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PRIMES AND RIGHT IDEALS IN RIGHT CONES

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A right cone H in a group G is a submonoid of G that generates G and $aH \not\subseteq bH$ for $a, b \in H$ implies $bH \subseteq aH$. With any right ideal $I \neq H$ of H a completely prime ideal $P_r(I)$ of H is associated and the set $\mathcal{P}(I)$ of right ideals I' of H with the same associated prime ideal $P' = P_r(I)$ is determined if $P' \supset P''$ is a right invariant segment in H. The set $\mathcal{P}(I)$ is also described if $P_r(I)$ is a limit prime.

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INTRODUCTION

The rank one cones of groups and, therefore, the rank one chain domains can be classified according to the structure of their ideals; they are invariant, nearly simple, or exceptional, where the last case splits further into infinitely many cases depending on the occurence of principal ideals (see [3]).

The same classification applies to right cones of groups, Theorem 1.5, and hence to right chain domains; here, however, it has not been proved that the classification is complete in the exceptional case.

In order to understand better the right ideals I of a right cone H in a group G, a completely prime ideal $P_r(I)$ is associated to I and, see Lemma 2.11, it is proved that there exists a right ideal I' with $P_r(I) \supseteq I' \supset P_2$ with $tI' = I$ for some $t \in H$ and for P_2 any completely prime ideal of H properly contained in $P_r(I)$. If $P_r(I) \supset P_2$ is a prime segment, i.e., there are no further completely prime ideals between $P_r(I)$ and P_2, then there corresponds a rank one right cone $H^0_{P_r(I)}$ in a subgroup of G to this segment and I' corresponds to a right ideal of $H^0_{P_r(I)}$ (see Proposition 2.16). The right ideals of $H^0_{P_r(I)}$ can be described in terms of real numbers if $H^0_{P_r(I)}$ is right invariant (see Theorem 4.2) and this leads to a description of right ideals I of H with $P = P_r(I)$ (see Theorem 4.3).

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Whereas real numbers correspond to the not finitely generated right ideals in this case, a second description of right ideals is given where in the same situation the right ideals $\neq aJ$ correspond to real numbers (Proposition 4.4). These right ideals I correspond also to limit sets $L \subseteq W(H) = \{aH \mid a \in H\}$ with $I = \bigcap_L aH$ with $aH \in L = \{aH \mid aH \supset I\}$. This description is more suitable for the discussion of I-compactness of a right chain domain R with right ideal I, and it is necessary to develop an arithmetic for right ideals using this description, see 4.7 and 5.6. Here we say that R is I-compact for the right ideal $I \neq R$ of R if the canonical map from R into $\lim_{\leftarrow} R/I_\lambda$ is onto for every family $\{I_\lambda \mid \lambda \in \Lambda\}$ of right ideals I_λ of R with $\bigcap_\lambda I_\lambda = I$ where $\lambda' > \lambda$ implies $I_{\lambda'} \subset I_\lambda$ (see [7]). This is equivalent to the completeness condition that every pseudoconvergent sequence in R with breadth I has a limit in R. Krull [12] and Ribenboim [13] have discussed commutative valuation rings that are P-compact for every prime ideal. This classical case corresponds to the case where the subset of all distinguished limit sets is considered. The arithmetic developed here leads to the construction of many other suitable subsets of $\mathscr{L}(H)$, see Definition 4.5.

The results just listed would be sufficient to describe and classify the right ideals of a right cone H with right invariant segments only and maximum condition on completely prime ideals. We are able, however, to also give a description of those right ideals I in an arbitrary right cone H with $P_r(I) = P = \bigcup_{i\in\Omega} P_i$, a limit prime of H, where $P_i \subset P$ is completely prime for all $i \in \Omega$, and hence no completely prime ideal P'' exists with $P \supset P''$ a prime segment. This is done in Section 3, see in particular Corallaries 3.5–3.7.

In Section 2 we attach to each right ideal $I \neq H$ of a right cone H the two completely prime ideals $W_r(I)$ and $P_r(I)$ and the completely prime right ideal $Q_r(I)$ with $W_r(I) \supseteq P_r(I) \supseteq Q_r(I) \supseteq I$. If H is a cone, then $W_r(I) = P_r(I) = Q_r(I)$, but $W_r(I) \supset P_r(I)$ is possible if H is a right cone only.

If $P_r(I) = P$ is a limit prime ideal or if $P \supset P''$ is a right invariant prime segment in the right cone H, then $P_r(I) = Q_r(I)$. If $P \supset P''$ is a simple or exceptional prime segment we do not know whether $P_r(I) = Q_r(I)$. However, $P_r(I) = Q_r(I)$ if I is a right ideal $\neq R$ in a right chain domain R; see Corollary 2.4 and Lemmas 2.6 and 2.7.

Some of the results in this article extend results for cones on the one hand, and right chain domains on the other, see [2]. However, new problems occur for right cones, and the results in Section 3 in particular are new even for commutative valution rings as far as we know.

1. RIGHT CONES

Definition 1.1. Let G be a group. A submonoid H of G is called a *right cone* of G if the following two conditions are satisfied:

(i) $G = \langle H \rangle$, i.e., G is generated by H; and
(ii) $a^{-1}b \notin H$ implies $b^{-1}a \in H$ for $a, b \in H$.

This is the same definition as given in [5]; however, here and throughout this article, we assume in addition that the right cone is contained in a group.

Right cones not only generalize right chain domains, but also the cones of ordered and right ordered groups: A group $(G, \leq_r)$ is right-ordered with H as the set of elements greater or equal than e if and only if $H \cup H^{-1} = G$, that is, H is a right and left cone, and if in addition $U(H) = H \cap H^{-1} = \{e\}$ for e the identity of G. The right order of G is then given by $a \leq_r b$ for $a, b \in G$ if and only if $ba^{-1} \in H$. This right order and the left order, which can be defined similarly, agree for G if and only if H is in addition invariant, that is, $g^{-1}Hg = H$ for all $g \in G$. Examples of right cones H with $U(H) = e$ which are not left cones have been constructed in some groups G (see [1, 6]) as an answer to a question of Frege [9]. It follows (see [11], Th. 8.2.1) that G can be right ordered if it contains such a right cone H, but right ordered groups can in general not be ordered.

A right cone H is called *right invariant* if $Ha \subseteq aH$ for all $a \in H$. The next examples given below illustrate two classes of right invariant right cones for which the theory of right ideals developed in this article is most complete.

Example 1.2. Let A and B be two ordered groups. The base group C of the wreath product $A \wr B$ is defined as the direct sum of B-indexed copies A_b of A ordered lexicographically; the b-component of an element $c \in C$ is denoted by c_b. Then $A \wr B$ is defined as the semidirect product of B with C, that is,

$$W = A \wr B = \{b'c \mid b' \in B, c \in C\},$$

where $c \cdot b' = b'c^{b'}$ and $(c^{b'})_b = c_{bb'^{-1}}$, and W is an ordered group under the lexicographical ordering.

(a) Let $A \cong B \cong \mathbb{Z}$, the infinite cyclic group. $H_1 = \{h = bc \in A \wr B$ with $b \geq e_B, a = c_{e_B} \geq e_A, c_{b'} = e_A$ for $b' < e_B\}$ is a right cone where $hH_1 = baH_1$. Here we identify $a \in A$ with $e_B c'$ where $c'_{e_B} = a$, $c'_b = e_A$ for $b \neq e_B$. $U(H_1) = \{e_B c = c \in H_1 \mid c_b = e_A$ for $b \leq e_B\}$. The right cone H_1 of W is right invariant, but is not a left cone, since for $b > e_B, a > e_A$, we have $H_1 b \not\subseteq H_1 a \not\subseteq H_1 b$. The semigroup of principal right ideals of H_1 is isomorphic to the semigroup of ordinal numbers $<\omega^2$. In particular, $aH_1 \cdot bH_1 = bH_1$ for all $e_A \leq a \in A, e_B < b \in B$. Using the wreath product W of finitely many copies of $(\mathbb{Z}, +)$ or another subgroup of $(\mathbb{R}, +)$, it is possible to construct right cones H of finite rank n (see the definition after Lemma 1.4) so that $aHbH = bH$ for any $a, b \in H$ with $bH \subseteq \bigcap_i (aH)^i$.

(b) Let H_1 be as in Example 1.2(a), then $J(H_1) = \{bac \in H_1 \mid b > e_B$ or $a > e_A$ with $c \in U(H_1)\}$ and $H_1 = J(H_1) \cup U(H_1)$. Let $\Pi = \{c \in U(H_1) \mid e_C \leq c\}$, where C is lexicographically ordered. Then $H_2 = J(H_1) \cup \Pi$ is a right cone of infinite rank, which is not a left cone and $U(H_2) = \{e_W\}$.

(c) Let K be an ordered field (K, P_K), V an ordered K-vector space, and $G = \{(k, a) \mid 0 < k \in K, a \in V\}$ and $(k, a)(k', a') = (kk', ka' + a)$ as operation. Note that $P_K \cup -P_K = K, P_K \cap -P_K = \{0_K\}, P_K + P_K \subseteq P_K$, and $P_K \cdot P_K \subseteq P_K$. Further note $P_V + (-P_V) = V$, $P_V \cap (-P_V) = \{0_V\}, P_K \cdot P_V \subseteq P_V$, and $P_V + P_V \subseteq P_V$. Then $H_3 = \{(k, a) \in G \mid k \geq 1$ and $a \geq 0_V$ if $k = 1\}$ is a right and left cone of G. For $K = \mathbb{Q}$, $V = \mathbb{Q}$ the cone H_3 has rank 2.

(d) In Example (c), let $(V', P_{V'})$ be an extension of (V, P_V), that is, $V' \supseteq V$, $P_{V'} \cap V = P_V$ for the ordered vector space V'. Let $\eta \in V' \backslash V$. Then $P_\eta = \{(k, a) \in$

$G \mid k\eta + a \geq \eta\}$ is a right and left cone of G. The element η defines a Dedekind cut $D_\eta = (L, U)$ of V with $L = \{a \in V \mid a < \eta\}$, $U = V \backslash L = \{a \in V \mid a > \eta\}$. Conversely, given any Dedekind cut $D = (L, U)$ of V, that is, $L \cup U = V$, $a \in L$, $b \in U$ implies $a < b$, then there exists an extension V' of V and $\eta \in V' \backslash V$ with $D = D_\eta$. The cone P_η is invariant if and only if either $L = V$ or if $U = V$. On the other hand, all prime segments (see Theorem 1.5) of P_η are simple if and only if $L \neq \emptyset \neq U$, L has no largest element, U has no smallest element and for every $v \in V$ with $v > 0$ there exists $w \in L$ with $w + v \in U$ (see [4, Theorem 4.8]).

Let H be a right cone in G. A nonempty subset I of H is called a *right ideal* of H if and only if $IH \subseteq I$; *left ideals* and *ideals* of H are defined similarly. An ideal $I \neq H$ of a right cone H is called a *prime ideal* if $aHb \subseteq I$, $a, b \in H$, implies $a \in I$ or $b \in I$. If this conclusion follows from $ab \in I$, then I is called *completely prime*. It can easily be checked that prime ideals (completely prime ideals) P are already characterized by the property that $X^2 \subseteq P$ ($x^2 \in P$) implies $X \subseteq P$ for an ideal X of H ($x \in P$ for $x \in H$). The group $U(H) = H \cap H^{-1}$ was defined earlier and $J(H) = H \backslash U(H)$, the *Jacobson radical* of H, is a completely prime ideal of H; we write $J(H) = J$ if there is no ambiguity.

In the next result a particular subset of right ideals of a right cone H, which plays a role in the following results, is characterized.

Lemma 1.3. *Let H be a right cone of a group G. The following conditions are equivalent for a right ideal $I \neq H$ of H:*

(a) $I = \bigcap_{i \in \Lambda} I_i$, $I_i \supset I$ *right ideals of H for some index set* Λ;
(b) $I \neq aJ$ *for* $a \in H$;
(c) $L_I = \{bH \mid b \in H$ *and* $bH \supset I\} \subseteq W(H) = \{aH \mid a \in H\}$ *has no last* (*smallest*) *element.*

Proof. Given (a), then I is certainly $\neq H$. If $I = aJ$ and $I_i \supset I = aJ$, then $I_i \supseteq aH$, since $aH \supset I_i$ implies $b = aj$ for any $b \in I_i$ and some $j \in J$; hence $aJ \supseteq I_i$—a contradiction which proves (b).

If we assume (b) and $bH \supset I$ and $I \neq bJ$, then $bH \supset bJ \supset I$, and there exists $b' = bj \in bJ \backslash I$ for some $j \in J$. Hence, $bH \supset b'H \supset I$, and the condition (c) is satisfied.

Finally, we assume (c). Then $I = \bigcap_{bH \supset I} bH$, since otherwise there exists $d \in \bigcap_{bH \supset I} bH \backslash I$, and by (c) there exists $d' \in H$ with $dH \supset d'H \supset I$ and $d \in d'H$, a contradiction. This proves (a). □

The following result will be used repeatedly to obtain completely prime ideals in a right cone P.

Lemma 1.4. *Let $I \neq H$ be an ideal of the right cone H. Then $\bigcap_{n \in \mathbb{N}} I^n = \emptyset$ or $\bigcap_{n \in \mathbb{N}} I^n = P \neq \emptyset$ is a completely prime ideal of H.*

Proof. Let $s \notin P$ and $sa \in P$ for elements $s, a \in H$. Then there exists $n_0 \in \mathbb{N}$ with $s \notin I^{n_0}$ and elements $c_i \in I$ for $i = 1, \ldots, n_0 + n$ with $sa = c_1 \ldots c_{n_0} c_{n_0+1} \ldots c_{n_0+n}$. Hence, $sb = c_1 \ldots c_{n_0}$ for some element $b \in H$ and $a = bc_{n_0+1} \ldots c_{n_0+n} \in I^n$ follows for any n, hence $a \in \bigcap_{n \in \mathbb{N}} I^n = P$, and P is completely prime. □

It follows immediately from this lemma that an idempotent ideal $I = I^2 \neq H$ in H is completely prime.

Let H be a right cone in the group G. Two completely prime ideals $P' \supset\!\!\!\cdot\, P''$ of H form a *prime segment* of H if no further completely prime ideal of H lies between P' and P''; if P' is a minimal completely prime ideal of H, then $P' \supset\!\!\!\cdot\, \emptyset = P''$ is also a prime segment of H. If $P' \supseteq I \supset P''$ for a right ideal I of H, then we say that I is in the prime segment $P' \supset\!\!\!\cdot\, P''$. The right cone H has *rank n* if H has exactly n completely prime ideals.

Let $P' \supset\!\!\!\cdot\, P''$ be a prime segment of the right cone H, and we consider

$$Q = \{a \in H \mid HaH \subset P'\}.$$

If $Q = P''$, then there is no ideal $\neq P'$ in the segment $P' \supset\!\!\!\cdot\, P''$, and we call the prime segment *simple* in this case.

Next we assume that $P' = P'^2$ and $P' \supset Q \supset P''$. Let $X \supset Q$ be an ideal of H, then $X \supseteq P'$ and $X^2 \supseteq P'^2 \supset Q$, which proves Q to be a prime but not a completely prime ideal of H. The prime segment $P' \supset\!\!\!\cdot\, P''$ is called *exceptional* in this case.

In the remaining case either $P'^2 \neq P'$ or $P'^2 = P'$ and $P' = Q$. Then there exists for any $a \in P' \backslash P''$ an ideal I of H with $a \in I$ and $\bigcap_{n\in\mathbb{N}} I^n = P''$ by Lemma 1.4. The segment $P' \supset\!\!\!\cdot\, P''$ is called *right invariant* in this case.

These observations can be summarized in the following result.

Theorem 1.5. *Let P be a completely prime ideal in the right cone H of the group G. Then exactly one of the following cases occurs*:

(i) *$P = \bigcup_{\lambda\in\Omega} P_i$ is a limit prime, where $P \supset P_\lambda$ are completely prime ideals of H for λ in the index set Ω*;

(ii) *There exists a completely prime ideal P'' of H or $P'' = \emptyset$, so that $P = P' \supset\!\!\!\cdot\, P''$ is a prime segment of H and*

(iia) *$P' \supset\!\!\!\cdot\, P''$ is right invariant, i.e., for $a \in P \backslash P''$, there exists an ideal I of H with $a \in I$ and $\bigcap_{n\in\mathbb{N}} I^n = P''$; or*

(iib) *$P' \supset\!\!\!\cdot\, P''$ is simple*; *or*

(iic) *$P' \supset\!\!\!\cdot\, P''$ is exceptional.*

See [8, Theorem 18] for an extension of these results to right chain semigroups.

Since every right ideal $I \neq H$ of H which is not completely prime in H determines a prime segment, it follows that there exists an exceptional segment $P' \supset\!\!\!\cdot\, P''$ with $P' \supset Q \supset P''$ for every prime ideal Q which is not completely prime.

If H is the positive cone of an ordered group, then all prime segments of H are invariant. The ordered group in Example 1.2(c) contains a right and left cone with a simple prime segment (see [4]). Examples of cones with exceptional prime segments are constructed in [3], where a complete classification of rank one cones is given based on the structure of the lattice of ideals. Examples illustrating all possible cases are constructed as cones in subgroups of the universal covering group of the $SL(2, \mathbb{R})$. We are not able to decide whether or not additional cases can occur in the exceptional case for right cones. In the final result of this section, several characterizations of a right invariant prime segment of a right cone H are given.

