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I-COMPACT RIGHT CHAIN DOMAINS

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Generalizing the concept of convergency to valued fields, Ostrowski in the 1930s introduced pseudo-convergent sequences. In the present paper we classify pseudo-convergent sequences in right chain domains R according to the prime ideal P associated to the breadth I of the sequence using an ideal theory developed for right cones in groups. The ring R is I-compact if every pseudo-convergent sequence in R with breadth I has a limit in R, and we construct right chain domains R which are I-compact only for right ideals I in particular subsets $\mathcal{B}$ of the set of all right ideals of R. Krull's perfect valuation rings and then Ribenboim's notion of a valuation ring complete par étages, where $\mathcal{B}$ is the minimal set containing the completely prime ideals in a commutative valuation ring, is a special case. For a non-discrete right invariant rank-one right chain domain R there are exactly two possibilities for the set $\mathcal{B}$ if the value group of R is the group of real numbers under addition, and there are infinitely many possibilities for $\mathcal{B}$ in all other cases.

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0. Introduction

Let R be a ring with identity and M be a right uniserial R-module, that is, a right R-module whose lattice of submodules is totally ordered by inclusion. A ring R which is right uniserial is also called a *right chain ring*.

We say that M is K-compact for a submodule K of M, if the natural mapping from M into the inverse limit of M/M_λ is onto for every family of submodules $M_\lambda, \lambda \in \Lambda$, of M with $M_\alpha \supseteq M_\beta$ for $\alpha \leq \beta$ in the index set Λ and $\bigcap_{\lambda\in\Lambda} M_\lambda = K$.

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The module M is called (almost) *compact* if it is K-compact for all (nonzero) submodules K of M. It is known that almost maximal (here, almost compact) commutative valuation domains are exactly the valuation domains for which finitely generated modules are direct sums of cyclic modules (see [3, 13, 15]).

A commutative discrete valuation ring V of rank-one is compact, if V is (0)-compact, that is, if and only if V is complete. This is no longer true if V is not discrete. However, a right chain domain R with $P \neq P^2$ for every nonzero completely prime ideal P of R is compact, if and only if R is P-compact for every completely prime ideal P of R, see [5] (also [17, 21] for the commutative case).

As another result of this type we recall, see [13], that a commutative chain ring R with zero divisors is also (0)-compact if R is I-compact for all ideals $I \neq (0), R$.

These results are the answers to special cases of the following problem: Assume that it is known that a right chain ring R is K-compact for every K in a subset $\mathcal{C}_0$ of the set of all right ideals of R. Determine the set $\mathcal{C} \supseteq \mathcal{C}_0$ of all right ideals I of R for which R must necessarily be I-compact.

Assume that the right chain domain R is I-compact for the right ideal I. It follows that R is $\phi(I)$-compact for any automorphism ϕ of R and I'-compact for any right ideal I' related to I. Here we say that I' is related to I and write $I \sim I'$ if $s^{-1}I = t^{-1}I'$ for some elements $s \in R\backslash I, t \in R\backslash I'$, see Proposition 1.4. We give an example of a chain domain R with right ideal I and an automorphism ϕ so that $\phi(I)$ is not related to I.

A right chain domain R is I-compact if and only if every pseudo-convergent sequence in R with breadth I has a limit in R; see Lemma 1.3. Here we consider only those right ideals I of R which are the intersection of right ideals I_λ that contain I properly. These are exactly the right ideals of R which are not equal to R or $cJ(R) \neq (0)$.

In Sec. 2, we describe the non-trivial right ideals $I \neq cJ(H)$ in the more general setting of right cones H in a group G in terms of limit sets $L = L_I = \{aH \mid aH \supset I \text{ for } a \in H\}$. We subdivide this class of right ideals further by singling out the distinguished right ideals cP for $c \in H$, and $P \neq J(H)$ completely prime or $\emptyset$, and also by considering associated prime ideals $P_r(I)$ of right ideals I. For limit sets L_I we define P-admissability for a completely prime ideal P as the property for $aH \supset I$ there exists $p \in P$ with $apH \supset I$. The relationship between these concepts is explained in Theorem 2.6.

For every ordered group G there exists a chain domain R with G as its group of values. In order to establish a similar relationship between right cones H in a group G and right chain domains R in a skew field F we consider an association as defined in Definition 3.1 between right cones. Associated right cones will not only have isomorphic lattices of right ideals, but the type of right ideal, whether it is an ideal, a prime ideal, or completely prime ideal, is preserved under this isomorphism; see Proposition 3.3 and its corollaries. Even though right chain domains R can be constructed so that H and $R\backslash\{0\}$ are associated for some classes of right cones (for example see [18], see [6, 14], or [11]), it remains an open problem whether even for any right ordered group G with corresponding cone Π, a chain domain R exists with Π associated to $R\backslash\{0\}$. In that case it follows immediately that the group ring $\mathbb{Q}[G]$ over G is embeddable into the skew field F which is the field of quotients of R.

We also do not know if conversely the embeddability of the group ring $\mathbb{Q}[G]$ into a skew field F guarantees the existence of a right chain domain R in F so that a right cone H of G is associated with $R\backslash\{0\}$. The question whether group rings $\mathbb{Q}[G]$ of right ordered groups G are embeddable into skew fields is known as "Malcev" problem and is solved in special cases only.

We do consider in Sec. 4 a special case of the problem where the right cone H of G is contained in the localization Π_P of the cone Π of non-negative elements of an ordered group (G, Π) so that $U(H) = U(\Pi_P) \cap H$. The right cones H in this class are not necessarily right invariant and even in the cases where H is right invariant, it does not follow that the semigroup $W(H) = (\{aH \mid a \in H\}, \cdot)$ of principal right ideals of H is embeddable into a group. However, it follows from Neumann's results in [18] that the ring $\hat{R} = K\{\{H\}\}$ of generalized power series $\alpha = \sum hk_h$, with $h \in H, h_k \in K$, and the support of $\alpha = \{h \mid k_h \neq 0\}$ well ordered in G, is associated with H for a skew field K.

In Sec. 5, we construct subrings $R_{\mathcal{B}}$ of $\hat{R}$ which are associated with H and are non-trivially compact exactly for the right ideals I of R with $L_{I\cap H} \in \mathcal{B} \subseteq \mathcal{L}(H)$, where $\mathcal{L}(H)$ is the set of all limit sets of H. The subsets $\mathcal{B}$ of $\mathcal{L}(H)$ satisfy four closure conditions (1)–(4) and the rings $R_{\mathcal{B}}$ are the subsets of $\hat{R}$ consisting of those elements α in $\hat{R}$ such that $\mathcal{L}(\alpha)$, the set of limit sets of H generated by the support of α, see Definition 5.1, is contained in $\mathcal{B}$. We prove this in Theorem 5.4 for right invariant H, and will consider the semiinvariant case in a later paper. We use the classification of limit points in semiinvariant right cones H, as given in [9], to obtain these results.

One example for $\mathcal{B}$ is the set $\mathcal{D} = \{L_{cP} \mid c \in H, J \neq P, \text{completely prime in } H\} \cup \{L_{\emptyset}\}$ of distinguished limit sets of H, see Proposition 5.5. The right chain rings $R_{\mathcal{D}}$ correspond to the perfect valuation rings as considered by Krull or the rings complete par étages defined by Ribenboim. Proposition 5.6 leads to another class of examples $R_{\mathcal{B}}$ where $\mathcal{B} \supset \mathcal{D}$ is determined by the value groups $\Phi_P(\overline{H_P^0}) \subseteq (\mathbb{R}^+, +)$ of the rank-one right invariant right chain cones $H_P^0 = H_P\backslash P''H_P$ for all completely prime ideals P which have a proper neighbor P'' in the chain of completely prime ideals of H.

Additional examples are given in Sec. 6. It may be of particular interest that examples for $\mathcal{B} \supset \mathcal{D}$ exist where all non-distinguished ideals determined by $\mathcal{B}$ have associated prime ideals that have no lower neighbor in the chain of prime ideals, that is, are limit primes, see Example 6.10.

The method of constructing valuation rings by applying certain conditions on the support of elements in $\hat{R}$ has been used in the commutative case in other instances. Fuchs, in [12] introduced κ-compact commutative valuation domains and constructed examples of such rings by restricting the cardinality of the support of elements in generalized power series rings. Osofsky in [19] constructed valuation domains R which admit non-standard valuation uniserial modules. The ring R is obtained as a subring of $\hat{R}$ consisting of all elements $\alpha \in \hat{R}$ such that the subgroup generated by the support of α can be generated by $\aleph$ elements for a certain regular infinite cardinal $\aleph$.

