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LOCALLY INVARIANT AND SEMI-INVARIANT RIGHT CONES

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A right cone of a group G is a submonoid H of G so that for $a, b \in H$ either $aH \subseteq bH$ or $bH \subset aH$ and $G = \{ab^{-1} \mid a, b \in H\}$. Valuation rings, right chain rings, the cones of right ordered groups provide examples. It is proved, see Theorem 17, that a semi-invariant right cone H with d.c.c. for prime ideals satisfies $Ha \subseteq aH$ for all $a \in H$, that is H is right invariant. Essential is the following Theorem 9: Let H be a locally invariant right cone in G with d.c.c. for prime ideals, and let $I \neq H$ be an ideal in H. Then $P_l(I) \subseteq P_r(I)$ for the associated left and right prime ideals of I.

Key Words: Locally invariant right cones; Ordered groups; Right chain rings; Right cones; Valuation rings.

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1. INTRODUCTION

Right cones generalize at the same time valuation rings and right chain domains as well as positive cones of ordered or right ordered groups. A chain domain R with skew field of quotients F corresponds to a valuation function from F into $\Gamma \cup \{\infty\}$ for an ordered group Γ if and only if $R = a^{-1}Ra$ for all $a \in F^*$, that is R is invariant, see Schilling [16]. The cone H of elements $h \geq e$ in a group G for a right order $\geq$ of G determines an order for G if and only if $g^{-1}Hg = H$ for all $g \in G$, again, if H is invariant under all inner automorphisms of G. We consider right cones H of a group G, see Definition 1, and prove that a semi-invariant right cone H with minimum condition for prime ideals is right invariant in the sense that $a^{-1}Ha \subseteq H$ for all $a \in H$ (Theorem 17). For the proof we use on the one hand the classification of prime segments of a right cone H, see Theorem 2, and on the other hand the prime ideals $P_r(I)$ and $P_l(I)$ associated with an ideal I in a locally invariant right cone H, see Definition 5. If H satisfies the minimum condition on prime ideals then it is proved that $P_l(I) \subseteq P_r(I)$, see Theorem 9. Some examples are given to illustrate the results.

For sets A and B the notation $A \subset B$ indicates the strict inclusion of A in B.

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2. RIGHT CONES AND PRIME SEGMENTS

Definition 1. Let G be a group. A submonoid H of G is a right cone of G if the following two conditions are satisfied:

(i) For $a, b \in H$ there is an element $c \in H$ with $a = bc$ or $b = ac$;
(ii) The group G is generated by H, i.e., $G = \{ab^{-1} \mid a, b \in H\}$.

In (ii) we use the fact that H is a right Ore submonoid of G by (i). Let H be a right cone of G, then $U(H) = H \cap H^{-1}$ is the subgroup of units of H.

A subring V of a field K is a valuation ring if and only if $V^* = V\backslash\{0\}$ is a (right) cone in the group $K^* = (K\backslash\{0\}, \cdot)$.

A group G is ordered if and only if G contains a right and left cone Π with $U(\Pi) = \{e\}$ which in addition is invariant, i.e., $\Pi a = a\Pi$ for all $a \in \Pi$. Similarly, a subring R of a skew field F is a right chain order of F if and only if R^* is a right cone of F^*. A group G contains a right cone H with $U(H) = \{e\}$, if and only if G contains a right and left cone H' with $U(H') = \{e\}$. (see Kopytov and Medvedev [12]). This is exactly the case when G is right ordered, but G may not be ordered in this case, (see also Conrad [7]).

Let H be a right cone in a group G. A nonempty subset I of H is a *right ideal* of H if $IH \subseteq I$, *left ideals* and *ideals* are defined similarly. An ideal I of H is called *prime* if $aHb \subseteq I$, $a, b \in H$, implies $a \in I$ or $b \in I$, and *completely prime* if this conclusion follows from $ab \in I$. The *maximal right ideal* $J(H) = H\backslash U(H)$ is a completely prime ideal for any right cone H with $U(H) \neq H$.

It can easily be checked that prime ideals, respectively, completely prime ideals P are already characterized by the property that $X^2 \subseteq P (x^2 \in P)$ implies $X \subseteq P (x \in P)$ with X an ideal.

If H is a right cone in G and $P' \supset P''$ are completely prime ideals of H so that there are no further completely prime ideals between P' and P'', then we write $P' \supset\!\!\!\cdot\; P''$ and say that $P' \supset\!\!\!\cdot\; P''$ is a *prime segment* of H. If P' is the minimal completely prime ideal of H, then $P' \supset\!\!\!\cdot\; \emptyset$ is also considered as a prime segment. We say that an element $a \in H$ is in the prime segment $P' \supset\!\!\!\cdot\; P''$ if $a \in P'\backslash P''$, and the right ideal I of H is in $P' \supset\!\!\!\cdot\; P''$ if $P' \supseteq I \supset P''$.