Proposition 1.6. *The following conditions are equivalent for a prime segment $P \supset\!\!\!\cdot\, P''$ of a right cone H:*

(a) *For every element $a \in P\backslash P''$, there exists an ideal I of H with $a \in I$ and $\bigcap_{n\in\mathbb{N}} I^n = P''$, that is, $P \supset\!\!\!\cdot\, P''$ is right invariant;*
(b) *$Pa \subseteq aP$ for all $a \in P\backslash P''$;*
(c) *For $a, b \in P\backslash P''$, there exists a natural number n with $a^n H \subseteq bH$.*

Proof. That (a) implies (b) is given in [5, p. 153].

If we assume (a) and $a, b \in P\backslash P''$, then there exists an ideal I of H with $a \in I$ and $\bigcap_{n\in\mathbb{N}} I^n = P''$. Hence there exists an n with $a^n H \subseteq I^n \subseteq bH$, which shows that (a) implies (c).

Assume (b) and that the segment $P \supset Q \supset P''$ is exceptional. Then for $a \in P\backslash Q$, we have $PaP = P(HaH)P \not\subseteq Q$, since Q is a prime ideal, and on the other hand $PaP \subseteq aP \subset P$. However, this is a contradiction since there is no ideal properly between P and the prime ideal Q. Similarly, $P \supset\!\!\!\cdot\, P''$ is not simple, since for $a \in P\backslash P''$, as before PaP is an ideal properly between P and P''. This shows that the prime segment $P \supset\!\!\!\cdot\, P'$ must be right invariant under the condition (b), i.e., (b) implies (a).

Finally, we assume (c) and want to show that the segment is then neither exceptional nor simple. If, on the contrary, $P \supset Q \supset P''$ is exceptional with Q a prime ideal, it follows from [5, Lemma 1.13], that there exists $a \in P\backslash Q$ with $\bigcap_{n\in\mathbb{N}} a^n H \supset Q$, and therefore, $a^n H \supset bH$ for all natural numbers n and $b \in Q\backslash P''$; the segment $P \supset\!\!\!\cdot\, P''$ is not exceptional.

If $P \neq P^2$, then the segment is certainly not simple. If $P = P^2$ and $a \in P\backslash P''$, then $P \supset aH \supset P''$ and $Ha^2H \supset P''$. It will be enough to show that $Ha^2H \subseteq aH$. If this is not true, then $ra^2s = a$ for some $r, s \in H$ and $a = (ra)as = raraass = (ra)^n as^n$. Hence, $a \in \bigcap_{n\in\mathbb{N}}(ra)^n H = P''$ by (c), a contradiction that shows $P \supset aH \supseteq Ha^2H \supset P''$, that is, that, $P \supset\!\!\!\cdot\, P''$ is not simple. □

2. RELATED RIGHT IDEALS

It was mentioned in the introduction that the development of the theory of right ideals in right cones was motivated to some degree by our attempts to understand at which right ideals I' a right chain ring R must be I'-compact, that is, R/I' is compact, if it is known that R is I-compact. This is certainly the case if I' is related to I according to the following definition.

Definition 2.1. Let H be a right cone in a group G, and let A and B be right ideals $\neq H$ in H. Then A is *related* to B, $A \sim B$, if there exist $s \in H\backslash A$, $t \in H\backslash B$ with $s^{-1}A = t^{-1}B$, where $s^{-1}A = \{s^{-1}a \mid a \in A\} = \{r \in H \mid sr \in A\}$ is a right ideal $\neq H$ in H.

In the case of chain domains R classes of related right ideals correspond to indecomposable injective right R-domains $E(R/I)$, the injective hull of R/I, see [14].

Remark. It follows that $A \sim B$ if and only if $B = ts^{-1}A$ for $s \in H\backslash A$ and $t \in H$, and that $\sim$ is an equivalence relation: If $B = ts^{-1}A$ and $C = wv^{-1}B$ for $s \in H\backslash A$, $v \in H\backslash B$, $t, w \in H$ and right ideals A, B, C of H which are not equal to H, then $C = wt_1s^{-1}A$ if $t = vt_1$, some $t_1 \in H$, and $C = w(sv_1)^{-1}A$ for $v = tv_1$, some $v_1 \in H$.

If $sv_1 \in A$, then $v_1 \in s^{-1}A = t^{-1}B$ and $tv_1 = v \in B$, a contradiction. Hence, $A \sim B$, $B \sim C$, implies $A \sim C$.

We consider the set $\{I_\lambda \mid \lambda \in \Lambda\}$ of all right ideals of the cone H which are related to the right ideal $I \neq H$ of H and make the following definition.

Definition 2.2. Let $I \neq H$ be a right ideal in the cone H in a group G. Then $P_r(I) = \bigcup_{I_\lambda \sim I} I_\lambda$, the union of all right ideals in H related to I, is the (right) *associated* (completely) *prime ideal* of I.

It appears to be appropriate to combine the proof that $P_r(I)$ is indeed a completely prime ideal with some results about the right ideal

$$Q_r(I) = \{p \in H \mid \exists s \in H \backslash I \text{ with } sp \in I\}.$$

In the case where R is a right chain domain and I is a right ideal of R, then $Q_r(I)$ is equal to the right ideal annihilator of R/I.

Proposition 2.3. *Let $I \neq H$ be a right ideal in the right cone H in a group G. Then*:

(i) $P_r(I) = \bigcup_{s \in H \backslash I,\, t \in H} ts^{-1}I$ *is a completely prime ideal of* H;
(ii) $Q_r(I) = \bigcup_{s \in H \backslash I} s^{-1}I$ *is a right ideal of* H *with* $H \backslash Q_r(I)$ *multiplicatively closed, so that* $L \supseteq Q_r(I)$ *for any ideal* L *implies* $L \supseteq P_r(I)$.

Proof. (i) Since I is a right ideal and $P_r(I) = \bigcup_{I_\lambda \sim I} I_\lambda = \bigcup_{s \in H \backslash I, t \in H} ts^{-1}I$, it follows that $P_r(I)$ is an ideal of H containing I. To show that $P_r(I)$ is completely prime, assume that $v \in H \backslash P_r(I)$ and $vx \in P_r(I)$ for some $x \in H$. Then $vx \in I_\lambda$ for some $I_\lambda \sim I$ and $x \in v^{-1}I_\lambda = I_\mu \sim I$ since $v \notin I_\lambda$. Hence, $x \in P_r(I)$. This argument proves (i).

To prove (ii) we observe that $Q_r(I) = \{p \in H \mid \exists s \in H \backslash I \text{ with } sp \in I\}$ is equal to $\bigcup_{s \in H \backslash I} s^{-1}I \subseteq P_r(I)$, a union of right ideals. If $L \supseteq Q_r(I)$ is an ideal, then $s^{-1}I \subseteq L$ for all $s \in H \backslash I$ and $ts^{-1}I \subseteq L$ for all $t \in H$. Hence, $P_r(I) \subseteq L$ by (i). To show that $Y = H \backslash Q_r(I)$ is multiplicatively closed, take $y_1, y_2 \in Y$, and assume that $ay_1y_2 \in I$. Then $ay_1 \in I$, and $a \in I$ implies $y_1y_2 \in Y$. □

We observe that Proposition 2.3(ii) implies that $P_r(I) \supseteq Q_r(I)$ and that $P_r(I) = Q_r(I)$ if and only if $Q_r(I)$ is (also) a left ideal. The following results will show that $P_r(I) = Q_r(I)$ under certain conditions. However, we do not know whether $P_r(I) = Q_r(I)$ is true in general.

Corollary 2.4. *Let $I \neq H$ be a right ideal in the right cone H in a group G. If $P_r(I) = \bigcup_{i \in \Omega} P_i$, $P_i \subset P_r(I)$, is the union of completely prime ideals $P_i \neq P_r(I)$ of H, and Ω a well-ordered index set, then $P_r(I) = Q_r(I)$.*

Proof. If $P_r(I) \supset Q_r(I)$, there exists $P_i \subset P_r(I)$ with $Q_r(I) \subseteq P_i$. By 2.3(ii) the contradiction $P_r(I) \subseteq P_i$ follows. □

The next result shows that $P_r(I) = Q_r(I)$ also in Case (iia) (in Theorem 1.5), where there exists a prime ideal P'' in H, so that $P_r(I) \supset P''$ is a right invariant prime segment.

Lemma 2.5. *Let Q be a right ideal in a right cone H in a group G, so that $H\backslash Q$ is multiplicatively closed. Assume that $P' \supseteq Q \supset P''$ for a right invariant prime segment $P' \supset P''$ of H. Then $Q = P'$.*

Proof. Assume $x \in Q\backslash P''$ and $r \in H$ with $rx \notin Q$. Then $rx \in P'$, and there exists $s_1 \in H$ with $rxs_1 = x$. It follows that $s_1 \in Q$ since $H\backslash Q$ is multiplicatively closed. Then there exists $s_2 \in Q$ with $rxs_2 = s_1$, and $(rx)^2 s_2 = (rx)s_1 = x$. Repeating this argument, there exists for any n an element $s_n \in Q$ with $(rx)^n s_n = x$.

Since the segment $P' \supset P''$ is right invariant, there exists an ideal $I \subseteq P'$ with $rx \in I$ and $\bigcap_{n\in\mathbb{N}} I^n = P''$ (see Proposition 1.6). Hence, $x \in I^n$ for all n and $x \in P''$, a contradiction that proves that Q is a completely prime ideal and the lemma. □

We consider two additional instances where $Q_r(I) = P_r(I)$.

To prove the next result, we introduce $W_r(I) = \{s \in H \mid Is \subset I\}$ for a right ideal $I \neq H$ in a right cone H. It follows that $W_r(I)$ is an ideal in H ($Is \subset I$ implies $Ihs \subseteq Is \subset I$ and $Ish \subset Ih \subseteq I$ for $s \in W_r(H)$, $h \in H$), and that $H\backslash W_r(I) = S_r(I)$ is multiplicatively closed, hence $W_r(I)$ is a completely prime ideal of H containing I.

The next results show that $W_r(I) = P_r(I) = Q_r(I)$ if H is a cone in G.

Lemma 2.6. *Let I be a right ideal $\neq H$ in the right cone H. Then*:

(i) $W_r(I) \supseteq P_r(I) \supseteq Q_r(I)$; *and*
(ii) *If H is a cone, then $W_r(I) = P_r(I) = Q_r(I)$.*

Proof. (i) To prove $H\backslash W_r(I) \subseteq H\backslash Q_r(I)$ assume that $Is = I$ and $bs = a \in I$ for $s, b \in H$. Then $a = a_1 s$ for $a_1 \in I$, $b = a_1$, and $s \in H\backslash Q_r(I)$ follows. Hence, $W_r(I) \supseteq Q_r(I)$ and $W_r(I) \supseteq P_r(I)$ by Proposition 2.3 since $W_r(I)$ is an ideal; this proves (i).

To prove (ii) assume that $s \in H\backslash Q_r(I)$, that H is a cone and that $a \in I$. Then $a = a_1 s$ for some $a_1 \in H$ and $a_1 \in I$ since $s \notin Q_r(I)$. It follows that $I = Is$, that $s \in H\backslash W_r(I)$, that $Q_r(I) \supseteq W_r(I)$, and hence $Q_r(I) = W_r(I)$. □

With $B = \langle b\rangle$ and $A = \langle a\rangle$ as in Example 1.2(a), we have $H_1 \supset aH_1 \supset bH_1$ as the chain of completely prime ideals with $W_r(aH_1) = P_r(aH_1) = Q_r(aH_1) = aH_1$, but $W_r(bH_1) = \{s \in H \mid bH_1 s \subset bH_1\} = aH_1 \supset P_r(bH_1) = Q_r(bH_1) = bH_1$.

Lemma 2.7. *Let $0 \neq I \neq R$ be a right ideal in a right chain domain R. Then $P_r(I) = Q_r(I)$.*

Proof. The statement of the lemma follows from Proposition 2.3 if we can show that a right ideal Q of R, for which $R\backslash Q$ is multiplicatively closed, is an ideal. This in turn follows immediately if we can show that in R a right ideal Q is an ideal if $U(R)Q \subseteq Q$: Then $u^{-1}(ua) = a \in Q$ for $a \in Q$, $u \in U(R)$, $Q \neq R$, implies $ua \in Q$ for all $u \in U(R)$, $a \in Q$.

Therefore, let A be a right ideal of R with $U(R)A \subseteq A$. We want to show that then A is an ideal. Let $a \in A$, $x \in R$, and assume $xa \notin A$, hence $xaj = a$ for some $j \in J(R)$. Then $xa(1+j) = (x+1)a$ and $x \in J(R)$, $x+1$, $1+j \in U(R)$ and $xa = (x+1)a(1+j)^{-1} \in A$. It follows that A is an ideal. □

Definition 2.2 implies that related right ideals have identical associated prime ideals. The set $\mathcal{P}(I)$ of right ideals I' with $P_r(I) = P_r(I')$ for a right ideal I of H is, therefore, the disjoint union of classes of related right ideals of H. Let $\mathcal{R}(I) = \{I' \text{ right ideals of } H \mid I \sim I'\}$, and hence $\mathcal{R}(I) \subseteq \mathcal{P}(I)$. For a given completely prime ideal P we like to describe the equivalence classes of related right ideals in $\mathcal{P}(P)$.

Lemma 2.8. *Let P be a completely prime ideal in a right cone H in a group G, and let $a \in H$. Then*:

(i) $\mathcal{R}(\{aP\}) = \{bP \mid b \in H\}$; *and*
(ii) $P_r(aP) = P$.

Proof. Since $a \notin aP$, it follows that $aP \sim a^{-1}aP = P$, and if $I' \sim P$, then $I' = bs^{-1}P$, $s \in H \backslash P$, $b \in H$. However, $s^{-1}P = P$, since $sa = p \in P$ implies $a \in P$, and $I' = bP$ follows. This proves (i) which implies (ii), by Definition 2.2. □

The equivalence class $\mathcal{R}(P) \subseteq \mathcal{P}(P)$ will play a special role in the remainder of this article, and we make the following definition.

Definition 2.9. The right ideals $I = aP$, $a \in H$, P a completely prime ideal $\neq J(H)$ of the right cone H are called *P-distinguished*.

It may be somewhat surprising that in general only $\mathcal{R}(\{aH\}) \subseteq \{bH \mid b \in J\}$ for $a \in J$.

Remark 2.10. The right cone H_1 in Example 1.2(a) can serve as an example where the right ideals $I \neq H_1$ have the form $I = b^m a^n H$, if $A = \langle a \rangle$ and $B = \langle b \rangle$ for $(m, n) \neq (0, 0)$, $m \geq 0 \leq n$. Then $\mathcal{R}(\{aH_1\}) = \{b^m a^n H_1 \mid n \geq 1\}$ with $P_r(aH_1) = aH_1$, and $\mathcal{R}(\{bH_1\}) = \{b^m H_1 \mid m \geq 1\}$ and $P_r(bH_1) = bH_1$.

It remains to show that in a right cone H with $I \sim aH$, $a \in J$, $I = ts^{-1}aH$ is indeed a principal right ideal. However, $s \notin aH$, hence $a = sa_1$ for some $a_1 \in J$ and $I = ts^{-1}aH = ta_1H = bH$ for $b = ta_1 \in H$.

The next result follows almost directly from the definition of $P_r(I)$ for a right ideal I of H, but will be very useful.

Lemma 2.11. *Let H be a right cone in a group G and I be a right ideal so that $P_r(I)$ strictly contains some completely prime ideal P of H. Then there exists a right ideal $I' \sim I$ with $P_r(I) \supseteq I' \supset P$ and $tI' = I$ for some $t \in H$; also $P_r(I) = P_r(I')$.*

Proof. Since $P_r(I) = \bigcup_{I_\lambda \sim I} I_\lambda$, there exists in H a right ideal $I_\lambda \sim I$ and $P_r(I) \supseteq I_\lambda \supset P$. Further, $I = ts^{-1}I_\lambda$ for some $s \in H \backslash I_\lambda$ and $t \in H$. Then $I' = s^{-1}I_\lambda \supset P$ and $I' \sim I$, $tI' = I$. That $P_r(I) = P_r(I')$ follows from the Definition 2.2. □

The following results show that the classes of related right ideals I of a right cone H with $P_r(I) = P$ correspond to classes of related right ideals of the localization H_P of H at P and sometimes to classes of related right ideals of a rank one cone.

We consider the following situation: I is a right ideal $\neq H$ in the right cone H in the group G with $P = P_r(I)$. The set $S = H \backslash P$ is then a right Ore set of H, and the localization $H_P = \{as^{-1} \mid a \in H,\ s \in S\}$ is a right cone in G with $J(H_P) = PH_P$.

In the next result, we give a new characterization of $P_r(I)$ for a right ideal I of H.