All rings in this paper have an identity and are associative, but are not necessarily commutative. We call a ring R a domain if R has no zero divisors. The Jacobson radical of the ring R is denoted by $J(R) = J$ whereas $U(R) = U$ is the group of units.

1. *I*-Compactness

Let R be a ring with identity. A right R-module M is called *uniserial* if $A \subseteq B$ or $B \subset A$ for any submodules A, B of M. An integral domain R is right uniserial if R is uniserial as a right R-module. This is the case if $aR \subseteq bR$ or $bR \subset aR$ for any elements $a, b \in R$. Such rings are called *right chain domains.*

Definition 1.1. A uniserial right R-module M is called I-compact for a submodule I of M if the canonical mapping from M to $\varprojlim M/M_\lambda$ is onto for any family of submodules $\{M_\lambda \,|\, \lambda \in \Lambda\}$, with $\bigcap_{\lambda\in\Lambda} M_\lambda = I$, and where $\lambda' > \lambda$ implies $M_{\lambda'} \subset M_\lambda$.

If for some submodule I of M in the above definition, it follows that $\bigcap_{\lambda\in\Lambda} M_\lambda = I$ for a family of submodules M_λ of M implies that $I = M_\lambda$ for some $\lambda \in \Lambda$, then the condition of the definition is satisfied trivially and we say that M is *trivially I-compact* in that case.

The I-compactness of M can be described with the help of pseudo-convergent sequences of elements in M.

If M is a uniserial right R-module, the set $W(M) = \{mR \,|\, 0 \neq m \in M\}$ is totally ordered under

$$mR \leq m'R \quad \text{if and only if } mR \supseteq m'R \quad \text{ for } m, m' \in M\backslash\{0\}.$$

There exists a mapping v from $M\backslash\{0\}$ onto $W(M)$ with $v(m) = mR$, and $v(m_1 \pm m_2) \geq \min\{v(m_1), v(m_2)\}$ follows for all $m_1, m_2, m_1 \pm m_2 \in M\backslash\{0\}$ with equality for $v(m_1) \neq v(m_2)$.

Definition 1.2. A sequence $(m_\rho)_{\rho\in\Lambda}$ of elements $m_\rho \in M$ and Λ a well-ordered index set with no last element is a *pseudo-convergent* (p.c.) *sequence* if $v(m_\tau - m_\sigma) > v(m_\sigma - m_\rho)$ for $\rho < \sigma < \tau \in \Lambda$.

It follows for such a p.c. sequence $(m_\rho)_{\rho\in\Lambda}$ that $v(m_\tau - m_\sigma) = v(m_{\sigma+1} - m_\sigma) = \gamma_\sigma$ for all $\tau > \sigma$ by using the above mentioned property of v.

The submodule $I = \{m \in M \,|\, v(m) \geq \gamma_\rho,\, \rho \in \Lambda\} = \bigcap_{\rho\in\Lambda} \gamma_\rho$ is called the *breadth* of the sequence $(m_\sigma)_{\sigma\in\Lambda}$, and an element $m \in M$ is called a *limit* of the p.c. sequence $(m_\sigma)_{\sigma\in\Lambda}$ if $v(m - m_\rho) = \gamma_\rho$ for all $\rho \in \Lambda$.

As for commutative valuation domains one obtains the following result (see [22, pp. 40 and 48; 4]).

Lemma 1.3. *Let M be a uniserial right R-module, I be a submodule of M. Then the following conditions are equivalent*:

(a) *M is I-compact.*
(b) *If $\{M_\lambda\}_{\lambda\in\Lambda}$ is a family of submodules of M with $\bigcap_{\lambda\in\Lambda} M_\lambda = I$, and $\lambda' > \lambda$ implies $M_{\lambda'} \subset M_\lambda$ and $\{a_\lambda\}_{\lambda\in\Lambda}$ is a set of elements of M with $(a_\lambda + M_\lambda) \cap (a_{\lambda'} + M_{\lambda'}) \neq \emptyset$ for $\lambda, \lambda' \in \Lambda$, then $\bigcap_{\lambda\in\Lambda}(a_\lambda + M_\lambda) \neq \emptyset$.*
(c) *Every p.c. sequence in M with breadth I has a limit in M.*

Proof. We only give a sketch of the proof. The condition (b) follows from (a) if the family $(a_\lambda + M_\lambda)_{\lambda\in\Lambda}$ is considered as an element of $\varprojlim M/M_\lambda$.

Let $(m_\sigma)_{\sigma\in\Lambda}$ be a p.c. sequence in M. Then $M_\sigma = \gamma_\sigma$ is a submodule of M and $M_{\sigma'} \subset M_\sigma$ for $\sigma' > \sigma \in \Lambda$.

By (b) there exists an element $m \in \bigcap_{\sigma\in\Lambda}(m_\sigma + M_\sigma)$, and hence m is a limit of $(m_\sigma)_{\sigma\in\Lambda}$ in M; condition (b) implies (c).

To prove that (c) implies (a) let $\tilde{m} = (m_\lambda + M_\lambda)_{\lambda\in\Lambda}$ be an element in $\varprojlim M/M_\lambda$ for a family $\{M_\lambda \,|\, \lambda \in \Lambda\}$ of submodules of M with $\bigcap_{\lambda\in\Lambda} M_\lambda = I$ and $M_{\lambda'} \subset M_\lambda$ for $\lambda' > \lambda \in \Lambda$.

To prove that there exists an element $m \in M$ with $\tilde{m} = (m + M_\lambda)_{\lambda\in\Lambda}$, we can assume that for no element m_τ we have $m_\tau + M_\lambda = m_\lambda + M_\lambda$ for all $\lambda \in \Lambda$; in particular we can assume that M is not trivially I-compact. It then requires a transfinite induction to show that there exists a well-ordered subset Λ_0 of Λ so that $(m_\lambda)_{\lambda\in\Lambda_0}$ is a p.c. sequence with I as its breadth. By assumption, this sequence has a limit $m \in M$ and $\tilde{m} = (m + M_\lambda)_{\lambda\in\Lambda}$ is the image of m in $\varprojlim M/M_\lambda$. □

We are interested in the set $\mathcal{C}(R) = \{I \,|\, R \text{ is } I\text{-compact}\}$ for a right chain domain R and I a right ideal of R, and in $\mathcal{C}(I) = \{I' \subseteq R \,|\, R \text{ is } I'\text{-compact in case } R \text{ is } I\text{-compact}\}$.

Two right ideals A, B in R are said to be *related*, $A \sim B$, (see [4, 9]) if $s^{-1}A = t^{-1}B$ for some $s \in R\backslash A$, and some $t \in R\backslash B$ with $s^{-1}A = \{r \in R \,|\, sr \in A\}$.

If $I \neq R$ is a right ideal in R, then $\mathrm{Rel}\,(I) = \{I' \subseteq R \,|\, I \sim I'\}$ is the set of right ideals I' in R related to I. Similarly, $\mathrm{Aut}\,(I) = \{\varphi(I) \,|\, \varphi \text{ automorphism of } R\}$ is the set of images of I under automorphisms φ of R.

The next proposition states the expected result that R is I' compact if R is I-compact and $I' \in \mathrm{Rel}\,(I)$ or $I' \in \mathrm{Aut}\,(I)$.

Proposition 1.4. *Let R be a right chain ring. Then*:

(a) $\mathcal{C}(I) \supseteq \mathrm{Rel}\,(I)$.
(b) $\mathcal{C}(I) \supseteq \mathrm{Aut}\,(I)$.

Proof. Statement (a) follows from the fact that for related right ideals A and B of R, the ring R is A-compact if and only if R is B-compact. To prove this, condition (b) in Lemma 1.3 can be used; this was done in [4, Theorem 4.15]. Condition (c) of Lemma 1.3 can be used to show that R is $\varphi(I)$-compact if R is I-compact and φ is an automorphism of R. □

The next example shows that conjugate right ideals are not necessarily related.

Example 1.5. Let $G = \{(b,a) \mid a,b \in \mathbb{Q},\, b > 0\}$ be the ordered group with $(b_1,a_1)(b_2,a_2) = (b_1b_2, b_1a_2 + a_1)$ as operation and lexicographical ordering. Then $H = \{(b,a) \mid 1 \leq b, a \in \mathbb{Q}, a \geq 0 \text{ if } b = 1\}$ is the positive cone of G and $R = K\{\{H\}\} = \{\alpha \mid \alpha = \sum hk_h \mid k_h \in K, h \in H, \operatorname{supp}(\alpha) \text{ well ordered}\}$ is an invariant chain domain, where $\operatorname{supp}(\alpha) = \{h \in H \mid k_h \neq 0\}$ and K is an arbitrary commutative field (see the construction of rings of this type in Sec. 4).