The next result (see Brungs and Törner [5]) shows that a prime segment of H falls into one of three categories; (see, among others, [1, 3, 8], for examples). An extension of this result can be found in [9].

Theorem 2. *For a prime segment $P' \supset\!\!\!\cdot\; P''$ of a right cone H in a group G one of the following alternatives occurs*:

(a) *The prime segment is right invariant, i.e., $P'a \subseteq aP'$ for all $a \in P'\backslash P''$*;
(b) *The prime segment is simple, i.e., there are no further ideals in $P' \supset\!\!\!\cdot\; P''$*;
(c) *The prime segment is exceptional, i.e., there exists a prime ideal Q in $P' \supset\!\!\!\cdot\; P''$ which is not completely prime. Then there are no further ideals between P' and Q and $\bigcap_{n\in\mathbb{N}} Q^n = P''$.*

3. LOCALLY INVARIANT RIGHT CONES

The right cone H in G is called *locally invariant* if all its segments $P' \supset\!\!\!\cdot\; P''$ are right invariant. In that case $a \in P'\backslash P''$ implies $\bigcap a^n H = P''$.

This is an immediate consequence of the following result (see [5]), Proof of Theorem 1.14).

Lemma 3. *Let a be an element in a right invariant prime segment $P' \supset P''$ of the right cone H of the group G. Then there exists an ideal $I \subseteq P'$ of H with $a \in I$ and $\bigcap_{n\in\mathbb{N}} I^n = P''$.*

It follows from Theorem 2 that a locally invariant right cone H does not contain prime ideals Q that are not completely prime, since such an ideal determines an exceptional segment.

The following result will be used in the proof of one of the main theorems.

Lemma 4. *If H is a locally invariant right cone of a group G, then $Ha^2 \subseteq aH$ for all $a \in H$.*

Proof. If $Ha^2 \not\subseteq aH$, there exist $r \in H$, $j \in J(H)$ with $ra^2j = a$. Substituting this expression for a back into this equation repeatedly one obtains

$$a = (ra)ra^2j^2 = (ra)^n aj^n \in (ra)^n H$$

for any $n \geq 1$. If ra is in the prime segment $P' \supset P''$, then it follows from Lemma 3 that the element a and hence ra is in P''. This is a contradiction which proves the statement of the lemma. □

If $J(H) = P$ is the only prime ideal of a locally invariant right cone H, then H is *right invariant*, i.e., $Ha \subseteq aH$ for all a in H. However, there exist locally invariant right cones with just two prime ideals which are not right invariant. Chain orders R in a division ring F finite dimensional over its center are locally invariant (see Gräter [10] or [11]), see also Example B.

4. ASSOCIATED PRIME IDEALS

To each right ideal I of H, we associate certain completely prime ideals.

Definition 5. Let $I \neq H$ be a right ideal in a right cone H in G. Then $P_r(I) = \{p \in H \mid \exists\, t \in H\backslash I \text{ with } tp \in I\}$. If I is an ideal, then $P_l(I) = \{p \in H \mid \exists\, t \in H\backslash I \text{ with } pt \in I\}$.

Proposition 6. *Let H be a right cone of the group G and $I \neq H$ a right ideal of H. Then*:

(i) *If I is an ideal of H, then $P_l(I)$ is a completely prime ideal of H containing I*;
(ii) *If H is locally invariant, then $P_r(I)$ is a completely prime ideal containing I.*

Proof. (i) It follows from the definition that $P_l(I)$ is a left ideal of H containing I and that $H\backslash P_l(I)$ is multiplicatively closed. To prove that $P_l(I)$ is a right ideal, let x be an element in $P_l(I)$ and $r \in H$. There exists $t \in H\backslash I$ with $xt \in I$. If $rt = tr'$, then $(xr)t = (xt)r' \in I$ with $t \notin I$ and $xr \in P_l(I)$. Otherwise, $rtr'' = t$ for some $r'' \in H$ and $xt = x(rtr'') = (xr)tr'' \in I$ with $tr'' \notin I$, since I is an ideal.

(ii) It follows from the definition that $P_r(I)$ is a right ideal of H containing I and $H\backslash P_r(I)$ is multiplicatively closed. To prove that $P_r(I)$ is also a left ideal, let $x \in P_r(I)$ with $tx \in I$ for some $t \in H\backslash I$, and let $r \in H$. Then $t^{-1}I = \{a \in H \mid ta \in I\}$ is a right ideal of H and $t^{-1}I$ is either a completely prime ideal or it defines a prime segment $P' \supset P''$ which contains $t^{-1}I$. Since $tx \in I$, it follows that $x \in t^{-1}I$. If $t^{-1}I$ is an ideal, then $rx \in t^{-1}I$ and $t(rx) \in I$. If $t^{-1}I \in P' \supset P''$ and $x \in P''$, then $rx \in P'' \subset t^{-1}I$, hence $trx \in I$ in both cases, and we obtain $rx \in P_r(I)$.