Proposition 2.12. *Let I be a right ideal $\neq H$ in a right cone H in a group G. Then $P_r(I) = P$ if and only if P is minimal in the set of completely prime ideals P' of H with $IH_{P'} \cap H = I$.*

Proof. Let $P_r(I) = P$ and $b = as^{-1} \in IH_P \cap H$ for $a \in I$, $s \in H \backslash P \subseteq H \backslash Q_r(I)$. Then $bs = a \in I$, and $b \in I$ follows since $s \notin Q_r(I)$. Hence, $IH_P \cap H = I$. To prove the statement about the minimality of P assume $P \supset \widetilde{P}$ for a completely prime ideal $\widetilde{P}$ of H. By Proposition 2.3(ii) it follows that $Q_r(I) \supset \widetilde{P}$, and there exists $t \in Q_r(I) \backslash \widetilde{P}$. Hence, $bt = a \in I$ for some element $b \in H \backslash I$. It follows that $b = at^{-1} \in IH_{\widetilde{P}} \cap H \not\subseteq I$ and $P = P_r(I)$ is minimal in the set of completely prime ideals P' of H with $IH_{P'} \cap H = I$.

Conversely, assume that $\widetilde{P}$ is minimal in the set of completely prime ideals P' of H with $IH_{P'} \cap H = I$. Then $P_r(I) \supseteq \widetilde{P}$ by the first part of the proof. By Proposition 2.3(ii) it is enough to show that $\widetilde{P} \supseteq Q_r(I)$ to conclude that $\widetilde{P} \supseteq P_r(I)$ and $\widetilde{P} = P_r(I)$. Assume that there exists an element $p \in Q_r(I) \backslash \widetilde{P}$. Then there exists $b \in H \backslash I$ with $bp = a \in I$ and $b = ap^{-1} \in IH_{\widetilde{P}} \cap H \not\subseteq I$. This is a contradiction that proves the proposition. □

Corollary 2.13. *Let H be a right cone in a group G with $P' \supset P''$ a prime segment of H and I a right ideal of H with $P' \supseteq I \supset P''$. Then $P_r(I) = P'$ if and only if $IH_{P'} \cap H = I$.*

Proof. If $P_r(I) = P'$, then $IH_{P'} \cap H = I$ by Proposition 2.12. Conversely, if $IH_{P'} \cap H = I$, then $P' \supseteq P_r(I)$. Since $P' \supseteq I \supset P''$, it follows that $P_r(I) \supseteq P'$, hence $P_r(I) = P'$ which proves the corollary. □

The next result guarantees that relatedness is preserved by localization.

Lemma 2.14. *Let P be a completely prime ideal in the right cone H in G.*

(i) *Then there exists a one to one correspondence between the set $\mathcal{P}(P)$ of right ideals I of H with $P_r(I) = P$ and the set $\mathcal{P}(PH_P)$ with $\phi(I) = IH_P$ for $I \in \mathcal{P}(P)$ and $\psi(\widetilde{I}) = \widetilde{I} \cap H$ for $\widetilde{I} \in \mathcal{P}(PH_P)$.*

(ii) *Two right ideals $I_1, I_2 \in \mathcal{P}(P)$ are related in H if and only if $\widetilde{I_1} = I_1H_P$, and $\widetilde{I_2} = I_2H_P$ are related in H_P.*

Proof. (i) Let $I \in \mathcal{P}(P)$ and $P = P_r(I) = \bigcup_{r \in H, s \in H \backslash I} rs^{-1}I$. We observe that $s \in H \backslash I$ implies $s \in H_P \backslash IH_P$, since otherwise $s = at^{-1}$ for $a \in I$, $t \in H \backslash P$, and $st = a \in I$ leads to a contradiction. Therefore, $PH_P \supseteq P_r(IH_P) \supseteq \bigcup_{s \in H \backslash P, r \in H} rs^{-1}IH_P \supseteq PH_P$ and $P_r(IH_P) = PH_P$. It follows from 2.12 that $\psi(\phi(I)) = IH_P \cap H = I$.

Next, let $\widetilde{I} \in \mathscr{P}(PH_P)$, and let $I = \widetilde{I} \cap H$. Then $\widetilde{I} = IH_P$, $IH_P \cap H = I$, and $P \supseteq P_r(I)$ by Proposition 2.12. Further,

$$P_r(\widetilde{I}) = \bigcup_{bs^{-1} \in H_P, vt^{-1} \in H_P \backslash IH_P} (bs^{-1})(vt^{-1})^{-1} IH_P = PH_P.$$

To prove that $P \subseteq P_r(I)$, assume that $p \in P$, and hence there exist $b \in H$, $v \in H \backslash I$, $s, t, r \in H \backslash P$, and $a \in I$ with $p = bs^{-1}tv^{-1}ar^{-1}$. It follows that $I_1 = tv^{-1}I \sim I$ and that $P_r(I_1) = P_r(I) \subseteq P$. Since $s \in H \backslash P$ implies $s \in H \backslash I_1$ and $bs^{-1}I_1 = I_2 \sim I_1 \sim I$. We obtain that $p = bs^{-1}tv^{-1}ar^{-1} \in I_2H_P \cap H = I_2$, since $P \supseteq P_r(I_2) = P_r(I)$ by Proposition 2.12. Hence, $p \in P_r(I) = \bigcup_{I_\lambda \sim I} I_\lambda$, and $P = P_r(I)$ follows.

To prove (ii) consider $I_1, I_2 \in \mathscr{P}(P)$. If $I_1 \sim I_2$, then $I_2 = rs^{-1}I_1$ for $s \in H \backslash I_1$, $r \in H$. Since then $s \in H_P \backslash I_1H_P$, it follows that $I_2H_P = rs^{-1}I_1H_P \sim H_1H_P$. If, on the other hand, $I_2H_P \sim I_1H_P$, then $I_2H_P = (bs^{-1}(vt^{-1})^{-1}I_1H_P$ for $s, t \in H \backslash P$, $vt^{-1} \in H_P \backslash I_1H_P$, hence $v \in H \backslash I_1$. In the proof of (i), we showed that $(bs^{-1})(vt^{-1})^{-1}I_1 = I_3 \sim I_1$ and $P_r(I_3) = P$. Hence, $I_2 = I_2H_P \cap H = I_3H_P \cap H = I_3$, and $I_1 \sim I_2$ follows. □

That localization preserves the right invariance of a prime segment is shown in the next result.

Proposition 2.15. *Let P a completely prime ideal in a right cone H. Then*:

(i) *There is a one-to-one correspondence between the set of completely prime ideals P' of H with $P' \subsetneq P$ and the set of completely prime ideals $\widetilde{P}$ of H_P with $\phi(P') = P'H_P$ and $\psi(\widetilde{P}) = \widetilde{P} \cap H$;*

(ii) *A prime ideal segment $P' \supset P''$ in H with $P \supseteq P'$ is invariant if and only if the corresponding prime segment $P'H_P \supset P''H_P$ is invariant.*

Proof. (i) If P' is a completely prime ideal in H with $P' \subseteq P$, then $S^{-1}P' = P'$, and $P'S^{-1}$ is a completely prime ideal in H_P with $P'S^{-1} \cap H = P'$.

Conversely, if $\widetilde{P}$ is a completely prime ideal in H_P, then $\widetilde{P} \cap H$ is a completely prime ideal in H which is contained in P and $(\widetilde{P} \cap H)S^{-1} = \widetilde{P}$. It follows that $P' \to P'S^{-1}$ defines a one-to-one mapping from the set of completely prime ideals of H which are contained in P' of H to the set of completely prime ideals of H_P.

The prime segments $P' \supset P''$ of H contained in P are, therefore, in one-to-one correspondence with the prime segments of H_P.

(ii) Assume that $P' \supset P''$ is right invariant. We want to prove that $P'S^{-1} \supset P''S^{-1}$ is right invariant in H_P by showing that for $as^{-1}, bt^{-1} \in P'S^{-1} \backslash P''S^{-1}$, there exists an n with $(as^{-1})^n H_P \subseteq bt^{-1}H_P$, using Proposition 1.6(c).

Since $s \in H \backslash P$, and $a \in P' \backslash P''$, there exists $q \in H$ with $a = sq$ and $q \in P' \backslash P''$ follows. Hence there exists a natural number $n - 1$ with $q^{n-1}H \subseteq bH$. It follows that $(as^{-1})^n H_P = aq^{n-1}s^{-1}H_P = aq^{n-1}H_P \subseteq q^{n-1}P'H_P \subseteq q^{n-1}H_P \subseteq bt^{-1}H_P$, where we use for the containment $aq^{n-1}H_P \subseteq q^{n-1}P'H_P$ the assumption $P'q^{n-1} \subseteq q^{n-1}P'$ (see 1.6(b)). Conversely, if $P'H_P \supset P''H_P$ is an invariant segment in H_P, and $a, b \in P' \backslash P''$, then $a, b \in P'H_P \backslash P''H_P$, and there exists an n with $a^nH_P \subset bH_P$. It follows that $a^nH \subset bH$ and that $P \ll \supset P''$ is an invariant segment in H. This proves the proposition. □

Let H be a right cone in a group G with completely prime ideal P. Then the complement $H^0 = H\backslash P$ of P in H is not only a submonoid of G, but a right cone in a subgroup of G.

Proposition 2.16. *Let $H \supset J(H) = P' \supset P''$ be a right cone in G with prime segment $J(H) = P' \supset P''$. Then the following holds for $H^0 = H\backslash P''$:*

(i) *H^0 is a right cone of rank one with $J(H^0) = J(H)\backslash P''$ in a subgroup of G;*
(ii) *There is a one-to-one correspondence ϕ between the set of right ideals I of H that contain P'' properly and the set of right ideals of H^0 with $\phi(I) = I^0 = I\backslash P''$;*
(iii) *The right ideal I of H with $I \supset P''$ is an ideal, a completely prime ideal, a prime ideal in H if and only if the corresponding property holds for I^0 in H^0;*
(iv) *Right ideals I_1 and I_2 of H containing P'' properly are related in H if and only if I_1^0 and I_2^0 are related in H^0.*

Proof. (i) If $a, b \in H^0 = H\backslash P''$, then $ab \in H\backslash P''$ and $a = bc$ or $b = ac$ for some $c \in H$. But then $c \notin P''$, and it follows that $H\backslash P'' = H^0$ is a right cone in the subgroup G^0 of G which is generated by H^0. That rank H^0 is one that follows from (ii) and (iii).

(ii) If $I \supset P''$ is a right ideal of H, then $I^0 = \phi(I) = I\backslash P''$ is a nonempty subset of H^0 with $I^0(H\backslash P'') \subseteq I\backslash P'' = I^0$, hence I^0 is a right ideal of H^0. The inverse ϕ^{-1} of ϕ maps a right ideal I^0 of H^0 to $\phi^{-1}(I^0) = I^0 \cup P''$; this is a right ideal of H since $(I^0 \cup P'')(H^0 \cup P'') \subseteq I^0 \cup P''$.

(iii) A similar argument as in (ii) shows that the right ideal $I \supset P''$ of H is an ideal of H if and only if $\phi(I)$ is an ideal. If $s_1, s_2 \in H\backslash I$ for the ideal $I \supset P''$ of H implies $s_1 s_2 \notin I$, then the same condition holds for I^0; it follows that I is completely prime in H if and only if I^0 is completely prime in H^0.

A similar argument where ideals $A \supset I$, $B \supset I$ in H replace the elements s_1, s_2 shows that I is prime in H if and only if I^0 is prime in H^0.

(iv) For $I \supset P''$, $s \in H\backslash I$, it follows that $s^{-1}I = s^{-1}(I^0 \cup P'') = s^{-1}I^0 \cup P''$ since for $p \in P''$, $s \notin P''$, and $s^{-1}p = a$ implies $p = sa$ and $a \in P''$. Therefore, $s_1^{-1}I_1 = s_2^{-1}I_2$ for right ideals $I_1 \supset P''$ in H, $s_i \in H\backslash I_i$, $i = 1, 2$, if and only if $s_1^{-1}I_1^0 = s_2^{-1}I_2^0$ in H^0; that is, $I_1 \sim I_2$ in H if and only if $I_1^0 \sim I_2^0$ in H^0. □

If we combine the results in Theorem 1.5, Lemmas 2.11 and 2.14, and Proposition 2.16, it follows that the classes of related right ideals in a right cone H with associated prime ideal $P_r(I) = P$ can be determined by describing equivalence classes of right ideals in the rank one cone H_P^0 unless P is a limit prime.

Theorem 2.17. *Let H be a right cone in the group G, P' a completely prime ideal in H. Then exactly one of the following four cases occurs:*

(i) *$P' = \bigcup_i P_i$ is a limit prime of H with $P' \supset P_i$ completely prime ideals in H;*
(ii) *There exists a completely prime ideal P'' of H or $P'' = \emptyset$ so that $P' \supset P''$ is a prime segment of H. Then there exists a one to one correspondence ψ between the set of equivalence classes of related right ideals I in H with $P_r(I) = P'$, and the set of equivalence classes of related right ideals in $H_{P'}^0 = H_{P'}\backslash P'' H_{P'}$, where $H_{P'}^0$ is a rank one right cone in a subgroup G_0 of G and:*

(iia) $H^0_{P'}$ *is right invariant, or*
(iib) $H^0_{P'}$ *is nearly simple, or*
(iic) $H^0_{P'}$ *is exceptional.*

Proof. The four alternatives, P' a limit prime, or $P' \supset P''$ a prime segment which is either right invariant, simple, or exceptional follow from Theorem 1.5. That in the last three cases $H^0_{P'}$ is a rank one cone so that its only prime segment is right invariant, simple or exceptional respectively, follows again from Theorem 1.5. The statement about classes of related right ideals follows from 2.11, 2.14, and 2.16. □

Remark 2.18. If $P' \supset P''$ is a prime segment in a right cone H as in 2.17(ii), then $\mathscr{P}(P') = \{c((I^0 \cup P''H_{P'}) \cap H) \mid c \in H,\ H^0_{P'} \neq I^0 \text{right ideal in} H^0_{P'}\}$.

It was proved in [5] that a rank one right cone H whose only prime segment is right invariant is itself right invariant in the sense that $Ha \subseteq aH$ for all $a \in H$, that is all right ideals of H are ideals. If the prime segment of the rank one right cone is simple, then H has only H itself and $J = J(H)$ as ideals, we say H is *nearly simple*. Finally, H is *exceptional* if H contains a prime ideal that is not completely prime.

The above results have reduced the determination of classes of related right ideals with the same associated prime ideal to the rank one case if H has no limit primes.

In the next section, we will gather some results about right ideals I with $P_r(I) = P$ a limit prime, and in the remaining sections we will consider right cones H in a group G with right invariant prime segments only, we then say that H is locally (right) invariant.

3. RIGHT IDEALS ASSOCIATED WITH LIMIT PRIMES

Let H be a right cone in a group G with a limit prime ideal $P = \bigcup_{i \in \Omega} P_i$ and Ω an index set, P_i completely prime ideals for all i. We can assume that Ω is well ordered with $P \supset P_j \supset P_i$ for $i < j \in \Omega$. We consider right ideals I of H with $P_r(I) = P$ and $I \neq cP$ for all $c \in H$, that is, I is associated with P, but not distinguished. We recall that $P_r(I) = Q_r(I)$ by Corollary 2.4.

Let λ be a limit ordinal with $\{j_k \mid k < \lambda\} = \Omega' \subseteq \Omega$ a subset of Ω with $j_i < j_k$ for $i < k < \lambda$. We consider the following condition:

$$\text{For } j_i < j_k \in \Omega' \text{ with } i < k < \lambda, \text{ there exist elements } h_{j_i}, h_{j_k} \in H \text{ with } h_{j_k} = h_{j_i} b_{j_i, j_k} \text{ for an element } b_{j_i, j_k} \in P_{j_k} \backslash P_{j_i}. \tag{1}$$

If condition (1) is satisfied, then $b_{j_i, j_k} \in P_{j_{i+1}} \backslash P_{j_i}$, since

$$h_{j_k} = h_{j_{i+1}} b_{j_{i+1}, j_k} = h_{j_i} b_{j_i, j_{i+1}} b_{j_{i+1}, j_k}, \qquad \text{if } j_k > j_{i+1},$$

and $b_{j_i, j_k} = b_{j_i, j_{i+1}} b_{j_{i+1}, j_k} \in P_{j_{i+1}} \backslash P_{j_i}$ follows.

Further, $h_{j_i}H \supset h_{j_k}H \supset h_{j_k}P_{j_k} = h_{j_i}b_{j_i,j_k}P_{j_k} \supset h_{j_i}P_{j_i}$, since $b_{j_i,j_k}H \supset P_{j_i} \subset P_{j_k}$ for $i < k < \lambda$ and P_{j_i} is completely prime. We can, therefore, form the two right ideals

$$I_\lambda = \bigcap_{k<\lambda} h_{j_k}H \supseteq \bigcup_{k<\lambda} h_{j_k}P_{j_k} = I_\lambda^0 \qquad \text{in } H.$$

Under these conditions, we obtain the following results.

Lemma 3.1. $P_r(I_\lambda) = \bigcup_{k<\lambda} P_{j_k} = P_r(I_\lambda^0)$.