To every $\alpha = \sum hk_h \neq 0$ in R we assign $v(\alpha) = \min\{h \mid k_h \neq 0\}$ and denote by $v(A) = \{v(\alpha) \mid 0 \neq \alpha \in A\}$ the upper class in H corresponding to an ideal $A \neq (0)$ in the invariant chain domain R.

Let A and B be two related ideals in R and let $v(A) = \Omega_1, v(B) = \Omega_2$ be the corresponding upper classes in H. Then $(b_1,a_1)^{-1}\Omega_1 = (b_2,a_2)^{-1}\Omega_2$ for certain elements $(b_1,a_1) \in H\backslash\Omega_1$ and $(b_2,a_2) \in H\backslash\Omega_2$, since $v(\alpha\beta) = v(\alpha)v(\beta)$ for nonzero elements α, β in R and A, B are related. Comparing the elements $(b_1,a_1), (b_2,a_2)$ in H we obtain an element $(s,t) \in H$ with $\Omega_1 = (s,t)\Omega_2$ or $\Omega_2 = (s,t)\Omega_1$. If both classes Ω_1 and Ω_2 contain elements with a least and identical first coordinate, we conclude that $(s,t) = (1,t)$.

Now let A be an ideal $\neq (0)$ in R with $v(A) = \Omega, x$ be an element in R with $v(x) = (b,a)$, then $Ax = xB$ for some ideal B of R, since the ring is invariant and $v(B) = \Omega'$ with $\Omega(b,a) = (b,a)\Omega'$, and $\Omega' = (b,a)^{-1}\Omega(b,a) = \{(z, \frac{1}{b}(za + y - a)) \mid (z,y) \in \Omega\}$ follows. For example, choose an element $\gamma \in \mathbb{R}\backslash\mathbb{Q}$, an element z_0 with $1 < z_0 \in \mathbb{Q}$, and set $\Omega = \{(z,y) \mid z \geq z_0, \text{ and } y \geq \gamma \text{ if } z = z_0\}$. It follows that both upper classes, Ω and Ω', contain elements with the same least first coordinate, namely z_0. The infimum of the set of elements y with $(z_0,y) \in \Omega$ is γ by definition and the infimum of the set of elements y with $(z_0,y) \in \Omega'$ is $\inf\{(z_0\frac{a}{b} + \frac{y}{b} - \frac{a}{b})\} = z_0\frac{a}{b} + \frac{\gamma}{b} - \frac{a}{b}$. However, for $v(x) = (b,a)$ with $b > 1$ neither the equality $\Omega = (1,t)\Omega'$ nor $\Omega' = (1,t)\Omega$ is possible for $0 < t \in \mathbb{Q}$. This shows that in this case B is conjugate to A, but not related to A.

We will classify p.c. sequences in right chain domains according to prime ideals associated to their breadth. The ideal theory used in this process has been developed in [7, 9] for the more general situation of right cones in groups, since a classification of right ideals is not only needed for right chain domains, but also for right cones, which under favorable circumstances play the role that cones of ordered groups play for invariant valuation rings.

In Sec. 4, we will use right cones in groups to construct I-compact right chain domains, whose right ideal structure is determined by the right ideal structure of the corresponding right cones.

2. *P*-Limit Sets for Right Cones

Right cones generalize the cones of ordered groups as well as right chain domains. We recall the next definition from [7, 9].

Definition 2.1. Let G be a group. A submonoid H of G is called a right cone of G if the following two conditions are satisfied:

(i) $G = \langle H\rangle$, that is, G is generated by H; and
(ii) $aH \subseteq bH$ or $bH \subset aH$ for any elements $a, b \in H$.

A right ideal of a right cone H is a non-empty subset I of H with $IH \subseteq I$; left ideals and ideals of H are defined similarly. A right cone $H \neq G$ of G has a unique maximal right ideal $J(H) = J = H\backslash U(H) \subset H$, where $U(H)$ is the subgroup of elements $u \in U$ which have an inverse $u^{-1} \in H$. If $aHb \subseteq I \neq H$ for an ideal I of H and $a, b \in H$ implies $a \in H$ or $b \in H$, then I is a prime ideal, and if this conclusion follows from $ab \in I$, then I is a completely prime ideal.

In the next result we characterize those right ideals I of a right cone H which are the intersection of right ideals properly containing I, that is, for a right chain domain R, those right ideals $I \neq R$ for which R is not trivially I-compact. It follows that R is trivially I-compact exactly for $I = cJ$ and $0 \neq c \in R$. It also follows that the subset $L_I = \{bH \mid b \in H \text{ and } bH \supset I\}$ of $W(H) = \{aH \mid a \in H\}$ for a right ideal I of H with $I \neq H$ has no last (smallest) element if and only if $I \neq aJ$ for $a \in J$.

We use the following.

Definition 2.2. Let H be a right cone of the group G and $W(H) = \{aH \mid a \in H\}$. A non-empty subset L of $W(H)$ is a limit set of H if and only if $bH \in L$ implies that $bH \supset b'H \in L$ for some $b' \in H$, and $aH \in L$ for any $a \in H$ with $aH \supseteq bH$.

It then follows from Lemma 2.3 that the set $\mathcal{L}(H)$ of limit sets L of H is empty if $H = G$ and is in one-to-one correspondence with the joint of the set of right ideals I of H with $I \neq H, I \neq cJ$ for $c \in H$ and $\{\emptyset\}$ for the empty set $\emptyset$ if $H \neq G$. The correspondence is given by $L \to I_L = \bigcap_{aH\in L} aH$ for $L \in \mathcal{L}(H)$ and the inverse mapping $I \to L_I = \{aH \mid aH \supset I\}$ with $L_\emptyset = W(H)$; hence, $L_{I_L} = L$ and $I = I_{L_I}, L_{I_{W(H)}} = W(H), \emptyset = I_{L_\emptyset}$.

We observe that $I_{L_{cJ}} = cH \supset cJ$ in the case $H \neq G$. It is convenient to include the possibility $P = \emptyset$ if we consider limit sets L_P for P completely prime.

Lemma 2.3. *Let H be a right cone in a group G. The following conditions are equivalent for a right ideal I of H:*

(a) $I = \bigcap I_i, I_i \supset I$ *right ideals of* H;
(b) $I \neq H$ *and* $I \neq aJ$ *for any* $a \in H$;
(c) $L_I = \{bH \mid b \in H \text{ and } bH \supset I\} \subseteq W(H) = \{aH \mid a \in H\}$ *has no last (smallest) element.*

The proof of this lemma was given in [9, Lemma 1.3].

In that paper we classified for some right cones H all limit sets $L \in \mathcal{L}(H)$.

By Lemma 2.3 we have, in any right cone $H \neq G$, on one hand the right ideals of the form cJ, for some $c \in H$, and on the other hand the right ideals I so that $L_I = \{aH \mid aH \supset I\}$ is a limit set of H. We want to subdivide the set of right ideals $I \neq H$ of a right cone H further:

Definition 2.4. Let H be a right cone and $I \neq H$ be a right ideal of H. Then:

(a) We say that I is *distinguished* if $I = cP$ for some $c \in H$ and P a completely prime ideal of H, we also consider $\emptyset$ as distinguished; the limit set L_I is distinguished if I is distinguished.
(b) Let I be not distinguished. We say that a limit set $L = L_I = \{aH \mid aH \supset I\}$ is P-*admissable* for a completely prime ideal P of H if $aH \in L$ implies that $apH \in L$ for some $p \in P$.
(c) Let I be not distinguished. We say that $L = L_I$ is a P-*limit set* of H if L is P-admissable but is not P'-admissable for any completely prime ideal P' with $P' \subset P$.

To formulate the next result that characterizes P-limit sets of H, we must recall the prime ideal associated with a right ideal I of H, see [9, Definition 2.2 and Results 2.3–2.7].

Definition 2.5. Let $I \neq H$ be a right ideal in the right cone H. Then $P_r(I) = \bigcup_{I_\lambda \sim I} I_\lambda$, the union of all right ideals I_λ in H related to I, is a completely prime ideal, and $Q_r(I) = \{p \in H \mid \exists s \in H\backslash I \text{ with } sp \in I\} \subseteq P_r(I)$.