Otherwise, $x \in t^{-1}I\backslash P''$, and we assume that $rx \notin P_r(I)$. Then $rxs_1 = x$ for some $s_1 \in H$ and $s_1 \in P_r(I)$ follows since $x \in P_r(I), rx \notin P_r(I)$, and $H\backslash P_r(I)$ is multiplicatively closed. Repeating this argument, we obtain $x = rxrxs_2 = \cdots = (rx)^n s_n$ for $s_n \in P_r(I)$ and $n \geq 1$. Since $x \in P'$, it follows $rx \in P'$ and $x \in \bigcap_{n\in\mathbb{N}}(rx)^n H = P''$ by Lemma 3. This is a contradiction that proves (ii). □

We observe that it follows directly from the definition that $P_r(P) = P_l(P) = P$ for a completely prime ideal P in the right cone H. We do not know whether $P_r(I)$ is a left ideal for I a right ideal in the right cone of a group G. However, $P_r(I)$ is a prime ideal if H is a right and left cone or if H equals R^* for a right chain domain R.

We call $P_l(I)$ the *left associated prime ideal* of the ideal I, and if $P_r(I)$ is a left ideal, then we call $P_r(I)$ the *right associated prime ideal* of I.

We need two more results about associated prime ideals.

Lemma 7. *Let H be a right cone of G and I an ideal of H in the right invariant prime segment $P' \supset P''$. Then $P_r(I) = P'$ implies $P_l(I) = P'$.*

Proof. Assume $s \in H\backslash P'$ and $sx \in I$. Then we have to show that $x \in I$. If $sxt = x$ for some $t \in H$, then $x \in I$ since $sx \in I$. If $sx = xt$ for $t \in H$ and $t \notin P_r(I) = P'$, then $x \in I$. We are left to consider $sx = xt$ with $t \in P'$. Then $s^n x = xt^n$ and $s^n q_n = x$ for $q_n \in H$ and $n \geq 1$. Hence, $s^n x = s^n q_n t^n$, $x = q_n t^n$ and we obtain $x \in P''$ if we apply Lemma 3 since there exists an ideal A of H with $t \in A$, $\bigcap_{n\in\mathbb{N}} A^n = P''$. It follows that $x \in I$ and the lemma is proved. □

The next result says something about the right associated prime ideal of the product of certain ideals.

Lemma 8. *Let H be a locally invariant right cone of the group G, let $I \neq H$ be a right ideal of H with $P_r(I) = P$ and Q a prime ideal of H with $Q \subseteq P$. Then $P_r(IQ) = Q$.*

Proof. We first consider the case $IQ = I$ and $P_r(IQ) = P_r(I) = P$ follows. We want to prove that in this situation $Q \subset P$ is not possible. Otherwise, there exists $p \in P\backslash Q, x \in H\backslash I$, and $xp = aq \in I = IQ$ for $a \in I, q \in Q$. Further, $a = xa_1$ for $a_1 \in H$ and $xp = aq = xa_1 q$ and $p = a_1 q \in Q$ follows. This contradiction proves the statement of the lemma in this case.

Now we assume $IQ \subset I$ and an element a exists with $a \in I\backslash IQ$, hence $aQ \subseteq IQ$. Conversely, if $xq \in IQ\backslash aQ, x \in I, q \in Q$, then $a = xs$ for $s \notin Q$. Hence, there exists $q_1 \in Q$ with $q = sq_1$. Therefore, $xq = xsq_1 = aq_1 \in aQ$, a contradiction that shows $IQ = aQ$.

To prove that $P_r(aQ) = Q$ we observe that $a \notin aQ$ and $Q \subseteq P_r(aQ)$. If $Q \subset P_r(aQ)$, there exist $s \notin Q, x \notin aQ$ with $xs \in aQ$, hence $xs = aq$ for some $q \in Q$.

If $ar_1 = x$ for $r_1 \in H$, then $ar_1 s = xs = aq$ and $r_1 s = q \in Q$, a contradiction since $s, r_1 \notin Q$. If $a = xr_2$ for $r_2 \in H$, then $aq = xs = xr_2 q$ and $s = r_2 q \in Q$, again a contradiction. It follows that $Q = P_r(aQ)$ and $P_r(IQ) = Q$ in all cases. □

5. LOCALLY INVARIANT RIGHT CONES WITH d.c.c. FOR PRIMES

Since right cones H are assumed to be locally invariant for the rest of this article, we will say "prime ideal" instead of "completely prime ideal."

We make one further assumption: The right cone H *does not have infinite strictly descending chains of prime ideals*, we say H satisfies the descending chain condition (*d.c.c.*) for prime ideals or the minimum condition for prime ideals.

The following result is essential for the proof of the main theorem.