Proof. Since $h_{j_k} \in H\backslash I_\lambda$ and $h_{j_k}P_{j_k} \subseteq I_\lambda^0 \subseteq I_\lambda$, we obtain $Q_r(I_\lambda) \supseteq \bigcup_{k<\lambda} P_{j_k} \subseteq Q_r(I_\lambda^0)$. To prove that $Q_r(I_\lambda) \subseteq \bigcup_{k<\lambda} P_{j_k}$, let $p \in Q_r(I_\lambda)$ and $h \in H\backslash I_\lambda$ with $hp \in I_\lambda$. Then there exists $k < \lambda$ with $hH \supset h_{j_k}H$ and $hpH \subset h_{j_{k+1}}H = h_{j_k}b_{j_k,j_{k+1}}H$. Therefore, $h_{j_k} = hw$ for some $w \in H$ and $h_{j_k}b_{j_k,j_{k+1}}v = hp$ for some $v \in H$. It follows that $hwb_{j_k,j_{k+1}}v = hp$ and $p = wb_{j_k,j_{k+1}}v \in P_{j_{k+1}}$. This shows that $Q_r(I_\lambda) \subseteq \bigcup_{k<\lambda} P_{j_k}$ and $Q_r(I_\lambda) = \bigcup_{k<\lambda} P_{j_k} = P_r(I_\lambda)$, since $Q_r(I_\lambda)$ is an ideal, see Proposition 2.3.

To prove that $Q_r(I_\lambda^0) \subseteq \bigcup_{k<\lambda} P_{j_k}$, let $p \in Q_r(I_\lambda^0)$ and $h \in H\backslash I_\lambda^0$ with $hp \in I_\lambda^0$. Then $hp = h_{j_k}p_{j_k}$ for some $k < \lambda$ and $p_{j_k} \in P_{j_k}$. If $h_{j_k} = hw$ for some $w \in H$, then $p = wp_{j_k} \in P_{j_k}$. If, on the other hand, $h = h_{j_k}w$, then $w \in H\backslash P_{j_k}$ and $h_{j_k}wp = h_{j_k}p_{j_k}$. It follows that $wp = p_{j_k} \in P_{j_k}$ and $p \in P_{j_k}$ since $w \notin P_{j_k}$. Hence, $Q_r(I_\lambda^0) \subseteq \bigcup_{k<\lambda} P_{j_k}$ and $Q_r(I_\lambda^0) = \bigcup_{k<\lambda} P_{j_k} = P_r(I_\lambda^0)$, see Proposition 2.3, which proves the lemma. □

Lemma 3.2. *Assume that H and $P = \bigcup_{i\in\Omega} P_i$, $\Omega' = \{j_k \mid k < \lambda\} \subseteq \Omega$ are given as above with elements h_{j_i}, b_{j_i,j_k} as in condition* (1). *Assume further that there exists in H a nondistinguished right ideal I with $P_r(I) = P \supset \bigcup_{k<\lambda} P_{j_k}$ and $I_\lambda \supseteq I \supseteq I_\lambda^0$. Then there exists $h_{j_\lambda} \in H\backslash I$ and $P \supset P_{j_\lambda} \supset \bigcup_{k<\lambda} P_{j_k}$ with $I_\lambda \supset h_{j_\lambda}H \supset I \supset h_{j_\lambda}P_{j_\lambda} \supset I_\lambda^0$ with $h_{j_\lambda} = h_{j_k}b_{j_k,j_\lambda}$ for $b_{j_k,j_\lambda} \in P_{j_\lambda}\backslash P_{j_k}$ and any $k < \lambda$.*

Proof. It follows from Lemma 3.1 that $P_r(I) \neq P_r(I_\lambda)$, hence $I_\lambda \supset I$, and there exists $d \in I_\lambda\backslash I$. Then $d = h_{j_k}v_{j_k}$ for any $k < \lambda$ and $v_{j_k} \in H\backslash P_{j_k}$. This implies $d(\bigcup_{k<\lambda} P_{j_k}) \subseteq I_\lambda^0 \subset I$. Since $dH \supset I$ and $P_r(I) = P$, there exists by Remark 3.3 below an element $p \in P$ with $dH \supset dpH \supset I$ since I not distinguished. There exist $j_\lambda < t_{\lambda+1} \in \Omega$ with $P \supset P_{t_{\lambda+1}} \supset P_{j_\lambda} \supset \bigcup_{k<\lambda} P_{j_k}$ since $P = \bigcup_{i\in\Omega} P_i \supset \bigcup_{k<\lambda} P_{j_k}$. For $q_{j_\lambda} \in P_{t_{\lambda+1}}\backslash P_{j_\lambda}$, there exists $h_{j_\lambda} \in H\backslash I$ with $h_{j_\lambda}q_{j_\lambda} \in I$, hence $h_{j_\lambda}P_{j_\lambda} \subset I$. If $dpH \subset h_{j_\lambda}H$, replace h_{j_λ} by dp, and we obtain: $h_{j_\lambda} = da_{j_\lambda} = h_{j_k}v_{j_k}a_{j_\lambda} = h_{j_k}b_{j_k,j_\lambda}$ with $b_{j_k,j_\lambda} = v_{j_k}a_{j_\lambda}$ for $a_{j_\lambda} \in P_{j_{\lambda'}}\backslash \bigcup_{k<\lambda} P_{j_k}$ for some $j_{\lambda'} \in \Omega$. We have $d = h_{j_{k+1}}v_{j_{k+1}} = h_{j_k}b_{j_k,j_{k+1}}v_{j_{k+1}} = h_{j_k}v_{j_k}$, and therefore, $v_{j_k} = b_{j_k,j_{k+1}}v_{j_{k+1}}$. Hence, $v_{j_k} \in P_{j_{k+1}}\backslash P_{j_k}$ and $b_{j_k,j_\lambda} = v_{j_k}a_{j_\lambda} \in P_{j_{k+1}}\backslash P_{j_k}$.

We obtain

$$I_\lambda \supset h_{j_\lambda}H \supset I \supset h_{j_\lambda}P_{j_\lambda} \supseteq I_\lambda^0$$

since $h_{j_\lambda}P_{j_\lambda} = h_{j_k}b_{j_k,j_\lambda}P_{j_\lambda} \supset h_{j_k}P_{j_k}$ for all $k < \lambda$ implies the last containement. This proves the lemma, since $h_{j_\lambda}P_{j_\lambda} \notin I_\lambda^0$ for $p_{j_\lambda} \in P_{j_\lambda}\backslash \bigcup_{k<\lambda} P_{j_\lambda}$. □

We used in the above proof the following result which will be applied repeatedly in this section.

Remark 3.3. If I is a nondistinguished right ideal in the right cone H, then $I = \bigcap_{a \in H \backslash I} aH$, and for every $a \in H \backslash I$, there exists a $p \in P_r(I)$ with $apH \supset I$.

Proof. The first part of this statement follows from 1.3. If $aH \supset I \ni b$, then $b = ac$ for some $c \in H$ and $c \in P_r(I)$ follows; hence $aP_r(I) \supseteq I$. Since I is not distinguished, we have $I \neq aP_r(I)$ and $aP_r(I) \supset I$. □

For the next result, we make the same assumptions as for Lemma 3.2 except that we now assume that $P_r(I) = \bigcup_{k<\lambda} P_{j_k} = \widetilde{P}$.

Lemma 3.4. *If I is a nondistinguished right ideal of H with $I_\lambda \supseteq I \supseteq I_\lambda^0$ and $P_r(I) = \bigcup_{k<\lambda} P_{j_k} = \widetilde{P}$, then*:

(i) $I_\lambda = I$; *and*
(ii) *Either $I_\lambda = I = I_\lambda^0$, or for every $d \in I \backslash I_\lambda^0$ we have $I = (dH_{\widetilde{P}} \cap H) \supset I_\lambda^0 = d\widetilde{P}$, where $H_{\widetilde{P}}$ is the localization of H at $\widetilde{P}$.*

Proof. (i) If there exists $d \in I_\lambda \backslash I$, then $d = h_{j_k} v_{j_k}$ for any $k < \lambda$ and $v_{j_k} \in H \backslash P_{j_k}$. Since $h_{j_k} P_{j_k} \subseteq I_\lambda^0$ it follows that $d(\bigcup_{k<\lambda} P_{j_k}) \subseteq I_\lambda^0 \subseteq I$. On the other hand, $dH \supset I$ and Remark 3.3 implies that there exists $p \in P_r(I)$ with $dpH \supset I$ which is a contradiction, since $P_r(I) = \bigcup_{k<\lambda} P_{j_k}$; this proves (i).

To prove (ii), we assume that $I_\lambda = I \supset I_\lambda^0$, and there exists $d \in I \backslash I_\lambda^0$. Hence, $d = h_{j_k} v_{j_k}$ for any $k < \lambda$ and some $v_{j_k} \in H$ which implies $d\widetilde{P} \subseteq I_\lambda^0$. Consider the localization $H_{\widetilde{P}}$ of H at $\widetilde{P}$, and we obtain

$$I = (IH_{\widetilde{P}} \cap H) \supseteq (dH_{\widetilde{P}} \cap H) \supseteq (I_\lambda^0 H_{\widetilde{P}} \cap H) = I_\lambda^0 \supseteq (d\widetilde{P} H_{\widetilde{P}} \cap H) = d\widetilde{P}.$$

The three equal signs in this list of containments follow since $P_r(I) = P_r(I_\lambda^0) = P_r(d\widetilde{P}) = \widetilde{P}$. For $d_1, d_2 \in I \backslash I_\lambda^0$, we have $d_i H_{\widetilde{P}} \cap H \supseteq I_\lambda^0 \supseteq d_i \widetilde{P}$ for $i = 1, 2$. Since $d_i H_{\widetilde{P}}$ and $d_i \widetilde{P} H_{\widetilde{P}}$ are neighbouring right ideals in $H_{\widetilde{P}}$, it follows that $d_1 H_{\widetilde{P}} \cap H = d_2 H_{\widetilde{P}} \cap H$ and $d_1 \widetilde{P} = d_2 \widetilde{P}$. Therefore, $I = dH_{\widetilde{P}} \cap H$ for any $d \in I \backslash I_\lambda^0$ and $I \supset I_\lambda^0 = d\widetilde{P}$, since $IH_{\widetilde{P}} = dH_{\widetilde{P}} \supset I_\lambda^0 H_{\widetilde{P}} = d\widetilde{P} H_{\widetilde{P}}$. This proves Lemma 3.4. □

Now consider a nondistinguished right ideal I in H with $P_r(I) = P = \bigcup_{i \in \Omega} P_i$, where H, P, Ω, and the P_i, $i \in \Omega$, are defined as before Lemma 3.1. Let $P_{t_2} \supset P_{j_1} \supset P_1$ for some $t_2, j_1 \in \Omega$. Then there exist $q_{j_1} \in P_{t_2} \backslash P_{j_1}$ and $h_{j_1} \in H \backslash I$ with $h_{j_1} q_{j_1} \in I$, and hence $h_{j_1} P_{j_1} \subseteq I$. Let $P_{j_2} = P_{t_2} \cup P_2$, and there exist $t_3 > j_2$ in Ω, $q_{j_2} \in P_{t_3} \backslash P_{j_2}$, and $h_{j_2} \in H \backslash I$ with $h_{j_2} q_{j_2} \in I$, hence $h_{j_2} P_{j_2} \subseteq I$. Since $h_{j_1} H \supset I$ and $P_r(I) = P$ and I nondistinguished, there exists by Remark 3.3 an element $p \in P$ with $h_{j_1} pH \supset I$. If $h_{j_1} pH \subset h_{j_2} H$, replace h_{j_2} by $h_{j_1} p$, and we obtain $h_{j_2} H \supset I \supset h_{j_2} P_{j_2}$ and $h_{j_2} = h_{j_1} a_{j_2}$ for $a_{j_2} \in P_{j_2'} \backslash P_{j_1}$ for some $j_2' > j_1$ in Ω; in addition, $h_{j_2} P_{j_2} = h_{j_1} a_{j_2} P_{j_2} \supset h_{j_1} P_{j_1}$. We set $P_{j_3} = P_{t_3} \cup P_{j_2'} \cup P_3$.

Assume that i is not a limit ordinal and $h_{j_{i-1}} H \supset h_{j_i} H \supset I \supset h_{j_i} P_{j_i} \supset h_{j_{i-1}} P_{j_{i-1}}$ have been obtained with $h_{j_i} = h_{j_{i-1}} a_{j_i}$ for $a_{j_i} \in P_{j_i'} \backslash P_{j_{i-1}}$ and $q_{j_i} \in P_{t_{i+1}} \backslash P_{j_i}$ with $h_{j_i} q_{j_i} \in I$ for $j_{i-1}, j_i, j_i', t_{i+1} \in \Omega$ and $h_{j_{i-1}}, h_{j_i} \in H \backslash I$. Then we define $P_{j_{i+1}} = P_{t_{i+1}} \cup P_{j_i'} \cup P_{i+1}$. There exists $t_{i+2} > j_{i+1}$ in Ω, $q_{j_{i+1}} \in P_{t_{i+2}} \backslash P_{j_{i+1}}$ and $h_{j_{i+1}} \in H \backslash I$ with

$h_{j_{i+1}} q_{j_{i+1}} \in I$ and hence $h_{j_{i+1}} P_{j_{i+1}} \subseteq I$. Since $h_{j_i} H \supset I$ and $P_r(I) = P$, there exists $p \in P$ with $h_{j_i} pH \supset I$. If $h_{j_i} pH \subset h_{j_{i+1}} H$, replace $h_{j_{i+1}}$ by $h_{j_i} p$, and we obtain

$$h_{j_{i-1}} H \supset h_{j_i} H \supset h_{j_{i+1}} H \supset I \supset h_{j_{i+1}} P_{j_{i+1}} \supset h_{j_i} P_{j_i} \supset h_{j_{i-1}} P_{j_{i-1}}$$

since $h_{j_{i+1}} = h_{j_i} a_{j_{i+1}}$ for $a_{j_{i+1}} \in P_{j'_{i+1}} \backslash P_{j_i}$ for some $j_i < j'_{i+1} \in \Omega$. We set $P_{j_{i+2}} = P_{t_{i+2}} \cup P_{j'_{i+1}} \cup P_{i+2}$.

We can now give a characterization of nondistinguished right ideals of H.

Corollary 3.5. *Let H, Ω, and $P = \bigcup_{i \in \Omega} P_i$ be given as before* 3.1. *Then a right ideal I of H has P as its associated prime ideal, that is $P_r(I) = P$, and is not distinguished, if and only if there exist a limit ordinal λ, a subset $\Omega' = \{j_k \mid k < \lambda\} \subseteq \Omega$, and elements $h_{j_k} \in H$ so that condition* (1) *holds with $I_\lambda = \bigcap_{k<\lambda} h_{j_k} H = I$ and $P = \bigcup_{k<\lambda} P_{j_k}$.*

Proof. Assume that λ and Ω' exist as in the statement of the Corollary 3.5. Then it follows from Lemma 3.1 that $P_r(I) = P$ for $I = I_\lambda$ and $P = \bigcup_{k<\lambda} P_{j_k}$. To show that I is nondistinguished, we assume that on the contrary $I = cP$. Then $c \notin I$, and there exists $k < \lambda$ with $h_{j_k} = cv$ for some $v \in H$; it follows that $h_{j_{k+1}} = cvb_{j_k, j_{k+1}} \in cP = I$, since $b_{j_k, j_{k+1}} \in P_{j_{k+1}} \subset P$. This is a contradiction that shows that I is not distinguished.

If conversely $P_r(I) = P$ for a nondistinguished right ideal I of H, then transfinite induction can be used to construct a subset $\Omega' = \{j_k \mid k < \lambda\} \subseteq \Omega$, $h_{j_k} \in H$, $k < \lambda$, for some limit ordinal λ so that condition (1) holds with $I_\lambda = \bigcap_{k<\lambda} h_{j_k} H = I$, and $P = \bigcup_{k<\lambda} P_{j_k}$.

The elements h_{j_1} and h_{j_i} are constructed as described after Lemma 3.4 in the case where i is not a limit ordinal and $h_{j_{i-1}}$ has already been constructed. In the case where i is a limit ordinal and the h_{j_k} for $k < i$ are given, then h_{j_i} is obtained as in Lemma 3.2. That this process ends for some limit ordinal λ with $\bigcap_{k<\lambda} h_{j_k} H = I_\lambda = I$ and $P = \bigcup_{k<\lambda} P_{j_k} = P_r(I)$ follows from Lemma 3.4. This proves the corollary. □

In the case where $\Omega = \mathbb{N}$, we obtain a more precise result.

Corollary 3.6. *Let H be a right cone in a group G with a limit prime ideal $P = \bigcup_{i \in \mathbb{N}} P_i$ where $P_i \subset P_j \subset P$ for $i < j \in \mathbb{N}$. Then a right ideal I of H is nondistinguished with $P_r(I) = P$ if and only if there exist an infinite subsequence $\Omega' = \{j_k \mid k \in \mathbb{N}\} \subseteq \mathbb{N}$ with $j_i < j_k$ for $i < k$ and elements $a_{j_i} \in H$, $a_{j_i} \in P_{j_i} \backslash P_{j_{i-1}}$ for $i \geq 2$ with $I = \bigcap_{n \in \mathbb{N}} I_n$ for $I_n = a_{j_1} \ldots a_{j_n} H$.*

Proof. This follows from Corollary 3.5: If I is nondistinguished with $P_r(I) = P$, then $h_{j_k} = a_{j_1} \ldots a_{j_k}$ with $a_{j_1} = h_{j_1}$, $b_{j_{i-1}, j_i} = a_{j_i}$ for $i \geq 2$, and $I = \bigcap_{k \in \mathbb{N}} h_{j_k} H$.