The definition, given in Sec. 1, for related right ideals extends directly to right cones H: Let I and I' be right ideals in the right cone H. Then I is related to $I', I \sim I'$, if and only if there exist $s \in H\backslash I$ and $t \in H\backslash I'$ with $s^{-1}I = t^{-1}I'$.

If $H = R\backslash\{0\}$ for a right chain domain R, then $P_r(I) = Q_r(I)$ for any right ideal $I \neq H$ of H; see [9, Lemma 2.7].

Similarly, if all prime segments of H are right invariant, see below for the definition, then again $P_r(I) = Q_r(I)$ for all right ideals $I \neq H$ of H. However, we do not know whether $P_r(I) = Q_r(I)$ in general.

Let H be a right cone. Then two completely prime ideals $P' \supset P''$ of H form a *prime segment* of H if there is no further completely prime ideal of H between P' and P''. If P' is the minimal completely prime ideal of H, then $P' \supset \emptyset$ is also a prime segment of H.

Such a prime segment is *right invariant* if $P'a \subseteq aP'$ for all $a \in P'\backslash P''$; see [9, Proposition 1.6] for other characterizations of right invariant prime segments, and Theorem 1.5 in this paper for the result that a prime segment is either right invariant, simple, or exceptional.

Theorem 2.6. *Let H be a right cone of a group G and P be a completely prime ideal of H. Then the following conditions are equivalent for a limit set $L \subset W(H)$*

with $I = I_L = \bigcap_{aH \in L} aH$ not distinguished and $P_r(I) = Q_r(I)$:

(a) $P_r(I) = P$.
(b) *L is a P-limit set.*
(c) *P is minimal among completely prime ideals P' of H with $I = IH_{P'} \cap H$, where $H_{P'} = \{as^{-1} \in G \,|\, a \in H, s \in H \backslash P'\}$ is the localization of H at P'.*

Proof. (a) $\Rightarrow$ (b) We assume (a) with $P_r(I) = P$ and $aH \in L$. Then $b \in I$ implies $b = ac$ for some $c \in H$ and by the definition of $P_r(I)$ it follows that $c \in P$; hence $I \subseteq aP$.

If $aP \subseteq I$, then $I = aP$ is distinguished and condition (b) does not hold. Otherwise $aP \supset I$ and $L = L_I$ is P-admissable. Assume $P \supset P'$ for a completely prime ideal P', then there exists $p \in P \backslash P'$ and $a \in H \backslash I$ with $ap \in I$. Since $pH \supset P'$, it follows that $aP' \subseteq I$, and L is not P'-admissable. This proves that (a) implies (b).

(b) $\Rightarrow$ (c) Assume that (b) holds for L, that $I = \bigcap_{aH \in L} aH$ and that $a \in (IH_P \cap H) \backslash I$. Then there exists $s \in H \backslash P$ with $as \in I$. However, by (b), there exists $p \in P$ with $ap \notin I$; a contradiction, since $p = sp_1$, for some $p_1 \in H$. Therefore $I = IH_P \cap H$, which proves the first part of (c).

Assume $P \supset \tilde{P}$ for a completely prime ideal $\tilde{P}$ of H with $I = IH_{\tilde{P}} \cap H$ and let $aH \supset I$ for some $a \in H$. Then $aH_{\tilde{P}} \supset IH_{\tilde{P}}$, since otherwise $aH_{\tilde{P}} = IH_{\tilde{P}}$ and $aH \subseteq aH_{\tilde{P}} \cap H = IH_{\tilde{P}} \cap H = I$. Therefore, $aH_{\tilde{P}} \supset a\tilde{P}H_{\tilde{P}} \supseteq IH_{\tilde{P}}$, since $J(H_{\tilde{P}}) = \tilde{P}H_{\tilde{P}}$. We show below that $a\tilde{P} = a\tilde{P}H_{\tilde{P}} \cap H$ and obtain:

$$a\tilde{P} = a\tilde{P}H_{\tilde{P}} \cap H \supseteq IH_{\tilde{P}} \cap H = I.$$

If $a\tilde{P} \subseteq I$, then $I = a\tilde{P}$ is distinguished. Hence $a\tilde{P} \supset I$ and there exists $\tilde{p} \in \tilde{P}$ with $a\tilde{p}H \supset I$. This means that L is $\tilde{P}$-admissable which contradicts (b) and shows that the assumption $I = IH_{\tilde{P}} \cap H$ is wrong.

It remains to prove that $a\tilde{P} = a\tilde{P}H_{\tilde{P}} \cap H$ and we consider an element $b \in a\tilde{P}H_{\tilde{P}} \cap H$. Then there exists $s \in H \backslash \tilde{P}$ with $bs = a\tilde{p}$ for some $\tilde{p} \in \tilde{P}$. If $b = ab_1$, for some $b_1 \in H$, then $b_1 s = \tilde{p}, b_1 \in \tilde{P}$ and $b = ab_1 \in a\tilde{P}$. Otherwise, $a = ba_1$, for some $a_1 \in H$, and $s = a_1\tilde{p} \in \tilde{P}$ follows. This is a contradiction, and this completes the proof that (b) implies (c).

(c) $\Rightarrow$ (a) To prove that (c) implies (a) we assume first that $I = IH_{P'} \cap H$ for a completely prime ideal P'. Then for $a, b \in H, s \in H \backslash P'$ and $b = as \in IH_{P'} \cap H = I$, it follows that $a \in I = IH_{P'} \cap H$. Hence, $H \backslash P' \subseteq H \backslash P_r(I)$, that is $P' \supseteq P_r(I)$. On the other hand, $I = IH_{P_r(I)} \cap H$, since $a = bs^{-1} \in IH_{P_r(I)} \cap H$ with $b \in I$, $s \in H \backslash P_r(I)$ implies $as = b \in I$ and hence $a \in I$. It therefore follows for the prime ideal P in (c) that $P = P_r(I)$, which proves (a). □

We observe that for a distinguished right ideal $I = cP$ for $c \in H$, P completely prime in H, we have $P_r(I) = Q_r(I) = P$, however L_I is not P-admissible, only P'-admissible for any completely prime ideal $P' \supset P$ of H. It follows that in the case where $P = \bigcap P_i$ with $P_i \supset P$ completely prime in H, there exists no minimal prime ideal P' so that L_P is P'-admissable. On the other hand, $cPH_P \cap H = cP$ for any completely prime ideal P in $H, c \in H$.

In [9, Corollary 3.4 and Theorem 4.6], the limit sets $L \in W(H)$ of a right cone H were classified further in case every prime segment of H is right invariant. These results will be used in Sec. 5 where right chain domains are constructed that are I-compact for certain right ideals I of R only. In the next section we consider the association between right chain domains and right cones.

3. Associated Right Cones

We want to construct right chain domains R by constructing first right cones H. In order that the properties of H are reflected in the properties of R there must be a close link between H and R, we will say H and R must be associated. This relationship between H and $R\backslash\{0\}$ can be defined more generally as an association between right cones.

Definition 3.1. Let H_1 be a submonoid of the group G_1, generating G_1, and let H_2 be a submonoid of a group G_2, generating G_2. We say that H_1 is *associated* with H_2 if the following conditions hold:

(i) $G_1 \subseteq G_2$;
(ii) $G_1 \cap H_2 = H_1$;
(iii) For $h_2 \in H_2$ exist $h_1 \in H_1$ and $u_2 \in U(H_2)$, the group of units of H_2, with $h_2 = h_1 u_2$;
(iv) For all $h_1 \in H_1$ we have $H_2 h_1 H_2 \cap G_1 = H_1 h_1 H_1$.

Remark. (a) If H_1 is associated with H_2 and H_1 is a right cone in G_1, then H_2 is a right cone in G_2 by (iii).

Conversely, if H_2 is a right cone in G_2 and h_1, h_1' are elements in H_1, then, say, $h_1 = h_1' h_2$ for some $h_2 \in H_2$. However, $h_2 = h_1'^{-1} h_1 \in G_1 \cap H_2 = H_1$ by (ii), and it follows that H_1 is a right cone.

(b) If H_1 is associated with H_2, then $G_1 \cap U(H_2) = U(H_1)$, since $g_1 \in G_1 \cap U(H_2)$ implies $g_1, g_1^{-1} \in G_1 \cap H_2 = H_1$ by (ii).

If H_1 is associated with H_2 or H_2 is associated with H_1 we say that H_1 and H_2 are *associated*. If H is a right cone in G and R a right Ore domain with F its skew field of quotients, we say that H and R are associated if H and $R\backslash\{0\} \subseteq F\backslash\{0\}$ are associated.