Theorem 9. *Let H be a locally invariant right cone in a group G with d.c.c. for prime ideals, and let $I \neq H$ be an ideal of H. Then $P_l(I) \subseteq P_r(I)$.*

Proof. If $I = P$ is prime, then we observed earlier that $P_l(I) = P_r(I)$, and we are done.

We can, therefore, assume that I is not a prime ideal, and hence there exists a prime segment $P' \supset\!\!\!\cdot\; P''$ of H with $P' \supset I \supset P''$, where $P' = \bigcap_{P \supseteq I} P$ with P prime and $P'' = \bigcup_{I \supset P} P$ with P prime.

We assume that the statement of the theorem is not true and consider as a counterexample an ideal I of H with $P_l(I) \supset P_r(I)$ and $P_r(I)$ minimal; as above $P' \supset I \supset P''$ for the prime segment $P' \supset\!\!\!\cdot\; P''$ of H containing I.

We compare the prime ideals $P_r(I)$ and P': If $P_r(I) = P'$, then $P_l(I) = P'$ by Lemma 7. Hence, we can assume $P_r(I) = P_1'' \supset P'$. Since H satisfies d.c.c. for prime ideals, there exists a prime ideal P_1' in H so that $P_l(I) \supseteq P_1' \supset\!\!\!\cdot\; P_1''$. We, therefore, have

$$P_l(I) \supseteq P_1' \supset\!\!\!\cdot\; P_1'' = P_r(I) \supset P' \supset\!\!\!\cdot\; P''$$

and $P' \supset I \supset P''$.

We choose an element $s \in P_1' \backslash P_1''$ and since $s \in P_l(I)$, there exists $x \in H \backslash I$ with $sx \in I$; it follows that $x \in P' \backslash I$. By discussing all possibilities, we will eventually reach the contradictory conclusion that $x \in I$, which proves the theorem.

We compare the elements sx and $x \in H$ and obtain the contradiction $x \in I$ in the first case where $sxt = x$ for some $t \in H$. We are left to consider the case $sx = xt$ for some $t \in H$. If $t \in P'$, there exists by the right invariance of the segment $P' \supset\!\!\!\cdot\; P''$ (see Lemma 3) a power n with $t^n H \subseteq x^3 H \subseteq P'$ and an element $r \in H$ with $t^n = x^3 r$. Therefore $s^n x = xt^n = x^4 r$ and $s^n v = x$ for some $v \in H$ since $s \notin P'$, $x \in P'$. Therefore, $s^n x = x^4 r = x(x^3 r) = s^n v x^2 xr = s^n x v' xr$, where the result $Ha^2 \subseteq aH$ (see Lemma 4) was used in the last equation to find an element $v' \in H$ with $vx^2 = xv'$.

By cancelling $s^n x$ we obtain $e = v'xr \in P' \subset H$, a contradiction. This shows that our assumption $t \in P'$ was wrong, and we must consider $sx = xt$ for $t \notin P'$. Again, using the assumption that H is locally right invariant and Lemma 3, we have $\bigcap_{n \in \mathbb{N}} t^n H = P_2 \supseteq P'$. It follows that $t \in P_r(I) \backslash P'$.

We want to show as our next result that $(P_1'' \backslash P_2)x \subseteq xH$ and recall $P_1'' = P_r(I)$. If this claim is not true, then there exists $p \in P_1'' \backslash P_2$ with $pxq = x$ for some $q \in J(H)$ and the segment $P_q' \supset\!\!\!\cdot\; P_q''$ with $q \in P_q' \backslash P_q''$ so that P_q' is minimal. Since $s \in P_1' \backslash P_1''$ and

$p \in P_1''$, there exists for every integer n an element $q_n \in H$ with $s^n q_n = p$ and $q_n \in P_1'' \backslash P_2$ follows.

We first assume that there exists $n \geq 1$ with $q_n x = x q_n'$ for some $q_n' \in H$. Then

$$x = pxq = s^n q_n xq = s^n x q_n' q = s^{n-1}(sx) q_n' q \in I,$$

since $sx \in I$, and the contradiction $x \in I$ follows.

We can, therefore, assume $q_n xH \supset xH$, hence $q_n x q_n' = x$ for $q_n' \in J(H)$ and all $n \geq 1$. At the beginning of the proof of the claim $p \in P_1'' \backslash P_2$ was chosen so that $pxq = x$ and P_q' minimal. Hence, $P'_{q_n'} \supseteq P_q'$ and there exists by Lemma 3 a natural number k with $q_n' H \supset q^k H$ and therefore $q_n' z = q^k$ for some $z \in J(H)$.