Conversely, the subset Ω' and the set of elements $\{h_{j_k} = a_{j_1} \ldots a_{j_k} \mid k \in \mathbb{N}\}$ satisfy the condition (1); hence I is a nondistinguished ideal with $P_r(I) = P$. □

Let H be a right cone in a group G and $P = \bigcup_{i \in \Omega} P_i$ be a limit prime ideal of H as before Lemma 3.1. It follows from Lemma 3.4 that the set of right ideals I of H with $P_r(I) = P$ can be divided into three classes:

(C1) $\{dP \mid d \in H\}$, the set of distinguished right ideals of H with P as associated prime ideal.

(C2) $\{I = dH_P \cap H \mid d \in H \text{ with } P_r(I) = P\}$.

A right ideal I of H is in this class if and only if there exists a subset Ω' of Ω and elements $h_{j_i} \in H\backslash I$, $j_i \in \Omega'$ as in Corollary 3.5 with $I = \bigcap_{j_i \in \Omega'} h_{j_i} H$ and $\bigcup_{j_i \in \Omega'} P_{j_i} = P$, but $I \supset \bigcup h_{j_i} P_{j_i}$. We note that in contrast to the case (C1), it is possible that $P_r(dH_P \cap H) \subset P$ for $d \in J(H)$, and it is, therefore, necessary to add the condition $P_r(I) = P$ in the case (C2).

(C3) $\{I \mid P_r(I) = P \text{ with } I \neq dP,\ I \neq dH_P \cap H \text{ for all } d \in H\}$.

A right ideal I is in this class if and only if

$$I = \bigcap_{j_i \in \Omega'} h_{j_i} H = \bigcup_{j_i \in \Omega'} h_{j_i} P_{j_i} \quad \text{for } \Omega' \subseteq \Omega, \qquad h_{j_i} \in H \quad \text{for } j_i \in \Omega', \quad \bigcup_{j_i \in \Omega'} P_{j_i} = P$$

as in Corollary 3.5 and also by Lemma 3.4.

Using the localization H_P of H at P and Lemma 2.14, one shows that these three classes are mutually disjoint.

In Corollary 3.5, we characterized the nondistinguished right ideas I with $P_r(I) = P$, and Lemma 3.4 was used to distinguish the elements in either (C2) or (C3).

In the next result we characterize those right ideals I with $P_r(I) = P$, which are in either the class (C1) or the class (C3).

Corollary 3.7. *Let H, Ω and $P = \bigcup_{i \in \Omega} P_i$ be given as before* 3.1. *Then*:

(a) *A right ideal I of H has P as its associated prime ideal, that is, $P_r(I) = P$, and $I \neq dH_P \cap H$ for all $d \in H$ if and only if there exists a limit ordinal λ, a subset $\Omega' = \{j_k | k < \lambda\} \subseteq \Omega$ and $h_{j_k} \in H\backslash I$ with $h_{j_k} = h_{j_i} b_{j_i, j_k}$ for $b_{j_i, j_k} \in H\backslash P_{j_i}$, $i < k < \lambda$, and*

$$I = \bigcup_{j_i \in \Omega'} h_{j_{i+1}} p_{j_{i+1}} H \quad \textit{for elements } p_{j_{i+1}} \in P_{j_{i+1}} \backslash P_{j_i} \quad \textit{and} \quad \bigcup_{j_i \in \Omega'} P_{j_i} = P.$$

(b) *The right ideal I of H in statement* (a) *is nondistinguished if and only if the elements $h_{j_i} \in H\backslash I$, $j_i \in \Omega'$, can be chosen so that $I = \bigcap_{j_i \in \Omega'} h_{j_i} H$ and $b_{j_i, j_k} \in P_{j_k} \backslash P_{j_i}$ for $i < k$.*

Proof. (a) If $P_r(I) = P$ and $I \neq dH_P \cap H$ for all $d \in H$, then I is either distinguished or in the class (C3) by the above observation.

If $I = cP = c \bigcup_{i \in \Omega} P_i$ is distinguished, then $\Omega' = \Omega$, $h_i = c \in H\backslash I$ for all $i \in \Omega$, and for every element $p_{i+1} \in P_{i+1} \backslash P_i$ we have $cP_i \subset cp_{i+1} H \subseteq cP_{i+1}$, and $I = \bigcup_{i \in \Omega} cp_{i+1} H$ follows with $b_{i,k} = 1$ for all $i \leq k \in \Omega$.

If I is in the class (C3), then by Lemma 3.4 and Corollary 3.5, we have a subset $\Omega' = \{j_k \mid k < \lambda\} \subseteq \Omega$ for a limit ordinal λ, elements $h_{j_i} \in H\backslash I$ for $j_i \in \Omega'$ with $h_{j_k} = h_{j_i} b_{j_i, j_k}$ for $b_{j_i, j_k} \in P_{j_k} \backslash P_{j_i}$, $i < k < \lambda$, and $I = \bigcap_{j_i \in \Omega'} h_{j_i} H = \bigcup_{j_i \in \Omega'} h_{j_i} P_{j_i} = \bigcup_{j_i \in \Omega'} h_{j_{i+1}} p_{j_{i+1}} H$ for any elements $p_{j_{i+1}} \in P_{j_{i+1}} \backslash P_{j_i}$. The last equation follows since $h_{j_i} P_{j_i} \subset h_{j_i} b_{j_i, j_{i+1}} p_{j_{i+1}} H = h_{j_{i+1}} p_{j_{i+1}} H \subseteq h_{j_{i+1}} P_{j_{i+1}}$.

To prove the converse, we have $h_{j_{i+1}} P_{j_i} \subseteq h_{j_{i+1}} p_{j_{i+1}} H \subseteq I$, $h_{j_{i+1}} \notin I$, and hence $P_{j_i} \subseteq Q_r(I)$, $P = \bigcup_{j_i \in \Omega'} P_{j_i} \subseteq Q_r(I)$. If $t \in Q_r(I)$ with $at \in I$ and $a \in H\backslash I$, then $at =$

$h_{j_{i+1}} p_{j_{i+1}} v$ for some $v \in H$ and some $j_{i+1} \in \Omega'$; since $a \in H \backslash I$, there exists $w \in H$ with $aw = h_{j_{i+2}} p_{j_{i+2}}$. It follows that

$$at = h_{j_{i+1}} p_{j_{i+1}} v = h_{j_{i+2}} p_{j_{i+2}} q_{j_{i+1}} v = awq_{j_{i+1}} v$$

for some $q_{j_{i+1}} \in P_{j_{i+1}}$, since $h_{j_{i+2}} p_{j_{i+2}} H = h_{j_{i+1}} b_{j_{i+1}, j_{i+2}} p_{j_{i+2}} H \supset h_{j_{i+1}} p_{j_{i+1}} H$ and

$$h_{j_{i+1}} b_{j_{i+1}, j_{i+2}} p_{j_{i+2}} q_{j_{i+1}} = h_{j_{i+1}} p_{j_{i+1}}$$

follows for some $q_{j_{i+1}} \in H$. Hence, $b_{j_{i+1}, j_{i+2}} p_{j_{i+2}} q_{j_{i+1}} = p_{j_{i+1}}$ which implies $q_{j_{i+1}} \in P_{j_{i+1}} \backslash P_{j_i}$. Therefore, $t = wq_{j_{i+1}} v \in P_{j_{i+1}}$, and $Q_r(I) \subseteq \bigcup_{j_i \in \Omega'} P_{j_i}$, $Q_r(I) = \bigcup_{j_i \in \Omega'} P_{j_i} = P_r(I)$ follows, since $Q_r(I)$ is an ideal.

It remains to prove that $I \neq dH_P \cap H$ for any $d \in H$. Otherwise, $I = dH_P \cap H$ and $IH_P = dH_P$ for some $d \in I$. Then $d = h_{j_{i+1}} p_{j_{i+1}} h$ for some $h \in H$, and

$$h_{j_{i+2}} p_{j_{i+2}} = das^{-1} = h_{j_{i+1}} p_{j_{i+1}} has^{-1} = h_{j_{i+2}} p_{j_{i+2}} q_{j_{i+1}} has^{-1}$$

for some $a \in H, s \in H \backslash P$, and $q_{j_{i+1}} \in P_{j_{i+1}}$ by a previous observation. From this follows the contradiction $s \in P_{j_{i+1}}$ which proves that $I \neq dH_P \cap H$ for any $d \in H$ and statement (a).

An application of Lemma 3.4 and Corollary 3.5 proves (b). □

Example 3.8. Let C_i be an ordered group for $i \in \mathbb{Z}$, and let $\widetilde{G} = \prod_{i \in \mathbb{Z}} C_i$ be the direct product of the C_i. Then $\widetilde{G}$ contains as a subgroup the group $\widehat{G} = \{\alpha = (a_i) \in \widetilde{G} \mid \operatorname{supp}(\alpha) \text{ well ordered}\}$ of elements $\alpha = (a_i) \in \widetilde{G}$ with $\operatorname{supp}(\alpha) =$ support of $\alpha = \{i \mid a_i \neq e_i\}$ well ordered, where e_i is the identity of C_i. The group $\widehat{G}$ is again ordered if we define $\alpha = (a_i) < \beta = (b_i)$ if and only if there exists an i_0 with $a_i = b_i$ for $i < i_0$ and $a_{i_0} < b_{i_0}$. For an element $\alpha \in \widehat{G}$, $\alpha = (a_i) \neq (e_i)$, we define the leading index $\operatorname{lind} \alpha = i_0 = \min\{i \mid a_i \neq e_i\}$ and then say that $a_{i_0} = \operatorname{lco}(\alpha)$ is the leading component of α. Then $\widehat{H} = \{\alpha \in \widehat{G} \mid \alpha = (e_i) \text{ or } \operatorname{lco}(\alpha) > e_{i_0} \text{ for } \operatorname{lind} \alpha = i_0\}$ is the cone of the order of $\widehat{G}$.

From now on we assume that $C_1 = (\mathbb{Q}, +)$ and $C_2 = (\mathbb{R}, +)$ and $C_i = (\mathbb{Z}, +)$ for $i \neq 1, 2$ where $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{Z}$ is the set of integers.

Next we show that $\widehat{P}_n = \{\alpha \in \widehat{H} \mid \operatorname{lind} \alpha \leq n\}$, $n \in \mathbb{Z}$ is a prime ideal in $\widehat{H}$ for every $n \in \mathbb{Z}$, and that

$$\ldots \widehat{P}_{-2} \subset \widehat{P}_{-1} \subset \widehat{P}_0 \subset \widehat{P}_1 \subset \cdots \subset \widehat{P}_n \subset \cdots \subset \bigcup_{n \in \mathbb{Z}} \widehat{P}_n = \widehat{P} = \widehat{J}(\widehat{H}) \subset H$$

is the chain of all prime ideals of $\widehat{H}$. Hence, $\widehat{P}$ is the only limit prime of $\widehat{H}$ and $\widehat{P}_n \supset \widehat{P}_{n-1}$ for $n \in \mathbb{Z}$ are the prime segments of $\widehat{H}$.

To prove that $\widehat{P}_n$ is (completely) prime, let $\alpha \in \widehat{P}_n$, $\beta \in \widehat{H}$, and $\alpha + \beta \in \widehat{P}_n$ follows since $\operatorname{lind}(\alpha + \beta) \leq n$, if $\gamma, \delta \in \widehat{H} \backslash \widehat{P}_n$, then $\operatorname{lind}(\gamma + \delta) > n$ and $\gamma + \delta \in \widehat{H} \backslash \widehat{P}_n$.

Let Q be any prime ideal of $\widehat{H}$, and we consider $\operatorname{ind}(Q) = \max\{\operatorname{lind}(\alpha) \mid \alpha \in Q\}$. Either $\operatorname{lind}(Q)$ does not exist, and $Q = \widehat{P}$ follows, or $\operatorname{lind}(Q) = n$ for some $n \in \mathbb{Z}$. We want to prove that $Q = \widehat{P}_n$ in this case, and $Q \subseteq \widehat{P}_n$ follows from the definitions.

To prove that $\widehat{P}_n \subseteq Q$, let $\beta = (b_i) \in \widehat{P}_n$, and we must consider the cases $n = 1$ or 2 and $n \neq 1, 2$ separately. Assume $n \neq 1, 2$. If $\operatorname{lind}(\beta) < n$, then $\beta \in Q$. If $\operatorname{lind}(\beta) = n$, then $b_n > 0$ for $b_n \in \mathbb{Z}$. Since $\operatorname{ind}(Q) = n$ there exists $\alpha = (a_i) \in Q$, $\operatorname{lind}(\alpha) = n$ and $0 < a_n \in \mathbb{Z}$. For $\delta = (d_i)$ with $d_n = 1$, $d_{n+1} < b_{n+1}$ and $d_i = 0$ for $i \neq n, n+1$ we have $\beta > \delta$ and $\delta + \delta + \cdots + \delta = (a_n + 1)\delta > \alpha$, and $\delta \in Q$ follows since Q is prime. Therefore, $\delta \in Q$, $\beta \in Q$, $\widehat{P}_n \subseteq Q$ and $\widehat{P}_n = Q$.

Now assume that $\operatorname{ind}(Q) = n = 1$ or 2 for a prime ideal Q of $\widehat{H}$. Then $Q \subseteq \widehat{P}_n$ and in order to prove that $\widehat{P}_n \subseteq Q$ assume $\beta \in \widehat{P}_n$. If $\operatorname{lind}(\beta) < n$, then $\beta \in Q$. If $\operatorname{lind}(\beta) = n$ with $0 \leq b_n \in \mathbb{R}$, then there exists $\epsilon \in \mathbb{Q}$ with $0 < \epsilon < b_n$ and $\delta = (d_i)$ with $d_n = \epsilon$ and $d_i = 0$ for $i \neq n$ is an element in $\widehat{H}$ with $\delta < \beta$. Since $\operatorname{ind}(Q) = n$, there exists $\alpha \in Q$ with $\operatorname{lind}(\alpha) = n$ and $a_n > 0$. There exists $k \in \mathbb{Z}$ with $a_n < k \cdot \epsilon$, and $\delta + \cdots + \delta = k \cdot \delta \in Q$ follows. Since Q is prime, we have $\delta \in Q$, $\beta \in Q$, and $\widehat{P}_n = Q$.

The ordered group $(\widehat{G}, \widehat{H})$ contains the subgroup $G = \sum_{i \in \mathbb{Z}} C_i$ with the cone $H = \widehat{H} \cap G$. The chain of prime ideals of H is

$$\cdots \subset P_{-2} \subset P_{-1} \subset P_0 \subset P_1 \subset \cdots \subset P_n \subset \cdots \bigcup_{n \in \mathbb{Z}} P_n = P \subset H$$

with $P_n = \widehat{P}_n \cap H$ for $n \in \mathbb{Z}$ and $P = \widehat{P} \cap H = \bigcup_{n \in \mathbb{Z}} P_n$.

Let I be an ideal in H and $P_r(I) = P$ or $P_r(I) = P_n$ where $P_n \supset P_{n-1}$ is a prime segment of H. In this last case, we obtain $H_{P_n} = P_n \cup U(H_{P_n})$ with $U(H_{P_n}) = \sum_{i > n} C_i$ considered as a subgroup of G, and $H_n = H^0_{P_n} = H_{P_n} \backslash P_{n-1} = (P_n \backslash P_{n-1}) \cup U(H_{P_n})$ is a rank one cone. There exists a monomorphism ϕ_n from the cone $\overline{H}_n = \{aH_n \mid a \in H_n\}$ into the semigroup $(\mathbb{R}^+, +)$ of non-negative real numbers under addition; the operation for $\overline{H}_n$ is defined by $aH_n bH_n = abH_n$.

For $n \neq 1, 2$ the cone H_n is discrete, hence $I = cP_n$ for some $c \in H$, and there exists only the one equivalence class of ideals I of H with $P_r(I) = P_n$, that is, $\mathscr{P}(P_n) = \{cP_n \mid c \in H\}$ for $n \neq 1, 2$. For $n = 1$, the image of ϕ_1 in $(\mathbb{R}^+, +)$ is $(\mathbb{Q}^+, +)$.

By 1.3 it follows that the ideals of $\overline{I} \neq H_1$ of H_1 are either distinguished or of the form $\overline{I} = \overline{I}_\rho = \bigcap_{\phi_1(aH_1) < \rho} aH_1$ for $0 \leq \rho \in \mathbb{R}$ and $\overline{I}_{\rho_1} \sim \overline{I}_{\rho_2}$ in H_1 for $\rho_1 \leq \rho_2 \in \mathbb{R} \backslash \{0\}$ if and only if $\rho_2 - \rho_1 \in \mathbb{Q}$, see Lemmas 4.1 and 4.2. It follows from 2.17, 2.11 that $\mathscr{P}(P_1) = \{cP_1 \mid c \in H\} \cup \{cI_\rho \mid c \in H, 0 < \rho \in \mathbb{R}\}$ for $I_\rho = \bigcap_{\phi_1(aH_1) < \rho} aH$. The subsets $\{cP_1 \mid c \in H\}$ and $\{c(dH_{P_1} \cap H) \mid c, \in H, d \in P_1 \backslash P_2\} = \{cI_\rho \mid c \in H, \rho \in \mathbb{Q}^+ \backslash \{0\}\}$ form two equivalence classes of ideals in $\mathscr{P}(P_1)$. The other equivalence classes contained in $\mathscr{P}(P_1)$ correspond to the intersection $(r + \mathbb{Q}) \cap \mathbb{R}^{>0}$ for a coset $r + \mathbb{Q} \neq \mathbb{Q}$ in $(\mathbb{R}, +)$ by Lemma 4.2. Infinitely many examples for such classes are given by $\sqrt{p} + \mathbb{Q} \cap \mathbb{R}^{>0}$ for p prime in $\mathbb{Z}$.