Definition 3.1 is a one-sided, a right-sided definition, necessitated by the fact that we deal with right cones. Nevertheless, we will just use the term *associated* as defined.

Lemma 3.2. *Let H_1 be associated with the right cone H_2. Then the right ideal I of H_2 is an ideal in H_2 if and only if $H_1 I \subseteq I$.*

Proof. Let $a = h_1u_2$ be an element in I with $h_1 \in H_1, u_2 \in U(H_2)$, and let $v \in H_2$. Then $va = vh_1u_2 = h_1'u_2'$ for some $h_1' \in H_1, u_2' \in U(H_2)$. Since $h_1' \in H_2h_1H_2 \cap H_1 = H_1h_1H_1$ (by Definition 3.1(iv)), it follows that $h_1' \in H_1h_1H_1 \subseteq H_1I = I$, which proves the lemma. □

We see in the next result that the ordered sets $W(H_1)$ and $W(H_2)$ are isomorphic if the right cones H_1 and H_2 are associated. Such an isomorphism alone is, in general, not sufficient to establish a correspondence between the ideals of H_1 and the ideals of H_2; however, such a correspondence follows from Definition 3.1(iv) and then from Lemma 3.2.

Proposition 3.3. *Let the right cone H_1 in G_1 be associated with the right cone H_2 in G_2. Then*:

(a) *If $h_2 = h_1u_2 = h_1'u_2' \in H_2$ for $h_1, h_1' \in H_1, u_2, u_2' \in U(H_2)$, then $h_1' = h_1u_1$ for some $u_1 \in U(H_1)$.*
(b) *There is a one-to-one correspondence α from the set $W(H_1)$ of principal right ideals of H_1 to the set $W(H_2)$ with $\alpha(h_1H_1) = h_1H_2$ and $\alpha^{-1}(h_2H_2) = h_2H_2 \cap H_1$.*
(c) *There is a one-to-one correspondence α from the set of right ideals I_0 of H_1 to the set of right ideals I of H_2 with $\alpha(I_0) = I_0H_2$ and $\alpha^{-1}(I) = I \cap H_1$.*
(d) *A right ideal I_0 of H_1 is right principal, is an ideal, is a completely prime ideal, is a prime ideal, if and only if $\alpha(I_0) = I_0H_2$ is a right ideal of H_2 of the same type.*
(e) *If the right ideals I_0 and I_0' are related in H_1 (see p. 5), then the right ideals $I = I_0H_2$ and $I' = I_0'H_2$ are related in H_2.*
(f) *$P_r(I_0)H_2 = P_r(I_0H_2)$ for any right ideal $I_0 \neq H_1$ of H_1.*

Proof. (a) If $h_2 = h_1u_2 = h_1'u_2'$, then $u_1 = h_1'^{-1}h_1 \in G_1 \cap U(H_2) = U(H_1)$, by Remark (b), and hence $h_1 = h_1'u_1$ with $u_1 \in U(H_1)$.

(b) It follows that α defines a mapping, since $h_1H_1 = h_1'H_1$ implies $h_1 = h_1'v$ for some $v \in U(H_1) \subseteq U(H_2)$ for $h_1, h_1' \in H_1$. It follows that $h_1H_2 = h_1'H_2$. The mapping α is onto since $h_2H_2 = \alpha(h_1H_1)$ for $h_2 \in H_2$ and $h_2 = h_1u_2$ with $h_1 \in H_1$ and $u_2 \in U(H_2)$, and α is one-to-one, since $\alpha(h_1H_1) = h_1H_2 = \alpha(h_1'H_1) = h_1'H_2$ implies $h_1H_1 = h_1'H_1$ by (a). Finally, if $h_2 = h_1u_2$ for $h_1 \in H_1, u_2 \in U(H_2)$, then $h_2H_2 = h_1H_2$ and $h_2H_2 \cap H_1 = h_1H_2 \cap H_1 = h_1H_1$ since $h_1' = h_1h_2'$ with $h_i' \in H_i$ for $i = 1, 2$, implies $h_2' \in H_1$.

(c) We prove (c) showing first that $I_0 = I_0H_2 \cap H_1$ and then that $(I \cap H_1)H_2 = I$. We have $I_0 \subseteq I_0H_2 \cap H_1$ and consider an element $h_1u_2 = h_1' \in I_0H_2 \cap H_1$ with $h_1 \in I_0, u_2 \in U(H_2)$. Then $u_2 \in G_1 \cap U(H_2) = U(H_1)$ and $h_1' \in I_0$.

To show that the second equation is true, let $h_2 \in I$ and $h_2 = h_1w_2$ for $w_2 \in U(H_2), h_1 \in H_1$. Then $h_1 = h_2w_2^{-1} \in I \cap H_1$ and $h_2 \in (I \cap H_1)H_2$.

(d) We saw in (b) that principal right ideals in H_1 correspond to principal right ideals of H_2. It follows from Lemma 3.2 that $\alpha(I_0) = I_0H_2$ is an ideal of H_2 if

I_0 is an ideal of H_1. Conversely, if I is an ideal in H_2, then $\alpha^{-1}(I) = I \cap H_1$ is an ideal in H_1 since $H_1 \subseteq H_2$.

Now we want to show that the ideal I in H_2 is completely prime if and only if the ideal $I_0 = I \cap H_1$ is completely prime in H_1. If I is completely prime and $h_1 h_1' \in I_0$ for $h_1, h_1' \in H_1$, then $h_1 h_1' \in I$ and $h_1 \notin I_0 = I \cap H_1$ implies $h_1' \in I_0$. Conversely, if we assume I_0 completely prime in H_1 and $h_2 h_2' \in I$ with $h_2 = h_1 u_2, h_2' = h_1' u_2'$ for $h_i, h_i' \in H_i, i = 1, 2, u_2, u_2' \in U(H_2)$ and $h_2 \notin I$, then $h_1 \notin I_0$. However, $u_2 h_1' u_2' = \tilde{h}_1 v$ for $\tilde{h}_1 \in H_1$ and $v \in U(H_2)$ and $h_1 \tilde{h}_1 \in I_0$. Since I_0 is completely prime, it follows that $\tilde{h}_1 \in I_0$ and then that $h_1' \in I$; hence, I is completely prime.

Finally, we want to show that an ideal I of H_2 is prime if and only if $I_0 = I \cap H_1$ is prime in H_1.

Assume that I is prime and that A_0, B_0 are ideals in H_1 properly containing I_0. Then $A = A_0 H_2, B = B_0 H_2$ and $AB = A_0 H_2 B_0 H_2 = A_0 B_0 H_2$ all contain I properly. Hence, $A_0 B_0$ contains I_0 properly, that is, I_0 is prime in H_1.

Conversely, if I_0 is prime in H_1, and A, B are ideals in H_2 that contain I properly, then $A_0 = A \cap H_1, B_0 = B \cap H_1$, and $A_0 B_0$ all contain I_0 properly. Hence, $AB \cap H_1 \supseteq A_0 B_0 \supset I_0$ and therefore $AB \supset I$, that is, I is prime in H_2.

(e) If $h_1^{-1} I_0 = h_1'^{-1} I_0'$ for $h_1 \in H_1 \backslash I_0, h_1' \in H_1 \backslash I_0'$, then $h_1 \in H_2 \backslash I, h_1' \in H_2 \backslash I'$ and $h_1^{-1} I_0 H_2 = h_1'^{-1} I_0' H_2$ shows that $I = I_0 H_2$ and $I' = I_0' H_2$ are related in H_2.

(f) $P_r(I_0) = \bigcup_{I_0' \sim I_0} I_0'$ by definition, and $P_r(I_0)$ is a completely prime ideal in H_1 for any right ideal $I_0 \neq H_1$ of H_1. By (e) it follows that $I_0' \sim I_0$ in H_1 implies that $I' = I_0' H_2$ is related to $I_0 H_2 = I$ in H_2 and by (d) it follows that

$$P = \bigcup_{I_0' \sim I_0} I_0' H_2 = \left(\bigcup_{I_0' \sim I_0} I_0' \right) H_2 \subseteq P_r(I_0 H_2)$$

is completely prime in H_2. If $P \subset P_r(I_0 H_2)$ then there exists by [9, Lemma 2.11] a right ideal $\tilde{I}$ of H_2 related to $I_0 H_2$ with $t \cdot \tilde{I} = I_0 H_2$ for some $t \in H_2$ and $P \subset \tilde{I} \subseteq P_r(I_0 H_2)$. We have $t = h_1 u_2$ for some $h_1 \in H_1$ and $u_2 \in U(H_2)$ and hence $I_0 H_2 = h_1 u_2 \tilde{I}$ and $u_2 \tilde{I} = h_1^{-1} I_0 H_2 \subseteq P$ since $h_1^{-1} I_0 \sim I_0$ in H_1. It follows that $\tilde{I} \subseteq u_2^{-1} P \subseteq P$, a contradiction that proves $P = P_r(I_0) H_2 = P_r(I_0 H_2)$, which is statement (f). □

Corollary 3.4. *Let the right cone H_1 be associated with the right cone H_2. If $P_r(I_0) = Q_r(I_0)$ for a right ideal I_0 of H_1, then $P_r(I_0 H_2) = Q_r(I_0 H_2)$.*

Proof. We have $P_r(I_0 H_2) = P_r(I_0) H_2 = Q_r(I_0) H_2 \subseteq Q_r(I_0 H_2) \subseteq P_r(I_0 H_2)$, where the first equation follows from Proposition 3.3(f), the second equation from the assumption, and the last containment from [9, Proposition 2.3].