We obtain

$$\begin{aligned} x = pxq = p^k x q^k &= p^{k-1} pxq^k = p^{k-1} s^n q_n x q^k \\ &= p^{k-1} s^n q_n x q_n' z = p^{k-1} s^{n-1}(sx) z \in I, \end{aligned}$$

where we used the equations $p = s^n q_n$, $q^k = q_n' z$, and $q_n x q_n' = x$. The contradiction shows that the above claim is true, and we can assume that

$$(P_1'' \backslash P_2) x \subseteq xH. \tag{1}$$

Again we pick $p \in P_1'' \backslash P_2$ and obtain elements q_n with $s^n q_n = p$ and $q_n \in P_1'' \backslash P_2$ follows for all $n \geq 1$, since $s \in P_1' \backslash P_1''$.

We obtain $px = s^n q_n x = s^n x q_n'$ for elements $q_n' \in H$ by Eq. (1) and $n \geq 1$. Since $sx = xt$, we have $s^n x = xt^n$ and $px = s^n x q_n' = xt^n q_n' = (xt)(t^{n-1} q_n')$ for all $n \geq 1$. Since $sx = xt \in I$ and $\bigcap_{m \in \mathbb{N}} t^m H = P_2$, it follows that $px \in IP_2$.

Since $P_2 \subset P_r(I)$, Lemma 8 can be applied to obtain $P_r(IP_2) = P_2$, and $P_l(IP_2) \subseteq P_r(IP_2)$ follows since I was a counter example to the statement in the theorem with $P_r(I)$ minimal. From $px \in IP_2$, $p \in P_1'' \backslash P_2$, and $P_l(IP_2) \subseteq P_2$, it follows that $x \in IP_2 \subseteq I$. Since all possible cases lead to contradictions, there is no counter example, and $P_l(I) \subseteq P_r(I)$ follows. □

Before we will prove that semi-invariant right cones H with d.c.c. for prime ideals are right invariant, we investigate the behaviour of prime ideals in locally invariant right cones under *conjugation*.

Let H be a locally invariant right cone in G and P a prime ideal in H with $Pa \subseteq aJ$ for some element $a \in H$. Then P^a is defined as *the intersection of all prime ideals* Q of H with $Pa \subseteq aQ$, that is P^a is the smallest prime ideal in H that contains $a^{-1}Pa$. Similarly, we define $I^a = I(P, a)$ as *the smallest ideal of* H with $a^{-1}Pa \subseteq I(P, a) = I^a$.

Proposition 10. *Let H be a locally invariant right cone, P a prime ideal, and a an element in H with $Pa \subseteq aJ$. Then*:

(i) *If $P = P^2$, then $P^a = I^a$*;
(ii) *Assume $ra = as$ for $r \in H \backslash P$, $s \in H$. Then $s \notin P^a$*;
(iii) *Assume $ras = a$ with $r \in H \backslash P$, $s \in H$. Then $s \notin P^a$.*

Proof. (i) Assume $I^a \subset P^a$. Since $P^n a = Pa \subseteq a(I^a)^n$ for all n, it follows that $Pa \subseteq aP_1$, for the prime ideal (see Lemma 3) $P_1 = \bigcap_{n\in\mathbb{N}}(I^a)^n \subset P^a$, a contradiction to the minimality of P^a.

(ii) If $ra = as$ for $r \in H\backslash P, s \in P^a$, there exists elements $p \in P, q \in P^a$ with $pa = aq$ and $sH \subset qH$ or sH and qH are in the same prime segment of H. By the local invariance of H there exists in any case an integer $n \geq 1$ with $s^n = qv$ for some $v \in J$. We also have $r^n a = as^n$ and $r^n w = p$ for an element $w \in P$, since $r \notin P$. Hence,

$$r^n a = as^n = aqv = pav = r^n wav = r^n aw'v,$$

where $wa = aw'$ for $w \in P, w' \in J$ is used in the last equation. The contradiction $e = w'v \in J$ proves the statement (ii).

(iii) If $ras = a, r \in H\backslash P, s \in P^a$. Then there exists, as in the proof of (ii), an element $p \in P, q \in P^a$ with $pa = aq$ and a natural number n with $s^n = qv$ for $v \in J$. Hence, $a = ras = r^n as^n = r^n aqv = r^n pav \in PaJ \subseteq aJ$, a contradiction, and (iii) follows. □

We consider a special case.

Lemma 11. *Let H be a locally invariant right cone and a an element in the segment $P \supset P'$ of G. Then $P^a = P$.*

Proof. It follows from Theorem 2 (a) that $Pa \subseteq aP$, and hence $P^a \subseteq P$. Since $a^2 \in Pa \subseteq aP^a$, we have $a \in P^a$ and $P^a = P$. □

Proposition 12. *Let H be a locally invariant right cone with a an element in H and P a prime ideal in H with $Pa \subseteq aJ$. Then*:

(i) *aP^a is an ideal*;
(ii) *$P_l(aP^a) = P$*;
(iii) *$P_r(aP^a) = P^a$*.