The previous discussion can be applied to $\mathscr{P}(P_2)$ with the result that $\mathscr{P}(P_2)$ consists of exactly two equivalence classes $\{cP_2 \mid c \in H\}$ and the class $\{c(dH_{P_2} \cap H) \mid c \in H, d \in P_2 \backslash P_3\}$ of locally principal ideals.

We now consider the set of ideals I in H with $P_r(I) = P = \bigcup_{n \in \mathbb{N}} P_n$. Then I is either of the form $= cP$, that is, I is distinguished, or $I = \bigcap a_{j_1} \ldots a_{j_n} H$ with $a_{j_i} \in H$, $a_{j_i} \in P_{j_i} \backslash P_{j_{i-1}}$ for $i \geq 2$ and $1 < j_1 < j_2 \leq \cdots \in \mathbb{Z}$ by Corollary 3.6. Since $a_{j_i} \in P_{j_i} \backslash P_{j_{i-1}}$ implies that $j_{i-1} < \operatorname{lind} a_{j_i} \leq j_i$, the element $\hat{g} = a_{j_1} + a_{j_2} + \cdots$ exists in $\widehat{H}$; only finitely many a_{j_i} contribute a nonzero element to the kth-component of $\hat{g}$ for any k. We claim that $I = I_{\hat{g}} = \{g \in H \mid g \geq \hat{g} \in \widehat{H}\}$.

We show first that $g \in I_{\hat{g}}$ implies $g \in I = \bigcap_{n\in\mathbb{N}} a_{j_1} + \cdots + a_{j_n} H$. Otherwise there exists an n and $u \in H$ with $a_{j_1} + \cdots + a_{j_n} = g + u$. However, $\hat{g} = a_{j_1} + \cdots + a_{j_n} + w$, and $g = \hat{g} + v$ for $w, v \in \widehat{H}$ implies $a_{j_1} + \cdots + a_{j_n} = a_{j_1} + \cdots + a_{j_n} + w + v + u$ which is possible only for $w = (0) = v = u$, $g = a_{j_1} + \cdots + a_{j_n}$ which is a contradiction. Hence, $I_{\hat{g}} \subseteq I$.

Conversely assume $g = (c_i) \in I$ and $g < \hat{g} = (d_i)$. Then there exists an index i_0 with $c_i = d_i$ for $i < i_0$ and $c_{i_0} < d_{i_0}$. However, $g = a_{j_1} + \cdots + a_{j_{i_0}} + v$ for $v \in H$ and $\hat{g} = g + w$ for $w \in \widehat{H}$ since $g < \hat{g}$. Hence, $\hat{g} = a_{j_1} + \cdots + a_{j_{i_0}} + v + w$ and $v + w = a_{j_{i_0+1}} + a_{j_{i_0+2}} + \cdots$, which implies $\operatorname{lind}(v + w) = \min\{\operatorname{lind}(v), \operatorname{lind}(w)\} > j_{i_0} \geq i_0$. It follows that $c_{i_0} = d_{i_0}$, and this contradiction proves that $I \subseteq I_{\hat{g}}$.

Since every element $\hat{g} \neq (0)$ in $\widehat{H}$ can be written in the form $\hat{g} = a_{j_1} + a_{j_2} + \ldots$ for $1 \leq j_1 < j_2 < \cdots$ with $a_{j_i} \in P_{j_i} \backslash P_{j_{i-1}}$ for $i \geq i \geq 2$, it follows that $\{g \in H \mid g \geq \hat{g}\} = I_{\hat{g}} = \bigcap_{n\in\mathbb{N}} a_{j_1} + \cdots + a_{j_n} + H$ is an ideal in $\mathscr{P}(P)$ and $\mathscr{P}(P) = \{cP \mid c \in H\} \cup \{I_{\hat{g}} \mid (0) \neq \hat{g} \in \widehat{H}\}$ follows. Two ideals $I_{\hat{g}_1}$ and $I_{\hat{g}_2}$ in $\mathscr{P}(P)$ are related in H if and only if $\hat{g}_2 = \hat{g}_1 + w$ for $w \in H$ if $\hat{g}_2 \geq \hat{g}_1$ in $\widehat{H}$. Hence $\mathscr{P}(P)$ contains infinitely many equivalence classes besides the classes $\{c + P \mid c \in H\}$ of distinguished ideals, and the class $\{I_{\hat{g}} \mid 0 \neq \hat{g} \in H\}$ of principal ideals. The ideals $I_{\widehat{g_n}}$ with $\widehat{g_n} = (\ldots, 0, 0, n, n, \ldots) = (d_i^{(n)})$ with $d_i^{(n)} = n$ for $i \geq 1$, $d_i^{(n)} = 0$ for $i < 1$ are nonequivalent representatives of distinct equivalence classes for related ideals contained in $\mathscr{P}(P)$ for $n = 1, 2, \ldots$.

We note that $I\widehat{g_1} \neq I\widehat{g_2}$ in H for $\widehat{g_1} \neq \widehat{g_2}$ in $\widehat{H}$ since H is dense in $\widehat{H}$. It also follows that the set $\mathscr{P}(\widehat{P})$ of ideals $\widehat{I}$ in $\widehat{H}$ with $P_r(\widehat{I}) = \widehat{P}$ consists of just two equivalence classes, i.e., the distinguished ideals and the principal ideals.

We can use Lemma 2.6 to show that $P_r(I_{\hat{g}}) = P$ for any $\hat{g}$ since $I_{\hat{g}} s = I_{\hat{g}}$ for $s \in H$ if and only if $s = (0)$.

4. RELATED RIGHT IDEALS IN LOCALLY INVARIANT RIGHT CONES

Let H be a right cone. We want to determine the classes of related right ideals I with $P_r(I) = P'$, and $P' \supset P''$ a right invariant prime segment of H. It follows from Theorem 2.17 and Proposition 2.14 that we can assume that H is a right invariant rank one right cone in a group with $P'' = \Phi$.

If $a \in H$ and $u \in U(H)$, then $ua = au'$, and for $v \in U(H)$ with $vu = 1$ it follows that $a = vua = av'u'$ for $u', v' \in H$, $v'u' = 1$, and $u' \in U(H)$; we obtain $U(H)a \subseteq aU(H)$.

It follows that $\overline{H} = H/U(H) = \{aU(H) \mid a \in H\}$ is a monoid with $aU(H)\, bU(H) = abU(H)$ and if $baU(H) = caU(H)$, then $b = cd$ or $c = bd$ and $daU(H) = aU(H)$ for $a, b, c \in H$, some $d \in H$. For $a \in H$, the set $\{d \mid d \in H,\ daU(H) = aU(H)\}$ is equal to $U(H)$ or to all of H since H is right invariant of rank one. However, this set can not be equal to H, hence $d \in U(H)$ follows, and $\overline{H}$ is right and left cancellative and a right cone in a group. We obtain the following result.

Lemma 4.1. *Let H be a right invariant rank one right cone in a group G, then $\overline{H} = H/U(H)$ exists and is a right cone in a group G_2. The semigroup $\overline{H}$ is right*

and left-ordered. There exists an order monomorphism ϕ from $\overline{H}$ into the semigroup $(\mathbb{R}^+, +)$ of nonnegative real numbers under addition.

Proof. It was shown above that the right cone $\overline{H} = H/U(H)$ exists. It follows that the identity $\overline{e} = U(H)$ of $\overline{H}$ is the only unit in $\overline{H}$, that $\overline{H}$ is right invariant, and that $aU(H) \geq bU(H)$ if and only if $a = bc$ for $a, b, c \in H$ defines a right and left (total) order in $\overline{H}$: It only remains to show that this is a right order. Therefore, assume $aU(H) \geq bU(H)$, that is, $a = bc$ for $a, b, c \in H$ and $d \in H$. Then $adU(H) = bcdU(H) = bdc'U(H) \geq bdU(H)$ for some element $c' \in H$. By Proposition 1.6(c) there exists for $a \in J(H)$ and $b \in H$ a natural number n wiht $a^n = bp$ for some $p \in J(H)$; hence $(aU(H))^n > bU(H)$. By Hölder's Theorem (see [10, p. 228]), there exists, therefore, an order preserving monomorphism ϕ from $\overline{H}$ into $(\mathbb{R}^+, +)$, the ordered monoid of non-negative real numbers under addition. □

The right cone $\overline{H}$ of rank one is order isomorphic to $\{aH \mid a \in H\}$ with $aHbH = abH$ defining the operation and $aH \geq bH$ if and only if $aH \subseteq bH$ for H a right invariant rank one right cone, since $aH = bH$ if and only if $aU(H) = bU(H)$ for $a, b \in H$. We define $\phi(aH) = \phi(aU(H)) \in \mathbb{R}$. It follows that $abH = aHbH = bHaH = baH$ for $a, b \in H$.

Then $\overline{H}$ has a smallest positive element if and only if $J(H) = aH$ is a principal right ideal. In this case, all right ideals $\neq H$ of H are positive powers $(aH)^n = a^nH$ of aH and form a single class of related ideals of H.

If $\overline{H}$ does not have a smallest positive element, then $\phi(\overline{H})$ is a dense submonoid of $(\mathbb{R}^+, +, \leq)$, and every real number $\rho \geq 0$ can be obtained as a limit of a sequence $\{s_i\}$ of elements $s_i = \phi(a_iH) \in \phi(\overline{H}) \subseteq \mathbb{R}^+$ with $\cdots s_i > s_{i+1} > \cdots > \rho$. For each $\rho \in \mathbb{R}^+$, we define a right ideal $I_\rho = \bigcup_{i\in\mathbb{N}} \phi^{-1}(s_i) = \bigcup_{i\in\mathbb{N}} a_iH$, and $I_\rho = \bigcup_{\rho<\phi(aH)} aH$ follows. Then I_ρ is not a principal right ideal of H.

Conversely, if I is a nonprincipal right ideal of H, then $I = I_\rho$ for $\rho = \inf\{\phi(aH) \mid a \in I\}$. Hence, if $\widehat{D}$ is the set of all nonprincipal right ideals I of H, the mapping $\psi : \widehat{D} \to \mathbb{R}^+$ defined by $\psi(I) = \inf\{\phi(aH) \mid a \in I\}$ is one-to-one and onto with $\psi(aJ(H)) = \phi(aH)$; in particular, $\psi(J(H)) = \phi(H) = 0$, where we assumed that $J(H)$ is not a principal right ideal.

A right ideal I of H is a nonprincipal right ideal if and only if $IJ = I$. Hence the set $\widehat{D}$ is in addition an ordered semigroup with multiplication of right ideals as operation and $I \geq I'$ if and only if $I \subseteq I'$ for $I, I' \in \widehat{D}$ defining the order. If $I = I_{\rho_1} = \bigcup_{i\in\mathbb{N}} a_iH$ and $I' = I_{\rho_2} = \bigcup_{i\in\mathbb{N}} b_iH$ are elements in $\widehat{D}$ with $a_i, b_i \in H$, $\lim \phi(a_iH) = \rho_1$, $\lim \phi(b_iH) = \rho_2$, then $I_{\rho_1}I_{\rho_2} = \bigcup_{i\in\mathbb{N}} a_iHb_iH$ with $\lim \phi(a_iHb_iH) = \rho_1 + \rho_2$, the right ideal $I_{\rho_1}I_{\rho_2}$ is in $\widehat{D}$ and $I_{\rho_1} \cdot I_{\rho_2} = I_{\rho_1+\rho_2}$. It follows that ψ is an order isomorphism from the ordered monoid $\widehat{D}$ onto $(\mathbb{R}^+, +, \leq)$. This proves part of the following theorem.

Theorem 4.2. *Let H be a right invariant right cone of rank one in a group G with $J(H)$ not a principal right ideal. Then*:

(i) *The set $\widehat{D}$ of nonprincipal right ideals I of H forms an ordered monoid which is order isomorphic under the mapping ψ to $(\mathbb{R}^+, +, \leq)$;*
(ii) *The monoid $\widehat{D}$ contains the dense monoid $D = \{aJ \mid a \in H\}$ and $\psi(aJ) = \phi(aH)$ for all $a \in H$;*

(iii) *The following conditions are equivalent for right ideals* $I_{\rho_1} \subset I_{\rho_2} \in \widehat{D}$:

(a) I_{ρ_1} *is related to* I_{ρ_2} *in* H;
(b) $I_{\rho_1} = aJI_{\rho_2}$ *for some* $a \in J(H)$;
(c) $\rho_1 - \rho_2 = \phi(aH)$ *for some* $a \in J(H)$.

Proof. The results (i) and (ii) were proved before the theorem was stated.

(iii) If $s^{-1}I_{\rho_1} = t^{-1}I_{\rho_2}$ for $s \in H\backslash I_{\rho_1}$, $t \in H\backslash I_{\rho_2}$ in (iii)(a), and if $s = ta$ for some $a \in H$, then $taH = atH$, $s = atu$ for $u \in U(H)$ and $I_{\rho_1} = atut^{-1}I_{\rho_2} = att^{-1}I_{\rho_2} = aI_{\rho_2}$ since $u \cdot I = I$ for $u \in U(H)$ and $I = t^{-1}I_{\rho_2}$ a right ideal of H by the right invariance of H. If $t = sb$ for $b \in J(H)$, then $t = bsv$ for some $v \in U(H)$, and the contradiction $I_{\rho_2} = bI_{\rho_1} \subseteq I_{\rho_1}$ follows. Hence, we are left with the case $I_{\rho_1} = aI_{\rho_2} = aJI_{\rho_2}$ which shows that (a) implies (b), where we use $JI_{\rho_2} = I_{\rho_2}J = I_{\rho_2}$ since $\widehat{D}$ is commutative.

Conversely, if $I_{\rho_1} = aJI_{\rho_2}$, then $aJI_{\rho_2} = aI_{\rho_2}$ and $a \notin I_{\rho_1}$ with $a^{-1}I_{\rho_1} = I_{\rho_2}$ and $I_{\rho_1} \sim I_{\rho_2}$; the conditions (a) and (b) are equivalent.

The equivalence of (b) and (c) follows if the mappings ψ and ψ^{-1} are applied and since $\psi(aJ) = \phi(aH)$ for $a \in H$. □

We are now in the position to classify the classes of related right ideals I of a cone H with associated prime ideal $P_r(I) = P' \supset P''$ and right invariant prime segment $P' \supset P''$.

Theorem 4.3. *Let* $H \supset P' \supset P''$ *be a right cone in a group* G *with a right invariant prime segment* $P' \supset P''$, *and let* $H' = H^0_{P'} = H_{P'}\backslash P''H_{P'}$, *where* $H_{P'}$ *is the localization of* H *at* P'.

(i) *The classes of related right ideals* I *of* H *with* $P_r(I) = P'$ *correspond to the classes of related right ideals in* H'.
(ii) *If* $P' \neq P'^2$, *then* $J(H') \neq J(H')^2$, $\phi(\overline{H'}) \cong \mathbb{N}_0$, *and there is exactly one class of related right ideals* I *in* H *with* $P_r(I) = P'$, *the class* $\{aP' \mid a \in H\}$ *of* P'-*distinguished right ideals of* H.
(iii) *If* $P' = P'^2$ *and* $\phi(\overline{H'}) = \mathbb{R}^+$, *then there are exactly two classes of related right ideals* I *of* H *with* $P_r(I) = P'$: *The class of* P'-*distinguished right ideals, and* $\{b(H \cap aH_{P'}) \mid a \in P'\backslash P'', b \in H\}$ *the class of right ideals in* H *related to right ideals in* H *that correspond to principal right ideals* $\neq H'$ *of* H'.
(iv) *If* $P' = P'^2$ *and* $\phi(\overline{H'}) \subset \mathbb{R}^+$, *then* $\{b(H \cap aH_{P'}) \mid a \in P'\backslash P'', b \in H\}$ *is one class of related right ideals* I *of* H *with* $P_r(I) = P'$, *and there are in addition infinitely many classes of such right ideals in* H, *and they correspond to the cosets of* $\psi(D) = \phi(\overline{H'})$ *in* $\mathbb{R}^+$.

Proof. (i) follows from Lemmas 2.11, 2.13, 2.14, Proposition 2.16(iv), and Theorem 2.17(iia).

(ii) $J(H') = P'H_{P'}\backslash P''H_{P'}$ and $P'^2 \neq P'$ if and only if $(P'H_{P'})^2 = P'^2H_{P'} \neq P'H_{P'}$. Hence, $J(H')^2 \neq J(H')$ if and only if $P'^2 \neq P'$ in H. In this case, $J(H') = aH'$ for some $a \in H$ is a principal right ideal in H', every right ideal I of H' has the form a^nH', and $\phi(\overline{H'}) \cong \mathbb{N}_0$.

There is exactly one class of related right ideals in H' and exactly one class, $\{bP' \mid b \in H\}$, of related right ideals in H with P' as associated prime ideal; the class of P'-distinguished right ideals, see Lemma 2.8.