To show that $Q_r(I_0) H_2 \subseteq Q_r(I_0 H_2)$ let $q \in Q_r(I_0)$ with $aq \in I_0$ for $a \in H_1 \backslash I_0$. Then $a \in H_2 \backslash I_0 H_2$ and $q \in Q_r(I_0 H_2)$ follows. □

If H_1 and H_2 are associated right cones, then prime segments of H_1 correspond to prime segments of H_2 of the same type, as is shown in the next result.

Corollary 3.5. *Let the right cone H_1 be associated with the right cone H_2. Then α defines a one-to-one correspondence between prime segments of H_1 and prime segments of H_2 such that corresponding segments are of the same type.*

Proof. It remains to show that corresponding prime segments are of the same type. By [9, Theorem 1.5] an exceptional prime segment is characterized by the existence of a prime ideal Q which is not completely prime, and a simple prime segment is characterized by the absence of any further ideals in this prime segment. All other prime segments are invariant. A discrete invariant prime segment $P_0' \supset P_0''$ in H_1 is characterized by the condition $P_0' \supset P_0'^2$, and then $P_0'H_2 \supset P_0'^2 H_2 = P_0'H_2P_0'H_2$ by Lemma 3.2; hence $P_0' \supset P_0''$ is discrete in H_1 if and only if $P_0'H_2 \supset P_0''H_2$ is discrete in H_2. □

Corollary 3.6. *Let the right cone H_1 be associated with the right cone H_2. Then*:

(a) *The mapping α defines a one-to-one correspondence between the set of distinguished right ideals of H_1 and the set of distinguished right ideals of H_2 given by $\alpha(cP_0) = cP_0H_2$ for $c \in H_1, P_0$ completely prime in H_1 and $\alpha(\emptyset) = \emptyset$.*

(b) *Assume $P_r(I_0) = Q_r(I_0)$ for any right ideal $I_0 \neq H_1$ of H_1. Then*: *The mapping α defines a one-to-one correspondence between the P_0-limit sets $L_0 \subseteq W(H_1)$ with $I_{L_0} = \bigcap_{a_0H_1 \in L_0} a_0H_1$ not distinguished and the set of P-limit sets $L \subseteq W(H_2)$ with $I_L = \bigcap_{aH_2 \in L} aH_2$ not distinguished, where P_0 is a completely prime ideal in $H_1, P = P_0H_2$ and $\alpha(a_0H_1) = a_0H_2$ for $a_0H_1 \in L_0$.*

Proof. Statement (a) follows since $\alpha(cP_0) = cP_0H_2$ and $P_0H_2 = P$ is completely prime in H_2 if and only if P_0 is completely prime in H_1 by Proposition 3.3(d).

If h_2P is distinguished in H_2 with P completely prime and $h_2 = h_1u_2 \in H_2$ for $h_1 \in H_1$ and $u_2 \in U(H_2)$, then $\alpha(h_1P_0) = h_1P_0H_2 = h_2P$ for $P_0 = P \cap H_1$ with P_0 completely prime in H_1 by Propositions 3.3(c) and 3.3(d).

We use Theorem 2.6 to prove statement (b). By assumption, $I_{L_0} = I_0$ is a not distinguished right ideal in H_1, and by Theorem 2.6, $P_r(I_0) = P_0$.

Then $\alpha(L_0) = \{a_0H_2 \mid a_0H_1 \in L_0\} = L \subseteq W(H_2)$, and $I_L = \bigcap_{a_0H_1 \in L_0} a_0H_2 = I_0H_2$. By Proposition 3.3(f) we have $P_r(I_L) = P_r(I_0)H_2$ and $I = I_0H_2$ is not distinguished since I_0 is not distinguished by (a). Finally, by the assumption and Corollary 3.4 we have $P_r(I) = Q_r(I)$. Conversely, if L is a P-limit set in $W(H_2)$ with $P = P_0H_2$ and I_L not distinguished, then $\alpha^{-1}(L) = \{aH_2 \cap H \mid aH_2 \in L\} = L_0$ is a P_0-limit set in H_1 with I_{L_0} not distinguished. □

4. Right Cones and Rings

We like to construct right chain domains R so that $\mathcal{C}(R)$, the set of right ideals I of R for which R is I-compact is a certain predescribed subset of the set of all right ideals of R. It follows from Proposition 1.4 that $\mathcal{C}(R)$ consists of classes of related right ideals of R and the results in [9, Secs. 3 and 4] will be useful for the description of such sets of right ideals in right cones H and hence in right chain domains R associated with H. We do not know the class of right cones that are associated with right chain domains, but, for the remainder of this paper, we will consider right cones in ordered groups and right chain domains associated with them.

Let (G, Π) be an ordered group G with cone $\Pi = \{g \in G \,|\, g \geq e\}$; hence $\Pi \cup \Pi^{-1} = G, \Pi \cap \Pi^{-1} = \{e\}$, and $g\Pi = \Pi g$ for all $g \in G$. Then there exists the generalized power series ring $F = K\{\{G\}\}$ over a skew field K with elements $\alpha = \sum g k_g$ for $g \in G, k_g \in K$ and support of $\alpha = \mathrm{supp}(\alpha) = \{g \,|\, k_g \neq 0\}$ well ordered in (G, Π).

The skew field F contains the chain domain $B = \{\alpha \in F \,|\, \mathrm{supp}(\alpha) \subseteq \Pi\}$ so that Π and B are associated, see also Corollary 4.4.

The next result describes the right cones H in G which we want to consider.

Proposition 4.1. *Let (G, Π) be an ordered group. The following conditions are equivalent for a right cone H of G:*

(a) *$aH \subset bH$ for $a, b \in H$ implies $b < a$ in (G, Π).*
(b) *There exists a prime ideal P in the cone Π of G with $H \subseteq \Pi_P$ and $U(H) = U(\Pi_P) \cap H$ for the localization $\Pi_P = \{as^{-1} | a \in \Pi, s \in \Pi \backslash P\}$ of Π at P.*
(c) *If $s \in H, s \notin \Pi$, then $s \in U(H)$.*

Proof. (a) $\Rightarrow$ (b) We assume (a) and let $S = \{s \in \Pi \,|\, s \leq u$ for some $u \in U(H)\}$. Then S is multiplicatively closed and $P = \Pi \backslash S$ is a completely prime ideal of Π. The localization Π_P of Π at P exists and $\Pi_P = \Pi \cup S^{-1}$. Then $H \subseteq \Pi_P$ since for $h \in H$ either $hH = H$ and $h \in U(H) \subseteq S \cup S^{-1} = U(\Pi_P)$, or $hH \subset H$ and $h > e$ by (a) and $h \in \Pi$ follows; that is $H \subseteq \Pi_P$. Therefore $U(H) \subseteq U(\Pi_P) \cap H$. If $h \in U(\Pi_P) \cap H$, then $h = s$ or $h = s^{-1}$ for some $s \in S$.

In the first case, $h = s \leq u \in U(H)$ and $H = uH \subseteq sH, s \in U(H)$, follows, using (a) again. In the second case, $h = s^{-1}$, the assumption $hH \subset H$ implies $h > e$, but $h = s^{-1} \leq e$. Hence, $hH = H$ and $h \in U(H)$ which proves (b).

(b) $\Rightarrow$ (c) If we assume (b) and $s \in H, s \notin \Pi$, then $s \in (\Pi \backslash P)^{-1}$, hence $s \in U(\Pi_P) \cap H = U(H)$ and (c) follows.

