there exists for any element $a \in H\backslash P$ a r.i.r. holoid $\widehat{H}$ containing H and of the same rank as H with $\{a^n \mid n \in \mathbb{N}\} \subseteq \mathrm{N(p)}$.

PROOF: It follows from Theorem 6.7 that $sp = p$ for every element $s \in S = H\backslash P$. Let $T = H\backslash aH$ and $F = \{s \in H \mid \text{there exists } r \in T : rs = a\} \cup \{e\}$. For any elements $s_1, s_2 \in$ either $s_1 = ts_2$ or $s_2 = ts_1$, since $r_1 s_1 = a = r_2 s_2$ and say, $r_1 = r_2 t$ for some $t \in H$ which implies $ts_1 = s_2$. For any $r \in T$ denote with $\bar{r}$ the unique element in H with $r\bar{r} = a$. We now adjoin to H elements $x_{j,-i,r}$ with $j, i \in \mathbb{N}$ and $r \in T$ with satisfy the following rules

(i) $x_{j,-i,r} = x_{j',-i',r'}$ if and only if $j = j', i = i'$ and $r = r'$.

(ii) $(x_{j_1,-i_1,r_1})(x_{j_2,-i_2,r_2}) = x_{j_1+j_2,-i_2,r_2}$

(iii) $Sx_{j,-i,r} = x_{j,-i,r}$

(iv) $x_{j,-i,r}p = p^{j+1}, px_{j,-i,r} = x_{j+1,-i,r}$

To define $x_{j,-i,r}b$ for $b \in S$ we consider two cases:

(α) If $\bar{r}a^{i-1}q = b$ then $x_{j,-i,r}b = p^j q$.

(β) If $\bar{r}a^{i-1} = bq$ then $r\bar{r}a^{i-1} = rbq = a^i$ and an integer k exists with $a^k s = rb$ and $s \in T$. We can assume that $q \neq e$ and $k < i$ follows. In this case we define $x_{j,-i,r}b = x_{j,-i+k,s}$.

Let $Q \subset P$ be the prime ideal directly below P or $Q = \emptyset$ if P is minimal. Then $H_1 = H\backslash Q \cup \{x_{j,-i,r} \mid j, i \in \mathbb{N}, \mathrm{r} \in \mathrm{T}\}$ is a r.i.r. holoid and $N(p) = \bigcup_{i=0}^{\infty} Fa^i$.

We must show that the conditions of Lemma 1.3 are satisfied. To check associativity, only one case presents any difficulties at all: $((x_{j,-i,r}b)c) = x_{j,-i,r}(bc)$ with $b, c \in S$. If $b = \bar{r}a^{i-1}q$ the equation is obvious. If $rb = a^k s$ for $k < i, s \in T$ and $c = \bar{s}a^{i-k-1}q'$ then $rbc = a^k s\bar{s}a^{i-k-1}q' = r\bar{r}a^{i-1}q'$ and both sides of the equation are equal to $p^j q'$. If $rb = a^k s$ as in the previous case and $sc = a^g t$ for $t \in T$ and $g < i - k$ then $rbs = a^{k+g}t$ and both sides are equal to $x_{j,-i+k+g,t}$ with $k + g < i$.

The order for the elements of $H\backslash Q$ in H_1 is the same as the order of those elements in H. In addition we have $p^{j-1}s < x_{j,-i,r} < p^j$ for every j, i in N, $r \in T$ and $s \in S$. Further, $x_{j,-i,r} < x_{j,-i',r'}$ if either $i > i'$ or $i = i'$ and $r < r'$ in H.

It can be checked that this order does indeed satisfy all the conditions of Lemma 1.3.

Finally, consider the set $M = \{(q, x_{j,-i,r}) \mid q \in Q, j, i \in N, r \in T\}$ and define an equivalence relation $(q_1, x_{j_1,-i_1,r_1}) = (q_2, x_{j_2,-i_2,r_2})$ if and only if

$q_1p^{j_1} = q_2p^{j_2}$ and $i_1 = i_2$, $r_1 = r_2$. The equivalence classes are denoted by $q_1x_{j_1,-i_1,r_1}$ or qx with $q \in Q$ and $x \in X = \{x_{j,-i,r} \mid j \in \mathbb{N}, i \in \mathbb{N}, r \in T\}$. Then $\widehat{H} = H_1 \cup Q \cup \{qx | q \in Q,\ x \in X\}$. We define multiplication in H by the following rules: For $q \in Q, x \in X$ let $q.x = qx$ and $x_{j,-i,r}q = p^jq$.

The order of H and H_1, both considered as subsets of H is extended to H and for elements qx the following holds:

(α) $qp^{j-1}s < qx_{j,-i,r} < qp^j$ for $q \in Q$ and all $s \in S$.