(iii) It follows from Remark 2.10 that a right ideal I' in H' related to a principal right ideal $aH' \neq H'$ is again a right principal ideal. Conversely, let $H' \supset bH' \supset aH'$, then $bc = a$ and $aH' = bcH' = cbH'$ since $\overline{H'}$ is commutative, and $bH' = c^{-1}aH' \sim aH'$ in H' follows where a, b, c are elements in H'. This shows that the proper principal right ideals in H' form a class of related right ideals. The class of related right ideals in H that corresponds to the class of proper principal right ideals of H' is then $\{b(H \cap aH_{P'}) \mid a \in P'\backslash P'', b \in H\}$ by Lemmas 2.11, 2.13, Proposition 2.16, and Theorem 2.17.

If the assumption $\phi(\overline{H'}) = \mathbb{R}^+$ holds, then there is on the one hand the related class of proper principal right ideals $\{aH' \mid a \in P'\backslash P''\}$ of H' and then, using 4.2(iii.c), a unique class of not finitely generated right ideals $\{aJ(H') \mid a \in H\backslash P''\}$ of H'. It follows from 2.11, 2.13, and 2.16 that $\{b(H \cap aH_{P'}) \mid b \in H, a \in P'\backslash P''\}$ is one class of related right ideals in H and $\{aP' \mid a \in H\}$, the class of P'-distinguished right ideals of H the other, corresponding to the two classes of H'.

(iv) It was shown in (iii) that the proper principal right ideals form a related class in H'. If we assume that there are only finitely many, say $n \geq 1$, classes of related nonprincipal right ideals in H', then we want to show that $\psi(D) = \mathbb{R}^+$; i.e., case (iii).

Hence, let I be a nonprincipal right ideal of H', and consider the powers $I, \ldots, I^{n+1}$ of I. By assumption, there exist m, k with $n + 1 \geq m > k \geq 1$ and I^m related to I^k. It follows from 4.2(iii) that $m\tau - k\tau \in \phi(\overline{H'}) = \psi(D)$ for $\psi(I) = \tau$, and $\psi(I^{m-k}) = (m-k)\tau \in \phi(\overline{H'})$; hence $\psi(I^{n!}) = n!\tau \in \phi(\overline{H'})$. Let t be any real number with $0 < t$. Then $r = t/n!$ satisfies $0 < r$, and there exists a right ideal I in $\widehat{D}$ with $\psi(I) = r$, $\psi(I^{n!}) = t \in \psi(D)$. It follows that $\psi(D) = \phi(\overline{H'}) = \mathbb{R}^+$. □

As in Lemma 4.1, let H be a right invariant rank one right cone with the additional assumption that $J = J(H)$ is not a principal right ideal. Then $\overline{H} = \{aH \mid a \in H\}$ with multiplication of right ideals as operation is order isomorphic under the mapping ϕ to a dense submonoid of $(\mathbb{R}^+, +)$. The order in $\overline{H}$ is given by $aH \leq bH$ if and only if $aH \supseteq bH$. We consider Dedekind cuts $\mathscr{C} = (L, U)$ of $\overline{H}$, that is, $L \cup U = \overline{H}$ and $aH < bH$ for $aH \in L$, $bH \in U$ with $L \neq \emptyset$. Let $I_L = \bigcap_{aH \in L} aH$ and $I_U = \bigcup_{bH \in U} bH$, and $I_L \supseteq I_U$ follows. We can distinguish three cases:

(i) L does not have a largest and U does not have a smallest element. Then $I_L = I_{L_\rho} = I_U = I_\rho$, where $\rho \in \mathbb{R}^+\backslash\phi(\overline{H})$ is the real number determined by $\mathscr{C}$, and I_ρ is the not finitely generated right of H introduced after Lemma 4.1, further $L = L_\rho = \{aH \mid \phi(aH) < \rho\}$. Then $I_L = I_U$ follows since $d \in I_L$ implies $\phi(dH) > \rho$ and $d \in I_U$.

(ii) L does not have a largest element, but U has a smallest element b_0H with $\phi(b_0H) = \rho \in \mathbb{R}^+$. Then $I_L = I_{L_\rho} = b_0H$ with $L_\rho = \{aH \mid \phi(aH) < \rho\}$, but $\rho \in \phi(\overline{H})$ in this case.

(iii) L has a largest element a_0H with $\rho = \phi(a_0H) \in \mathbb{R}^+$, and U does not have a smallest element. Then $I_L = a_0H \supset I_U = a_0J = I_\rho$.

Whereas for the results in Theorems 4.2 and 4.3, we divided the set of right ideals of H into principal right ideals and the not finitely generated right ideals I_ρ for $\rho \in \mathbb{R}^+$, we can now also divide the same set of right ideals into $\{H\} \cup \{aJ \mid a \in H\}$ on the one hand, and the set $\{I_{L_\rho} = \bigcap_{\phi(aH)<\rho} aH \mid \rho \in \mathbb{R}^+\backslash\{0\}\}$ on the other hand. We have the following proposition.

Proposition 4.4. *Let H be a right invariant rank one right cone with $J(H) = J$ not a principal right ideal. Then*:

(i) *A right ideal $I \neq H$ of H is either of the form aJ for some $a \in H$ or of the form $I = I_{L_\rho} = \bigcap_{\phi(aH)<\rho} aH$ for some $0 < \rho \in \mathbb{R}^+$ with $L_\rho = \{aH \mid \phi(aH) < \rho\}$;*
(ii) $L_{\rho_1} \cdot L_{\rho_2} = \{aHbH \mid aH \in L_{\rho_1}, bH \in L_{\rho_2}\} = L_{\rho_1+\rho_2}$ *for* $0 \neq \rho_1, \rho_2 \in \mathbb{R}^+$;
(iii) $I_{L_{\rho_1}} \cdot I_{L_{\rho_2}} = I_{L_{\rho_1+\rho_2}} J$ *if* $\rho_1 + \rho_2 \in \phi(\overline{H})$ *and* $\rho_1, \rho_2 \in \mathbb{R}\backslash\phi(\overline{H})$, *and* $I_{L_{\rho_1}} \cdot I_{L_{\rho_2}} = I_{L_{\rho_1+\rho_2}}$ *in all other cases.*

Proof. (i) follows from Lemma 1.3 and also from the above discussion.

The statement (ii) follows from properties of the real numbers. To prove (iii), we observe that $\bigcap_{\phi(aH)<\rho} aH = I_{L_\rho}$ is equal to the set of elements $d \in H$ with $\phi(dH) \geq \rho$. Hence, $I_{L_\rho} = b_0H$ with $\phi(b_0H) = \rho$ if $\rho \in \phi(\overline{H})$ (case (ii) in the above discussion) and $I_{L_\rho} = \{d \in H \mid \phi(dH) > \rho\}$ if $\rho \in \mathbb{R}^+\backslash\phi(\overline{H})$, (case (i)). From this it follows that $I_{L_{\rho_1}} \cdot I_{L_{\rho_2}} = I_{L_{\rho_1+\rho_2}} J$ if $\rho_1 + \rho_2 \in \phi(\overline{H})$ and $\rho_1, \rho_2 \in \mathbb{R}^+\backslash\phi(\overline{H})$: If $v \in I_{L_{\rho_1}}$, $w \in I_{L_{\rho_2}}$, then $\phi(vwH) > \rho_1 + \rho_2$ and $vw \in I_{L_{\rho_1+\rho_2}} J$. Conversely, for $h \in I_{L_{\rho_1+\rho_2}} J$, we have $\phi(hH) - \rho_1 - \rho_2 = \epsilon > 0$, and elements v, w exist in H with $\rho_1 < \phi(vH) < \rho_1 + \epsilon/2$, $\rho_2 < \phi(wH) < \rho_2 + \epsilon/2$. Then $v \in I_{L_{\rho_1}}$, $w \in I_{L_{\rho_2}}$ with $vwH \supset hH$ and $h \in I_{L_{\rho_1}} I_{L_{\rho_2}}$ follows. Similar arguments show that $I_{L_{\rho_1}} I_{L_{\rho_2}} = I_{L_{\rho_1+\rho_2}}$ in all other cases. □

In the next definition, we single out those initial segments L of $W(H)$ which have no last element.

Definition 4.5. Let H be a right cone. A limit set of H is a nonempty subset L of $W(H) = \{aH \mid a \in H\}$ so that $aH \in L$ implies $bH \in L$ for any $b \in H$ with $bH \supseteq aH$, and that there exists $a'H \in L$ for some $a'H \in W(H)$ and $aH \supset a'H$. The set $\mathscr{L}(H)$ is the set of all limit sets of H.

Assume that $N = \{I_i \mid i \in \Omega\}$ is a set of right ideals of H so that for any $I_i \in N$ there exists $I_j \in N$ with $I_i \supset I_j$. Then $\langle N \rangle = \{aH \in W(H) \mid \exists I_i \in N \text{ with } aH \supseteq N_i\}$ is the limit set generated by N. If $L \in \mathscr{L}(H)$ and $t \in H$, then $tL = \langle \{taH \mid aH \in L\} \rangle$ is also a limit set.

It follows from Lemma 1.3 that for a limit set $L \in \mathscr{L}(H)$ either $I_L = \bigcap aH$, $aH \in L$, is the empty set $\emptyset$ or is a right ideal of H not equal to H or $cJ(H)$ for any $c \in H$.

Conversely, if $I \neq H, \neq cJ(H)$ for all $c \in H$, then the right ideal I defines a limit set $L_I = \{aH \in W(H) \mid I \subset aH\}$ and $I_{L_I} = I$.

For the remainder of this section, let H be a right cone with right invariant prime segments only, that is, H is locally invariant. We recall that $\text{spec}(H) = \{P \mid P \text{ completely prime ideal in } H\}$, including $P = \emptyset$, is the totally ordered set of completely prime ideals of H, and $\emptyset$. We denote with $\text{spec}_0(H) = \{P \mid \emptyset \neq P \in$

spec(H), P not a limit prime} the set of elements P in spec(H), for which there exists P'' in spec(H) with $P = P' \supset P''$ a prime segment of H.

Let $P \in \operatorname{spec}(H)$. If $P \neq J(H)$, we define the limit set $L_P = L_{0,P} = \{aH \in W(H) \mid P \subset aH\}$. That L_P is indeed a limit set follows since $H \supset aH \supset P$ implies $aH \supset a^2H \supset P$. A limit set L is called *distinguished* if $L = tL_P$ for some $t \in H$ and a completely prime ideal $P \neq J(H)$ of H. If there exists a prime segment $P = P' \supset P''$ in H, then $H_P^0 = H_P \backslash P''H_P$ is a rank one right invariant right cone, and by Lemma 4.1 there exists a monomorphism Φ_P from $\overline{H_P^0} = \{aH_P^0 \mid a \in H_P^0\}$ into the semigroup $(\mathbb{R}^+, +)$ of nonnegative real numbers under addition. If $P \neq J$, in spec(H) and $P \neq P^2$ or P is a limit prime, we define $V_P = \{0\} \subseteq \mathbb{R}$. If $P = P^2$ and $P \in \operatorname{spec}_0(H)$, then $\Phi_P(\overline{H_P^0})$ is dense in $\mathbb{R}^+$. In that case, we define $V_P = \mathbb{R}^+$. If $P = J = J^2$ and $J \in \operatorname{spec}_0(H)$, then $V_P = \mathbb{R}^+ \backslash \{0\}$.

For each element $0 < \rho \in V_P$, $P \in \operatorname{spec}(H)$, there exists the subset $L_{\rho,P} = \{aH \in W(H) \mid \Phi_P(aH_P^0) < \rho\}$ of $W(H)$. Since $\Phi_P(\overline{H_P^0})$ is dense in $\mathbb{R}^+$ in this case, there exists for $a \in H \in L_{\rho,P}$ an $a'H \in W(H)$ with $\Phi_P(aH_P^0) < \Phi_P(a'H_P^0) < \rho$, hence $aH \supset a'H \in L_{\rho,P}$ which shows that the sets $L_{\rho,P}$ are limit sets of H for all $P \in \operatorname{spec}(H)$ and all $\rho \in V_P$.

We have $P_r(tP) = P$ for any $t \in H$ and any $\emptyset \neq P \in \operatorname{spec}(H)$ by 2.8(ii) and will prove below, see Proof of Theorem 4.6, that $P_r(I_{L_{\rho,P}}) = P$ for $0 < \rho \in V_\rho$; we set $P_r(\emptyset) = \emptyset$.

In the next result we show that the set $\{tL_{\rho,P} \mid t \in H, P \in \operatorname{spec}(H), \rho \in V_P\}$ describes all distinguished limit sets and all those limit sets L with $P_r(I_L) = P \in \operatorname{spec}_0(H)$, that is, $P = P' \supset P''$ for a prime segment $P' \supset P''$ of H.

Theorem 4.6. *Let H be a locally invariant right cone, and $L \in \mathcal{L}(H)$ a limit set of H with L either distinguished or $P_r(I_L)$ not a limit prime. Then $L = t \cdot L_{\rho,P}$ for some $t \in H$, $P_r(I_L) = P \in \operatorname{spec}(H)$, and $\rho \in V_P$. Conversely, the subsets $L = tL_{\rho,P}$ are in $\mathcal{L}(H)$ for any $t \in H$, $P \in \operatorname{spec}(H)$, and $\rho \in V_P$ with $P_r(I_{L_{\rho,P}}) = P$.*

Proof. It was shown above that $L = L_{0,P}$ is a limit set of H and hence $t \cdot L_{0,P}$, $t \in H$, $J \neq P \in \operatorname{spec}(H)$ is a limit set for any $t \in H$. We are left with those limit sets L that are not distinguished, but $P_r(I_L) = P = P' \supset P''$ for a prime segment $P' \supset P''$ of H. By 2.11 there exists a right ideal I' of H with $P' \supseteq I' \supset P''$ and an element $t \in H$ with $tI' = I$ for $I = I_L$, where $P_r(I') = P' = P = P_r(I)$. Since $I \neq cP'$ it follows that $I' \neq dP'$ for all $c, d \in H$, and since $I'H_P \cap H = I'$, by 2.13, it follows that $I'H_P \neq H_P$ and $\neq dJ(H_P) = dPH_P$. Since $s^{-1}PH_P = PH_P$ for $s \in H \backslash P$, it follows from Lemma 1.3 applied to H_P that $I'H_P = \bigcap a'H_P$ with the intersection taken over $a'H_P$, $a' \in H$, and $a'H_P \supset I'H_P$, which implies $a'H \supset I'$.

Conversely, if $a'H \supset I'$ for some $a' \in H$, and $a'H_P = I'H_P$, then $I' = I'H_P \cap H = a'H_P \cap H \supseteq a'H$, by Lemma 2.13, a contradiction. It follows that $L_{I'} = \{a'H \in W(H) \mid a'H \supset I'\} = L_{\rho,P}$ for $\rho = \sup\{\Phi_P(a'H_P^0) \mid a'H \supset I'\}$ and that $L = tL_{\rho,P}$.

To complete the proof it remains to show that $P_r(I_{L_{\rho,P}}) = P$ for $P \in \operatorname{spec}(H)$ and $0 \neq \rho \in V_P$ (that $P_r(I_{L_{0,P}}) = P$ was observed above).

If $s \in H \backslash P$, $as \in I'$ for some $a \in H$, then either $a \in P'' \subseteq I'$ or $\Phi_P(asH_P^0) = \Phi_P(aH_P^0) \geq \rho$ and $a \in I'$, hence $P_r(I') \subseteq P$. If $p \in P$, then either $p \in P'' \subseteq I' \subseteq P_r(I')$ or $p \in P' \backslash P''$ for $P = P' \supset P''$. Then $\Phi_P(pH_P^0) = \epsilon$ for some $0 < \epsilon \in \mathbb{R}$. Then there exists $a \in H$ with $\Phi_P(aH_P^0) < \rho$ but $\Phi_P(apH_P^0) > \rho$, that is, $a \in H \backslash I'$, but $ap \in I'$. Hence, $p \in P_r(I')$ and $P_r(I') = P$ follows. $\square$

In the next result, we consider properties of $L \cdot L' = \langle\{aHbH \mid aH \in L,\ bH \in L'\}\rangle$ for limit sets L, L' as considered in Theorem 4.6.

Proposition 4.7. *Let H be a locally invariant right cone. Then*:

(i) $L_{\rho_1,P_1} \cdot L_{\rho_2,P_2} = \begin{cases} L_{\rho_1+\rho_2,P_1} & \text{for } P_1 = P_2 \\ L_{\rho_1,P_1} & \text{for } P_1 \subset P_2 \\ L_{\rho_2,P_2} & \text{for } P_1 \supset P_2, \end{cases}$ *where* $P_i \in \operatorname{spec}(H)$ *and* $\rho_i \in V_{P_i}$;

(ii) *Let $t \in H$, $L_{\rho,P} \in \mathscr{L}(H)$. Then $tL_{\rho,P} = \langle L_{\rho,P}tH\rangle = L_{\rho,P}$ for $t \notin P$, and $tL_{\rho,P} = \langle L_{\rho,P}tH\rangle = L_{\rho+\Phi_P(tH_P^0),P}$ for $0 < \rho$ and $t \in P'\backslash P''$, where $P = P'$ and $P' \supset P''$ is a prime segment of H.*

Proof. (i) To prove that $L_{\rho_1,P_1}L_{\rho_2,P_2}$ is contained in the right-hand side, we consider an element $cH \in L_{\rho_1,P_1}L_{\rho_2,P_2}$, which means $cH \supseteq aHbH$ for some $aH \in L_{\rho_1,P_1}$, $bH \in L_{\rho_2,P_2}$ and must show that $cH \in L_{\rho_1+\rho_2,P_1}$ for $P_1 = P_2$, that $cH \in L_{\rho_1,P_1}$ for $P_1 \subset P_2$, and $cH \in L_{\rho_2,P_2}$ for $P_2 \subset P_1$. This can be done by distinguishing various cases depending on whether $\rho_1 = 0$ or $\rho_2 = 0$ and whether P_1 or P_2 are limit primes. If P_i is not a limit prime, then $H^0_{P_i}$ exists, is right invariant, and Proposition 4.4 can be used.