(c) $\Rightarrow$ (a) Finally, if (c) holds and $aH \subset bH$ for $a, b \in H$, then $a = bj$ for $j \in J(H)$, hence $j \in \Pi \backslash \{e\}$ and $b < a$, which proves statement (a). □

Right cones, as described in the previous result, have been constructed to provide an answer to a question of Frege [1, 8], and also to construct right noetherian right chain domains R associated with H so that R has an arbitrary number of prime segments (see [6, 10, 14]). The group G generated by the right cone H is the wreath product of ordered groups in these cases; see also [9, Example 1.2] and further examples in Sec. 6 of this paper. We give a typical example.

Example 4.2. Let $G_1 = \langle x \rangle$ and $G_2 = \langle y \rangle$ be two infinite cyclic groups. The direct sum $B = \bigoplus_{i \in \mathbb{Z}} B_i, B_i = G_1$ for all i is called the base group and B is lexicographically ordered by $b = (b_i) > b' = (b'_i)$ for $b, b' \in B$ if and only if for $i_0 = \min\{i \mid b_i \neq b'_i\}$ and $b_{i_0} = x^{n_{i_0}}, b'_{i_0} = x^{n'_{i_0}}$ we have $n_{i_0} > n'_{i_0}$. The group B also admits an automorphism σ with $\sigma(b) = b'$ with $b = (b_i)$ and $b' = (b'_i)$ where $b'_i = b_{i-1}$. Then $G = \{y^n b \mid n \in \mathbb{Z}, b \in B\}$ with $by = y\sigma(b)$ is an ordered group with $\Pi = \{y^n b \in G \mid n \geq 0 \text{ and if } n = 0, \text{ then } b \geq e_B \in B\}$ as the cone of non-negative elements in G.

The group G_1 is embedded into G if we identify x^n in G_1 with $b = (b_i)$ in G where $b_0 = x^n$ and $b_i = x^0$ for $i \neq 0$. The subset $S = \{y^0 b \mid b_i = x^0 \text{ for } i \leq 0\}$ of Π is multiplicatively closed and $s_1 s_2 = s \in S$ for $s_1, s_2 \in \Pi$ implies $s_1, s_2 \in S$. Hence, $P = \Pi \backslash S$ is a completely prime ideal of Π. We consider $H = \{y^n b \in G \mid n \geq 0, b_i = x^0 \text{ for } i < 0, b_0 = x^m \text{ for } m \geq 0\}$.

If $h = y^n b$ with $b_0 = x^m$ for $m \geq 0$, then $hH = y^n x^m H$ and $y^{n_1} x^{m_1} H \supset y^{n_2} x^{m_2} H$ if and only if $n_1 < n_2$ or $n_1 = n_2$ and $m_1 < m_2$. It follows that H is a right invariant right cone of G with exactly two prime segments, $xH \supset yH$ and $yH \supset \emptyset$. Since $Hy \not\subseteq Hyx \not\subseteq Hy$, it follows that H is not a left cone of G.

The cone Π is invariant and has infinitely many prime segments. The localization Π_P of Π at P is a cone of G, that is right invariant and has infinitely many prime segments. Since $\Pi_P y s^{-1} \not\subseteq \Pi_P y$ for $s = (b_i)$ with $b_1 = x$ and $b_i = x^0$ for $i \neq 1$, it follows that Π_p is not left invariant.

The condition (b) of Lemma 4.1 is satisfied, since $H \subseteq \Pi_P$ and $U(\Pi_P) = S \cup S^{-1} = U(H)$.

Let H be a right cone in the ordered group (G, Π) that satisfies the equivalent conditions in Proposition 4.1. Then $F = K\{\{G\}\}, K$ a skew field, contains the subring

$$\hat{R} = \{\alpha \in F \mid \operatorname{supp}(\alpha) \subseteq H\}$$

and $\hat{R}$ contains $K[H]$ the semigroup ring of H over K of elements $\alpha \in \hat{R}$ with finite support.

Lemma 4.3. *Let H be a right cone in the ordered group (G, Π) that satisfies the equivalent conditions in Proposition 4.1. Then the following conditions are equivalent for a subring R of $\hat{R}$ that contains $K[H]$:*

(a) *H is associated with R.*
(b) *R satisfies the following conditions:*

  (b-i) *Every nonzero element $r \in R$ can be written as $r = h_0 k_0 (1 - m)$ with $h_0 \in H, 0 \neq k_0 \in K$ and $m = \sum h_i k_i \in R$ with $e < h_i \in H$ for all h_i, and $k_i \in K$.*

  (b-ii) *If $0 \neq m = \sum h_i k_i \in R, e < h_i \in H, k_i \in K$ for all i, then*

$$(1 - m)^{-1} = \sum_{t=0}^{\infty} m^t \in R.$$

Proof. We assume (a) and $R \cap G = H$ follows immediately. To prove (b-i) we write $0 \neq r = hu$ for some $h \in H, u \in U(R)$. Then $u = g_0k_0 + g_1k_1' + g_2k_2' + \cdots$ for $g_0 < g_i < g_j \in H$ if $1 \leq i < j, k_0 \neq 0, k_i' \in K$ and g_0 must be a unit in H since $u \in U(R)$.

It follows that $r = hu = hg_0k_0[1+g_0^{-1}g_1k_0^{-1}k_1'+g_0^{-1}g_2k_0^{-1}k_2'+\cdots] = h_0k_0(1-m)$ with $h_0 = hg_0$ and $m = \sum h_ik_i \in R$, where $e < g_0^{-1}g_i = h_i \in H$ and $k_i = -k_0^{-1}k_i' \in K$ for $i \geq 1$; as in (b-i).

To show that (a) implies (b-ii) we write $1 - m = hu$ for $h \in H, u \in U(R)$. Then $1 = hu + m$ and there must be an element $h' \in \mathrm{supp}(u)$ with $hh' = 1$; hence $h \in U(H), 1 - m \in U(R)$ and therefore $(1-m)^{-1} = \sum_{t=0}^{\infty} m^t$, that is (b-ii) follows from (a).

Now we assume (b) and show that the conditions in Definition 3.1 are satisfied for H and $R\backslash\{0\}$. Since $K[H] \subseteq R \subseteq \hat{R}$, it follows that $R \cap G = H$, and from (b) that $r = h_0k_0(1-m) = h_0u$ for $r \neq 0$ in R and $u = k_0(1-m) \in U(R)$ with $0 \neq k_0 \in K$.

Finally, $HhH \subseteq RhR \cap H$ for $h \in H$. Conversely, $(\sum h_ik_i)h(\sum g_jt_j) = g \in RhR \cap H$ for $h_i, h, g_j, g \in H$ and $k_i, t_j \in K$ implies $g = h_ihg_j$ for some i, j and $RhR \cap H \subseteq HhH$ follows. □

The conditions under (b) in Lemma 4.3 are satisfied in particular for $\hat{R}$:

Corollary 4.4. *H is associated with $\hat{R}$.*

It follows from Remark (a) after Definition 3.1 that $\hat{R}$ is a right chain domain.

Proposition 4.5. *Let H be a right cone of (G, Π) as in Proposition 4.1 and H be associated with a subring R of $\hat{R}$ as in Definition 3.1. Then:*

(i) *H and R are both locally right invariant.*
(ii) *If Π_P is right invariant, then H is right invariant.*
(iii) *H is right invariant if and only if R is right invariant.*
(iv) *If R is in addition a left chain domain, then $H = \Pi_P$.*

Proof. (i) The prime segments $P_1 \supset P_2$ of Π_P are exactly the prime segments of Π with $P \supseteq P_1$ and they are right invariant by [9, Proposition 2.15]. Let $P' \supset P''$ be a prime segment of H with $a \in P'\backslash P''$. Since $a \in H$ is a nonunit in Π_P, it determines a prime segment $P_1 \supset P_2$ of Π_P with $a \in P_1\backslash P_2$ and this segment is right invariant. It follows that $\tilde{P}_1 = P_1 \cap H$ and $\tilde{P}_2 = P_2 \cap H$ are completely prime ideals in H with $a \in \tilde{P}_1\backslash\tilde{P}_2$ and $\tilde{P}_2 = \emptyset$ a possibility. If $c \in \tilde{P}_1\backslash\tilde{P}_2$, then there exists by [9, Theorem 1.5(iia)] an ideal I of Π_P in P_1 with $c \in I$ and $\bigcap_n I^n = P_2$. Hence, $c \in I_0 = I \bigcap H$ and I_0 is an ideal in H with $\bigcap_n I_0^n = \tilde{P}_2$. Therefore, $P' = \tilde{P}_1 \supset \tilde{P}_2 = P''$ is a right invariant prime segment of H, which proves (i), if we also apply Corollary 3.5 to H and R.