Proof. (i) Let aq be an element in aP^a and $r \in H$. If $ra = ar_1$ for $r_1 \in H$, then $raq \in aP^a$. Otherwise, there exists $r \in H\backslash P$ with $rar_2 = a$ for $r_2 \in H$ and $r_2 \notin P^a$ by Proposition 10 (iii). Since $q \in P^a$, there exists $q_1 \in P^a$ with $q = r_2 q_1$. Then $raq = rar_2 q_1 = aq_1 \in aP^a$. This proves (i).

(ii) Since $Pa \subseteq aP^a \subseteq aJ$, it follows that $a \notin aP^a$ and $P \subseteq P_l(aP^a)$. If $s \in H\backslash P$ and $sx = aq$ for $q \in P^a$ we must show that $x \in aP^a$. We compare xH and aH and assume first that $xr_1 = a$ for some $r_1 \in H$. Then $sa = sxr_1 = aqr_1$ which contradicts Proposition 10 (ii).

Hence, we assume $x = ar_2$, and we are done if $r_2 \in P^a$. We are left with the case $x = ar_2, r_2 \in H\backslash P^a$. Now we compare aH and saH. If $sa = as'$, then $s' \notin P^a$ by Proposition 10, and $aq = sx = sar_2 = as'r_2$ implies $q = s'r_2 \notin P^a$, but $q \in P^a$, a contradiction.

In the remaining case $x = ar_2, r_2 \notin P^a$ and $sas_2 = a$ and $s_2 \notin P^a$ by Proposition 10 (iii). Then $aq = sx = sar_2$. Finally, we compare r_2H and s_2H. If $r_2 = s_2t$, then $t \notin P^a$ and $aq = sas_2t = at$ implies $t = q \in P^a$, a contradiction. If $s_2 = r_2t'$, then $aqt' = sar_2t' = sas_2 = a$ implies $qt' = e \in P^a$, a contradiction. This proves (ii).

The statement in (iii) follows from the proof of Lemma 8, where it was shown that $P_r(aP) = P$ for $a \in H$ and P a prime ideal. □

The next result, similar to Lemma 11, makes a statement about the relationship between P and P^a.

Theorem 13. *Let H be a locally invariant cone with d.c.c. for prime ideals in the group G, and let a be an element, P a prime ideal in H with $Pa \subseteq aJ$. Then $P \subseteq P^a$.*

Proof. We have $Pa \subseteq aP^a$ and $I = aP^a$ is an ideal of H by Proposition 12 (i). If we apply Theorem 9 to I, using also Proposition 12, we obtain $P = P_l(I) \subseteq P_r(I) = P^a$, the statement of the theorem. □

6. SEMI-INVARIANT RIGHT CONES

We observed earlier and will show in Example B, Section 7, that locally invariant cones H with finitely many prime ideals are not necessarily right invariant. The situation is different for semi-invariant cones using the following definition.

Definition 14. A right cone H in a group G is semi-invariant if and only if $Ja \subseteq aH$ for all $a \in H$.

This condition was introduced by Mathiak [13] for chain domains, see also Brungs and Gräter [4].

Lemma 15. *Let H be a right cone in a group G. Then*:

(i) *H is semi-invariant if and only if $Ha \subseteq aH$ or $Ja \subseteq aJ$ for any $a \in H$*;
(ii) *If H is semi-invariant, then H is locally invariant*;
(iii) *If H is a right and left cone of G, then H is semi-invariant if and only if $Ha \subseteq aH$ or $aH \subseteq Ha$ for all $a \in H$.*

Proof. (i) We assume $Ja \subseteq aH$ and $Ha \not\subseteq aH$, and we want to prove $Ja \subseteq aJ$. If the last statement is not true, then there exists $j \in J$ and $u \in U = U(H)$ with $ja = au$. However, there exists $r \in H$ with $ra \notin aH$ and hence $rja = rau \notin aH$, a contradiction that proves (i).

(ii) If H is semi-invariant, but not locally invariant, then H has a prime segment $P' \supset P''$ which is either simple or exceptional with prime ideal Q and $P' \supset Q \supset P''$ with no further ideal between P' and Q (see Theorem 2). For $a \in P' \backslash P''$ or $a \in P' \backslash Q$ in the second case, $I = JaJ$ satisfies $I = JaJ \subseteq aJ \subset aH$ since H is semi-invariant. Hence, $I \subset P'$, and since both P'' and Q are prime ideals and $I = J(HaH)J$, it is not possible that $I \subseteq P''$ or $I \subseteq Q$ in the second case. It follows that H has right invariant segments only.