(β) $q_1x_{j_1,-i_1,r_1} < q_2x_{j_2,-i_2,r_2}$ if and only if either $q_1p^{j_1} < q_2p^{j_2}$ or $q_1p^{j_1} = q_2p^{j_2}$ and then in addition either $i_1 > i_2$ or $i_1 = i_2$ but $r_1 < r_2$.

It is not difficult to check that all conditions in Lemma 1.3 are satisfied by $\widehat{H}$ and we will only show that for every pair of elements $\hat{h}_1 = q_1x_{j_1,-i_1,r_1}$ and $\hat{h}_2 = q_2x_{j_2,-i_2,r_2}$ with $\hat{h}_1 < \hat{h}_2$ there is an element $\hat{h} \in \widehat{H}$ with $\hat{h}_1\hat{h} = \hat{h}_2$. We proof the following auxiliary result first.

Let $q_1p^{j_1}v = q_2p^{j_2}$ for $q_i \in Q, j_1, j_2 \geq 1$ and $v \neq e$ in H. Then either $v = p^k$ for some $k \geq 1$ or $v = qp^{j_2}$. To prove this we compare $q_1p^{j_1}$ and q_2. In the first case we have $q_1p^{j_1}w = q_2$ for some $w \in H$. Then $v = wp^{j_2}$ and we are done. In the other case $q_1p^{j_1} = q_2w$ and $wv = p^{j_2}$ follows. This implies that $w = p^{k_1}s_1$, $v = p^{k_2}s_2$ for $s_1, s_2 \in S$ and $p^{k_1}s_1p^{k_2}s_2 = p^{j_2}$. We conclude $k_1 + k_2 = j_2, s_2 = e$. This proves the auxiliary result.

Now let $\hat{h}_1, \hat{h}_2$ be as above. Then either $q_1p^{j_1} < q_2p^{j_2}$ or $q_1p^{j_1} = q_2p^{j_2}$ with either $i_1 > i_2$ or $i_1 = i_2$ and $r_1 < r_2$. Let $i_1 < i_2$ in the second case and we can choose $\hat{h} = \bar{r}_1a^{i_1-i_2-1}r_2$ or let $i_1 = i_2$, $r_1h = r_2$ and we can choose $\hat{h} = h$.

We now consider the first case $q_1p^{j_1} < q_2p^{j_2}$ and apply the auxiliary result. If $v = p^k$ for some $k \geq 1$ then $q_1x_{j_1,-i_1,r_1}x_{k,-i_2,r_2} = q_1x_{j_1+k,-i_2,r_2} = q_2x_{j_2,-i_2,r_2}$ since $q_1p^{j_1+k} = q_2p^{j_2}$. If $v = qp^{j_2}$ then $(q_1x_{j_1,-i_1,r_1})(qx_{j_2,-i_2,r_2}) = q_1p^{j_1}qx_{j_2,-i_2,r_2} = q_2x_{j_2,-i_2,r_2}$ since $q_1p^{j_1}qp^{j_2} = q_2p^{j_2}$. ∎

We conclude with a final example:

Example 7.2 Consider the semigroup H with identity e generated by p_1, p_2 and $p_ip_{i-2}^{-k}$ where $i = 3, \cdots, n, \cdots$, $k = 0, 1, 2, \cdots$ with the following relations:

$$
\begin{aligned}
(p_jp_{j-2}^{m})(p_kp_{k-2}^{-r}) &= p_kp_{k-2}^{-r} && if\ j < k-1 \\
&= p_kp_{k-2}^{-r+1} && if\ j = k-1 \\
&= p_k^2p_{k-2}^{-r} && if\ j = k \\
&= p_{k+1}p_kp_{k-2}^{-r-m} && if\ j = k+1 \\
&= p_jp_{j-2}^{-m+1}p_{j-4}^{-r} && if\ j = k+2 \\
&= p_jp_{j-2}^{-m}p_kp_{k-2}^{-r} && if\ j > k+2
\end{aligned}
$$

This defines an associative operation and every element h can be written in the form $h = p_n^{e_n} p_{n-1}^{e_{n-1}} \cdots p_1^{e_1}$ where $e_n \geq 0$ and $e_i < 0$ is only allowed if for some $j > i$ we have $e_j > 0$ and for $e_s \neq 0$ with $j > s \geq i$; if follows that $e_s < 0$ and $s \equiv j \bmod 2$; in particular $i \equiv j \bmod 2$. Using this standard form one can order the elements of H lexicographically. The conditions of Lemma 1.3 can be checked. We illustrate results about denominator sets and solution sets.

The denominator set $N(p^7)$ for example is equal to $N(p^7) = \{p_1^n, p_3^k p_1^{-m}, p_5^r p_3^{-s}$ with $n, k, m, r, s \geq 0$ and $k = 1$ if $m > 0, r = 1$ if $s > 0\}$. The solution set $L(p^7, p^3) = \{x \in H \mid xp_3 = p_7\} = \{p_7 p_3^{-1} p_1^k \mid k \in \mathbb{Z}\}$.

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