We will give details for the case $P = P_1 = P_2$ with $\rho_1 \neq 0 \neq \rho_2$. Since $cH \supseteq aHbH$ for $\phi_P(aH_P^0) < \rho_1$, $\phi_P(bH_P^0) < \rho_2$, it follows that $\phi_P(cH_P^0) \leq \phi_P(abH_P^0) < \rho_1 + \rho_2$ and that $cH \in L_{\rho_1+\rho_2,P}$. To prove the converse containment in this case, let $dH \in L_{\rho_1+\rho_2,P}$. Then $\phi_P(dH_P^0) < \rho_1 + \rho_2$ and $dH_P^0 \supset aH_P^0bH_P^0$ with $a, b \in H$ and $\phi_P(aH_P^0) < \rho_1$, $\phi_P(bH_P^0) < \rho_2$ by Proposition 4.4.

Using $dH_P = dH_P^0 \cup P''H_P$, where $P \supset P''$ is a prime segment, it follows that $dH_P \supset aH_P \cdot bH_P$, $dH \supseteq aHbH$, and finally $L_{\rho_1+\rho_2,P} \subseteq L_{\rho_1,P}L_{\rho_2,P}$ and $L_{\rho_1,P} \cdot L_{\rho_2,P} = L_{\rho_1+\rho_2,P}$.

The statement (ii) is proved with arguments that are similar to the ones used in the proof of (i). □

We conclude this Section 4 with a slightly different characterization of the limit sets $L_{\rho,P}$.

Lemma 4.8. *Let H be a locally invariant right cone and $L \in \mathscr{L}(H)$. Then $L = L_{\rho,P} = \{aH \mid \phi_P(aH_P)^0 < \rho\}$ for some $P \in \operatorname{spec}_0(H)$ and $0 < \rho \in V_\rho$ if and only if for each $aH \in L$ exists $p \in P$ with $apH \in L$ and $aH \supset bH \supset P''$ for some fixed $b \in H$, where $P = P' \supset P''$ is a prime segment.*

Proof. (i) If $L = L_{\rho,P}$ and $aH \in L$, then there exists $a' \in H$ with $aH_P^0 \supset a'H_P^0$ and $\phi_P(aH_P^0) < \phi_P(a'H_P^0) < \rho$. It follows that $a'H = apH$ for some $p \in P$ and that for any $b \in H$ with $\phi_P(bH_P^0) > \rho$, we have $aH \supset bH \supset P''$ for all $aH \in L$. Conversely, if L satisfies the conditions in the lemma, then $\{aH_P^0 \mid aH \in L\}$ is a limit set $\neq W(H_P^0)$ of H_P^0, and $L = L_{\rho,P}$ for some $0 < \rho \in V_P$ follows. □

The sets $L_{\rho,P}tH$ of principal right ideals of H for arbitrary $t \in H$ are considered in the next section under the additional assumption that H is right invariant, see Proposition 5.6.

5. RIGHT INVARIANT RIGHT CONES

In this section, we assume that H is a right invariant right cone.

The totally ordered sets $\mathrm{spec}(H)$, $\mathrm{spec}_0(H)$, the prime segments $P = P' \supset P''$ for $P \in \mathrm{spec}_0(H)$, the rank one right invariant right cones $H_P^0 = H_P \backslash P'' H_P$, the value sets $W_P = \Phi_P(\overline{H}_P^0) \subseteq \mathbb{R}^+$ for $P \in \mathrm{spec}_0(H)$, and $V_P \subseteq \mathbb{R}^+$ for $P \in \mathrm{spec}(H)$ are defined as in Section 3 after Definition 4.5.

Here we define the segments $C_P = P \backslash P''$ for $P \in \mathrm{spec}_0(H)$ with $P = P' \supset P''$ a prime segment and also call $C_H = H \backslash J(H) = U(H)$ a segment; we define $\mathrm{spec}_0(H)^* = \mathrm{spec}_0(H) \cup \{H\}$. On $\mathrm{spec}(H)$, $\mathrm{spec}_0(H)^*$ we will use the order defined by $P_1 < P_2$ if and only if $P_1 \supset P_2$ for $P_i \in \mathrm{spec}(H)$ or in $\mathrm{spec}_0(H)^*$. Since H is right invariant, there exists for every $c, a \in H$ an element $c' \in H$ with $ca = ac'$.

Lemma 5.1. *Let H be a right invariant right cone, $a \in H$, and C a segment of H. Then there exists a segment $\widehat{C}$ of H with $Ca \subseteq a\widehat{C}$.*

Proof. We show first that $U \cdot a \subseteq aU$: For $u \in U$ there exists $v \in U$ with $uv = vu = 1$, and for $u', v' \in H$ with $ua = au'$, $va = av'$ it follows that $a = uva = au'v'$ and $u'v' = v'u' = 1$; hence $u' \in U$.

Next, let $c_1, c_2 \in C \neq U$, and we can assume that $c_2 = c_1 d$ for some $d \in H$. Since the prime segment $P' \supset P''$ with $C = P' \backslash P''$ is right invariant, there exists n with $c_1^n = c_2 b$ for some $b \in H$; Proposition 1.6(c).

For $c_1 a = ac_1'$, $c_2 a = ac_2' = ac_1' d'$ with $c_1', c_2', d' \in H$, it follows that $c_2' H \subseteq c_1' H$ and that $c_1' \in C_{\widehat{P}} = \widehat{P'} \backslash \widehat{P''}$ for some segment $C_{\widehat{P}}$. If $c_2' \in \widehat{P''}$, then $c_1^n a = ac_1'^n = ac_2' b'$ and $c_1'^n = c_2' b'$ for some $b' \in H$, and the contradiction $c_1'^n \in \widehat{P''}$ proves the lemma. □

Definition 5.2. If $H, a \in H$, and C_P with $C_P a \subseteq aC_{\widehat{P}}$ are as in Lemma 5.1, we define a function $f : \mathrm{spec}_0(H)^* \longrightarrow \mathrm{spec}_0(H)^*$ with $f(a, P) = \widehat{P}$.

Lemma 5.3. *Let $a \in H$. Then*:

(i) *If $P_1 < P_2$ for $P_1, P_2 \in \mathrm{spec}_0(H)^*$, then $f(a, P_1) \leq f(a, P_2)$;*
(ii) *If $P_1 < P_2$ for $P_1, P_2 \in \mathrm{spec}_0(H)^*$ and $f(a, P_1) \neq H$, then $f(a, P_1) < f(a, P_2)$.*

Proof. Let $c_1 \in C_{P_1}$, $c_2 \in C_{P_2}$, and $c_2 = c_1 w$ for some $w \in H$. Then $c_2' = c_1' w'$ for $c_j a = ac_j'$, $wa = aw'$ for $c_j', w' \in H$, $j = 1, 2$. Hence, $f(a, P_1) \leq f(a, P_2)$, which proves (i).

To prove (ii), we assume in addition that $f(a, P_1) = f(a, P_2) = \widehat{P} \neq H$. Then again by 1.6(c), we have $c_1'^n = c_2' d'$ for some n and $d' \in H$. But also $c_2 = c_1^{n+1} b$ for some $b \in H$ since $P_1 < P_2$ and $c_2 a = ac_1'^{n+1} b' = ac_1'^n c_1' b' = ac_2' d' c_1' b' = c_2 a d' c_1' b'$ for some $b' \in H$. It follows that $d' c_1' b' = 1$, hence $c_1' \in U$, a contradiction that proves the lemma. □

Corollary 5.4. *Let H be as in* 5.1 *with maximum condition on prime ideals in H. Assume that for some $a \in H$ and $P_1, P_2 \in \mathrm{spec}_0(H)^*$, we have $f(a, P_1) = P_2 < P_1$. Then there exists n with $f(a^n, P_1) = H$.*

Proof. We define inductively $f(a, P_{k-1}) = P_k$ for $k \geq 2$ and $P_{k-1} > P_k$ follows if $P_k = f(a, P_{k-1}) = f(a^{k-1}, P_1) \neq H$. Since the assumption about prime ideals implies that $\mathrm{spec}_0(H)^*$ is well-ordered, it follows that $f(a^n, P_1) = H$ for some n. □

Corollary 5.5. *Let H be a right cone as in* 5.1 *with minimum condition on completely prime ideals. Then $f(a, P) \leq P$ for $a \in H$, $P \in \mathrm{spec}_0(H)$.*

Proof. If $f(a, P_1) = P_2 > P_1$, then $P_k = f(a, P_{k-1}) > P_{k-1}$ for $k = 2, 3, \ldots$ by 5.3(ii) and $P_1 \supset P_2 \supset \cdots$ follows. A contradiction that proves the corollary. □

We return to the description of $L_{\rho,P}tH$ for $t \in H$, $P \in \mathrm{spec}(H)$, $\rho \in V_P$. In Lemma 4.7(ii), this was done for $t \notin P$ and for $t \notin P''$ if $P \in \mathrm{spec}_0(H)$ with $P = P' \supset P''$ a prime segment. We will use the above results, and it follows that $L_{\rho,P}tH = tH$ does not generate a limit set of H if $P \in \mathrm{spec}_0(H)$ with $f(t, P) = H$.

Proposition 5.6. *Let H be as in* 5.1, *let t be an element in H, and let L be a limit set of H with $L = L_{\rho,P}$, $P \in \mathrm{spec}(H)$ and $\rho \in V_P$. Then*:

(i) *$L_{\rho,P}tH = \langle\{tH\}\rangle$ is not a limit set if $aHtH = tH$ for all $aH \in L_{\rho,P}$*;
(ii) *If $L_{0,P}tH \neq \langle\{tH\}\rangle$, then $L_{0,P} \cdot tH = tL_{0,\widehat{P}}$ for some $\widehat{P} \in \mathrm{spec}(H)$ and $\widehat{P} \in \mathrm{spec}_0(H)$ if $P \in \mathrm{spec}_0(H)$*;
(iii) *If $0 < \rho \in V_P$ and $f(t, P) = \widehat{P} \subset H$, then $L_{\rho,P}tH = tL_{\hat{\rho},\widehat{P}}$ for some $0 < \hat{\rho} \in V_{\widehat{P}}$*;
(iv) *If in* (iii) *the number $\rho \in W_P = \phi_P(\overline{H}_P^0)$, then $\hat{\rho} \in \phi_{\widehat{P}}(H_{\widehat{P}}^0) = W_{\widehat{P}}$.*

Proof. (i) It follows that $at = tu$ for some $aH \in L_{\rho,P}$ and $u \in U(H)$. Hence, $L_{\rho,P}tH = \langle\{tH\}\rangle$ has tH as smallest element, it is not a limit set.

(ii) If there exists $a \in H$ with $aH \in L_{0,P}$ with $at = ta'$ and $a' \notin U(H)$, then $bt = tb'$ with $b' \notin U(H)$ for all $b \in H$ with $aH \supseteq bH \in L_{0,P}$. We have in a first case $P = \bigcap P_i$ for $P_i \in \mathrm{spec}_0(H)$ and $P_i \supset P$. Then there exists an i with $f(t, P_i) = \widehat{P}_i \neq H$ and $\widehat{P}_i \in \mathrm{spec}_0(H)$, and $\widehat{P}_j \subset \widehat{P}_i$ for $P_j \subset P_i$ by Lemma 5.3. It follows that $L_{0,P}tH = tL_{0,\widehat{P}}$ for $\widehat{P} = \bigcap \widehat{P}_i$.

In the remaining case, there exists a prime segment $\widetilde{P} \supset P$ in H and $f(t, \widetilde{P}) = \widehat{\widetilde{P}} \in \mathrm{spec}_0(H)$ with $\widehat{\widetilde{P}} \supset \widehat{P}$ a prime segment. Since for any $a \in \widetilde{P} \backslash P$ and $b' \in \widehat{\widetilde{P}} \backslash \widehat{P}$, there exists an n with $a'^n H \subset b'H$ for $a' \in H$ with $at = ta'$, it follows that $L_{0,P}tH = tL_{0,\widehat{P}}$, which proves the first part of (ii).

If P is in $\mathrm{spec}_0(H)$, and $P = P' \supset P''$ is a prime segment, then $f(t, P') = \widehat{P} \supset \widehat{P''} = f(t, P'')$ is a prime segment in H and $\mathrm{spec}_0(H)$.

To prove (iii), we have $P \in \mathrm{spec}_0(H)$ and $f(t, P) = \widehat{P} \in \mathrm{spec}_0(H)$. Further, there exists $b \in C_P$ with $\phi_P(bH_P^0) > \rho$, hence $aH \supset bH$ for all $aH \in L_{\rho,P}$. Therefore, $L_{\rho,P}tH = tL_{\hat{\rho},\widehat{P}}$ for some $\hat{\rho} \in V_{\widehat{P}}$ with $\hat{\rho} = \sup\{\phi_{\widehat{P}}(a'H_{\widehat{P}}^0) \mid aH \in L_{\rho,P};\ at = ta'\} \leq \phi_{\widehat{P}}(b'H_{\widehat{P}}^0)$ for $b' \in H$ with $bt = tb'$.

(iv) If $0 < \rho = \Phi_P(bH_P^0) \in V_P$ under the assumptions as in (iii) for an element $b \in H$, we have $L_{\rho,P}tH = tL_{\hat{\rho},\widehat{P}}$ for $\widehat{P} \in \mathrm{spec}_0(H)$ and $0 < \hat{\rho} \in V_{\widehat{P}}$.

We want to show that $\hat{\rho} = \phi_{\widehat{P}}(b'H^0_{\widehat{P}}) = \rho'$ for $b' \in H$ with $bt = tb'$ and $\hat{\rho} \leq \rho'$ follows immediately. We assume that $0 < \sigma' < \rho'$ for $\sigma' \in \mathbb{R}^+$ and want to show that there exists $cH \in L_{\rho,P}$ with $ct = tc'$ for $c' \in H$ and $\sigma' < \phi_{\widehat{P}}(c'H^0_{\widehat{P}}) < \rho'$.

Let $\tau' = \rho' - \sigma' \in \mathbb{R}^+$, and there exists n with $(n-1)\tau' < \rho' \leq n\tau'$. In $L_{\rho,P}$, there exist elements $a_0H \supset a_1H \supset a_2H \supset \cdots \supset a_kH$ with $a_k = a_{k-1}v_k = a_{k-2}v_{k-1}v_k = \cdots = a_0v_1 \ldots v_k$ for $v_i \in P$ for $i = 1, \ldots, k$, and therefore,

$$\phi_P(a_0H^0_P) < \phi_P(a_1H^0_P) < \cdots < \phi_P(a_kH^0_P) = \phi_P(a_0H^0_P) + \sum_{i=1}^{k} \phi_P(v_iH^0_P) < \rho.$$

We pick $k = n+1$ and $vH \in L_{\rho,P}$ with $\phi_P(vH^0_P) = \min\{\phi_P(v_iH^0_P) \mid i = 1, \ldots, k\}$. Let $vt = tv'$ for $v' \in H$, and $0 < k\phi_P(vH^0_P) < \rho = \phi_P(bH^0_P)$ follows; hence, $0 < k\phi_{\widehat{P}}(v'H^0_{\widehat{P}}) < \rho' = \phi_{\widehat{P}}(b'H^0_{\widehat{P}})$. However, for $\delta = \phi_{\widehat{P}}(v'H^0_{\widehat{P}})$, there exists $m \geq k > n$ with $m\delta < \rho' \leq (m+1)\delta$. Since $\rho' \leq n\tau'$, it follows that $\sigma' < m\delta < \rho'$ for $m\delta = \phi_{\widehat{P}}(v'mH^0_{\widehat{P}})$, where $cH = v^mH \in L_{\rho,P}$. This proves $\hat{\rho} = \rho'$ and the lemma. □

This result together with 4.4 and 4.7 makes it possible to compute $\langle t_1L_{\rho_1,P_1}t_2L_{\rho_2,P_2}\rangle$ if enough information is available for $t^{-1}Ht$ for $t \in H$ and H a right invariant right cone. We single out the following result that will be used in the construction of I-compact right chain rings.

Corollary 5.7. *Let H be as in* 5.1. *Then*:

(i) *The set $\mathcal{D} = \{tL_P \mid t \in H,\ P \in \operatorname{spec}(H)\}$ is closed under multiplication of limit sets*;
(ii) *The set $\{L_{\rho,P} \mid P \in \operatorname{spec}(H),\ \rho \in \phi_P(\overline{H}^0_P) = W_P\}$ is closed under multiplication of limit sets.*

The proof follows from 4.4, 4.7, and 5.6.

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