(iii) H is semi-invariant if and only if $Ha \subseteq aH$ or $Ja \subseteq aJ$ for any $a \in H$ by (i). We show first that $Ja \subseteq aJ$ if and only if $aH \subseteq Ha$ if H is a right and left cone. Assume $Ja \subseteq aJ$ and $aH \not\subseteq Ha$. Then $Har \supset Ha$ for some $r \in H$ and $a = jar = aj'r$ for $j, j' \in J$. This contradiction shows that $aH \subseteq Ha$. If we assume conversely that $aH \subseteq Ha$ and $Ja \not\subseteq aJ$, then there exists $j \in J$ with $jaH \supset aJ$, hence $jaH \supseteq aH$ and $a = jar = jr'a$ for some $r, r' \in H$. This is a contradiction that proves $Ja \subseteq aJ$.

For (ii) see also Corollaries 20 and 21 in Ferrero, Mazurek, Sant'Ana [9]. □

We now characterize semi-invariant right cones as localizations of right invariant right cones.

Theorem 16. *Let H be a semi-invariant right cone in a group G. Then there exists a right invariant right cone A and a prime ideal $P = A\backslash S$ in A such that $H = AS^{-1}$.*

Proof. Let $A = \bigcap_{d\in H} dHd^{-1}$ and $J = J(H)$. Then A is a submonoid of G contained in H. Since $Jd \subseteq dH$ for all $d \in H$, it follows that $J \subset A$. Let $u \in U(H)\backslash(U(H) \cap A)$. We claim that $u^{-1} \in A$. If both elements $u, u^{-1} \notin A$, then there exist elements $a, b \in H$ and $r, s \in J$ with $a = uar$ and $b = u^{-1}bs$. We compare aH and bH.

If $aq = b$ for $q \in H$, then $aq = uarq = uaqr'$ with $r' \in H$ since $Jq \subseteq qH$. Hence, $br' = aqr' = u^{-1}b$, and the contradiction $b = u^{-1}bs = br's$ follows. The remaining case $b = aq'$ for $q' \in H$ is treated similarly, and the claim follows. If $r, r' \in A$, then $rc = r'$ or $r'c = r$ for some $c \in H$. Then $c \in J \subseteq A$ or $c \in U(H)$ and c or $c^{-1} \in A$. Hence, A is a right cone in G. If $r \in A$, and $h \in H$, then $rh = hr'$ and $r' = h^{-1}rh \in A$ follows, hence A is right invariant. Finally, $H = AS^{-1}$ for $S = A\backslash J$ and the prime ideal $J = P$ of A. □

The next result shows that certain semi-invariant right cones H are right invariant.

Theorem 17. *A semi-invariant right cone H in G with d.c.c. for prime ideals is right invariant, that is $Ha \subseteq aH$ for all $a \in H$.*

Proof. We assume $Ha \not\subseteq aH$ for an element $a \in H$, and $Ja \subseteq aJ$ follows by Lemma 15 (i).

We consider the ideal HaH. If $HaH = J$, then $P_l(HaH) = J = P_r(HaH)$. Otherwise, there exists $j \in J\backslash HaH$ and an element b with $jb = a$. If $b = r_1ar_2 \in HaH$, it follows that $a = jb = jr_1ar_2 \in aJ$, since $Ja \subseteq aJ$. Hence, $b \notin HaH$ and $j \in P_l(HaH)$ which implies $J = P_l(HaH)$ and $J = P_r(HaH)$ by Theorem 9. Since $Ha \not\subseteq aH$ and $Ja \subseteq aJ$, there exists $u \in U(H)$ and $s \in J$ with $a = uas = u^2as^2$. After replacing s by s^2 and u by u^2 if necessary, we therefore, know that there exists $t \in J$ with $tv = s$ for $v \in J$. Since $t \in J = P_r(HaH)$, there exists $b \in H\backslash HaH$ with $bt = r_1ar_2 \in HaH$. If r_1 is a nonunit, then $r_1a = ar_1'$ for some $r_1' \in H$ and $bt = ar_1'r_2$. If r_1 is a unit in H, then we rename $r_1^{-1}b$ as b again and obtain $bt = ar_2$, hence $bt = ar$ for $r \in H$ in every case.

There exists an element $w \in J$ with $bw = ua$ where $a = uas$ from above. Hence $a = uas = bws = bwtv = btw'v = arw'v$ where $wt = tw'$ for some $w' \in H$ since H is semi-invariant. The contradiction $a \in aJ$ finally proves that our first assumption was wrong: the right cone H is right invariant. □

7. EXAMPLES

We first give an example of a semi-invariant cone H in a group G that is not invariant.

Example A. Let $F = \mathbb{R}(t)$ be the field of rational functions in one indeterminate t over the real numbers with

$$(r_n t^n + \cdots + r_1 t + r_0)(s_m t^m + \cdots + s_1 t + s_0)^{-1} > 0 \text{ if and only } r_n s_m > 0$$

defining an order on F. Then $G = \{(a, b) \mid a, b \in \mathbb{R}(t), a > 0\}$ with

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)$$

as operation is an ordered group with $\Pi = \{(a, b) \mid a = 1 \text{ and } b \geq 0 \text{ or } a > 1 \text{ and } b \in F\}$ as the invariant cone of elements $g \geq e$ in G.

Then $H = \{x \in G \mid \exists\, r \in \mathbb{R} \text{ with } x \geq (1, r)\}$ is an overcone of Π in G, hence is semi-invariant, but H is not left invariant:

$$(t, 0)(1, -1)(t, 0)^{-1} = (1, -t) \notin H \text{ and } H(t, 0) \subset H(t, 0)(1, -1).$$

The cone Π neither satisfies the minimum condition for prime ideals nor the maximum condition.

The cone Π contains the ideal $I = \{x \in G \mid \exists r \in \mathbb{R}^+ \text{ with } (t^2, rt) < x\}$, and we observe that $P_l(I) \neq P_r(I)$:

In Π we have $(t^2, 0)(1, t^{-1} + 1) = (t^2, t + t^2) \in I$. Since $(t^2, 0) \notin I$, it follows that $(1, t^{-1} + 1) \in P_r(I)$. However, if $(1, t^{-1} + 1)(t^2, a) = (t^2, a + t^{-1} + 1) \in I$, then $a + t^{-1} + 1 \geq rt$ for some $r > 0$ in $\mathbb{R}$. Hence, $rt \leq a + t^{-1} + 1 < a + 1 + 1 < a + \frac{r}{2} t$ and $\frac{r}{2} t < a$ follows, which implies $(t^2, a) \in I$ and $(a, t^{-1} + 1) \notin P_l(I)$; hence $P_l(I) \subset P_r(I)$, see also Radó [15].

Next, we give an example of a locally invariant chain domain with exactly two nonzero prime ideals which is not invariant.

Example B. Let $\mathbb{Q}(i) \cong \mathbb{Q}[x]/(x^2 + 1)$ and σ the $\mathbb{Q}$-automorphism of $\mathbb{Q}(i)$ that maps i to $-i$. Then $F = \mathbb{Q}(i)((x, \sigma))$ the skew field of skew Laurent series $\alpha = \sum_{k \geq -m}^{\infty} x^k a_k$, $a_k \in \mathbb{Q}(i)$ has dimension 4 over its center $K = \mathbb{Q}((x^2))$ and contains the chain order $R = \mathbb{Z}[i]_{(2-i)} + x\mathbb{Q}(i)[[x, \sigma]]$. Here, $\mathbb{Q}(i)[[x, \sigma]]$ is the power series ring of all elements $\alpha \in F$ with $m = 0$, and $\mathbb{Z}[i]_{(2-i)}$ is the localization of $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, the ring of Gaussian integers, at the prime ideal $(2 - i)\mathbb{Z}[i]$. The complete prime ideals of R are $J = J(R) = (2 - i)\mathbb{Z}[i]_{(2-i)} + x\mathbb{Q}(i)[[x, \sigma]]$ and $P_1 = x\mathbb{Q}(i)[[x, \sigma]]$. The two segments $J \supset P_1$ and $P_1 \supset (0)$ are both right invariant, however $x^{-1}Rx = R'$ with $R' = (2 + i)\mathbb{Z}_{(2+i)} + x\mathbb{Q}(i)[[x, \sigma]] \not\subseteq R$ since for example $(2 - i)^{-1} \in R'$. All chain domains R in finite dimensional division algebras F are locally invariant. More examples can be found in Brungs and Gräter [2, 3].

We give an example of a right cone H in a group G that is not a left cone, see Brungs and Gräter [2] and see Brungs and Törner [6] for additional examples.

Example C. Let $A = \langle t \rangle$ and $Z = \langle z \rangle$ be both infinite cyclic groups. Let $B = \sum_{i \in \mathbb{Z}} A_i$ be the direct sum of cyclic groups $A_i = A$. The group B contains the subgroup $B_0 = \sum_{i\geq 0} A_i$ and B admits an automorphism σ with $\sigma(b) = b'$ and $(b')_i = (b)_{i-1}$ where $(b)_i \in A$ is the i-component of $b \in B$. The semidirect product $G = \{z^k b \mid k \in \mathbb{Z}, b \in B\}$ of Z and B with $bz = z\sigma(b)$ defining the operation is an ordered group (using lexicographic ordering) isomorphic to the wreath product $A \wr Z$. The group G contains the submonoid $H = \{z^k b \mid b \in B_0$ and k > 0 or k = 0 and $b \geq e$ in $B_0\}$.

Then H is a right cone of G since H generates G and $h_1 \geq h_2$ for elements $h_1, h_2 \in H$ implies $h_1 = h_2 h$ for $h \in H$. However, H is not a left cone, since for the elements $h_1 = z$ and $h_2 = zb$ in H with $(b)_0 = t$, $(b)_i = t^0$ for $i \neq 0$, there is no element in $h \in H$ with $hh_1 = h_2$ or $hh_2 = h_1$.

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