To prove (ii) we assume that Π_P is right invariant, but $haw = a$ for some $h, a \in H$ and $w \in J(H)$. Then there exists $p \in \Pi_P$ with $ha = ap$ and hence $p^{-1} = w \in U(\Pi_P) \cap H = U(H)$. This contradiction proves (ii).

Statement (iii) follows from Proposition 3.3(b) and (d).

For a proof of (iv) we assume that R is in addition a left chain domain and consider an element $p \in \Pi_P$. Since H is a right cone in G we have $p = ab^{-1}$ for some $a, b \in H$. Then either $Rb \supseteq Ra$ in R or $Ra \supset Rb$. In the first case $a = rb$ for some $r \in R$ and $r = ab^{-1} \in R \cap G = H$ and $r = p = ab^{-1} \in H$ follows. In the case $Ra \supset Rb$ we have $b = ra$ for some $r \in R$ and $r = ba^{-1} \in R \cap G$ implies $r \in H$ and $r \in J(H)$. But then $p = ab^{-1} \in \Pi_P, r = ba^{-1} \in U(\Pi_P) \cap H = U(H)$, a contradiction, which proves $p \in H$ and $H = \Pi_P$. □

5. Subrings of Malcev–Neumann Rings

We consider in this section right cones H in an ordered group (G, Π) as defined in Proposition 4.1 and certain subsets $\mathcal{B}$ of the set of limit sets $\mathcal{L}(H)$ of H. By considering the subring $R_{\mathcal{B}}$ of all elements $\alpha \in \hat{R} = K\{\{H\}\}$ whose support generates only limit sets in $\mathcal{B}$, we can show that $R_{\mathcal{B}}$ is I-compact exactly for the right ideals I in $R_{\mathcal{B}}$ that correspond to the right ideals I_L in H for $L \in \mathcal{B}$.

Hence, let H be a right cone in an ordered group (G, Π) that satisfies the equivalent conditions in Proposition 4.1. Then H is locally right invariant by Proposition 4.5 and $P_r(I) = Q_r(I)$ for every right ideal $I \neq H$ of H by [9, Corollary 2.4 and Lemma 2.5]. We can therefore apply Theorem 2.6 to characterize P-limit sets of H for a completely prime ideal P of H.

With $\mathcal{L}(H)$ we will denote the set of all limit sets of H and with $\langle N \rangle$ the limit set $\{aH \in W(H) \,|\, aH \leq bH, \text{ some } bH \in N\}$ of H generated by N if N is a subset of $W(H)$ without a last element.

Definition 5.1. Let H be a right cone in (G, Π) as in Proposition 4.1 and $\alpha \in \hat{R} = K\{\{H\}\}$. Then the set of limit sets of α is $\mathcal{L}(\alpha) = \{L \in \mathcal{L}(H) \,|\, \langle L \cap \{h_i H \,|\, h_i \in \text{supp }(\alpha)\}\rangle = L\}$.

In the following we will define subrings R of $\hat{R}$ so that H is associated with R by imposing restrictions on the set $\mathcal{L}(\alpha)$ for $\alpha \in R$.

We will consider the case where H is in addition right invariant; the right chain domains associated with an H as in Proposition 4.1 that is not right invariant will be obtained as localizations of right invariant right chain domains. We will deal with this case in a separate paper.

The following assumptions will therefore be made for the results in this section.

(G, Π) is an ordered group, P a completely prime ideal of Π and $H \subseteq \Pi_P$ a right invariant right cone of G with $U(\Pi_P) \cap H = U(H)$. It then follows by Corollary 4.4 and Proposition 3.3 that H is associated with the right invariant right chain domain $\hat{R} = K\{\{H\}\}$.

Lemma 5.2. *Let H be as above and $\alpha, \beta \in \hat{R}$. Then*:

(i) $\mathcal{L}(\alpha \pm \beta) \subseteq \mathcal{L}(\alpha) \cup \mathcal{L}(\beta)$;

(ii) $$\mathcal{L}(\alpha\beta) \subseteq \{\langle a_i y H \mid a_i H \in L' \in \mathcal{L}(\alpha); y \in \operatorname{supp}(\beta)\rangle\} \cup \{\langle xL''\rangle \mid L'' \in \mathcal{L}(\beta), x \in \operatorname{supp}(\alpha)\} \cup \{\langle L_1 L_2\rangle \mid L_1 \in \mathcal{L}(\alpha), L_2 \in \mathcal{L}(\beta)\}.$$

Proof. (i) Let L be an element in $\mathcal{L}(\alpha \pm \beta)$. Then there exists a sequence (t_i) of elements either in $\operatorname{supp}(\alpha)$ or $\operatorname{supp}(\beta)$ so that $L = \langle\{t_i H\}\rangle$. It follows that at least one of the two subsequence of $\{t_i H\}$ consisting of those $t_i H$ with $t_i \in \operatorname{supp}(\alpha)$ on the one hand and of those $t_i H$ with $t_i \in \operatorname{supp}(\beta)$ on the other hand will generate L.

(ii) We assume that $T = L \cap \{hH \mid h \in \operatorname{supp}(\alpha\beta)\}$ generates L, that is $\langle T\rangle = L \in \mathcal{L}(\alpha\beta)$. Let $u_1 H < u_2 H < \cdots < u_\rho H < u_{\rho+1} H < \cdots <$ be a cofinal sequence of elements $u_i H$ in L. Then there exists $a_1 b_1 H$ in T with $u_1 H < a_1 b_1 H$, $a_1 \in \operatorname{supp}(\alpha)$, $b_1 \in \operatorname{supp}(\beta)$ and a_1 minimal with respect to the order in G. We define by transfinite induction a sequence of elements $a_\rho b_\rho H \in T$ with $u_\rho H < a_\rho b_\rho H$ and $a_\rho \leq a_\sigma$ for $\rho < \sigma$.

If ρ is a limit ordinal, then either $\langle\{a_\lambda b_\lambda H \mid \lambda < \rho\}\rangle = L$ or there exists $a_\rho b_\rho H \in T$ with $a_\lambda b_\lambda H < a_\rho b_\rho H$ for all $\lambda < \rho$, and in addition $u_\rho H < a_\rho b_\rho H, a_\rho \in \operatorname{supp}(\alpha), b_\rho \in \operatorname{supp}(\beta)$ and a_ρ minimal.

If ρ has a predecessor $\rho - 1$, then there exists $a_\rho b_\rho H \in T$ with $a_{\rho-1} b_{\rho-1} H < a_\rho b_\rho H, u_\rho H < a_\rho b_\rho H, a_\rho \in \operatorname{supp}(\alpha), b_\rho \in \operatorname{supp}(\beta)$ and a_ρ minimal. We then obtain a subsequence $\{a_\lambda b_\lambda H\}$ of T cofinal in L with $a_\lambda \leq a_\rho$ for $\lambda \leq \rho$.

Next we apply the same process to this sequence $\{a_\lambda b_\lambda H\}$ where we choose $a'_1 b'_1 H \in \{a_\lambda b_\lambda H\}$ with $a_1 b_1 H < a'_1 b'_1 H$ and b'_1 minimal, $a'_2 b'_2 H \in \{a_\lambda b_\lambda H\}$ with $a'_1 b'_1 H < a'_2 b'_2 H, a_2 b_2 H < a'_2 b'_2 H$ and b'_2 minimal, apply transfinite induction and obtain a subsequence $\{a'_i b'_i H\}$ of T cofinal in L with $a'_\lambda \leq a'_\rho$ and $b'_\lambda \leq b'_\rho$ for $\lambda \leq \rho$.

The following three possibilities occur: The sequence $\{a'_1 H\}$ has a final element xH, then $L = \langle\{xHb'_i H = xb'_i H\}\rangle = \langle xHL''\rangle = \langle xL''\rangle$ with the limit set $L'' = \langle\{b'_i H\}\rangle \in L(\beta)$ and $x \in \operatorname{supp}(\alpha)$.

Similarly, if the sequence $\{b'_i H\}$ has a final element, say yH, then $L = \langle\{a'_i y H\}\rangle = \langle L' y H\rangle$ for the limit set $L' = \langle\{a'_i H\}\rangle \in \mathcal{L}(\alpha)$ and $y \in \operatorname{supp}(\beta)$.

Finally, there is the possibility that $L_1 = \langle\{a'_i H\}\rangle \in \mathcal{L}(\alpha)$ and $L_2 = \langle\{b'_i H\}\rangle \in \mathcal{L}(\beta)$ and $L = \langle L_1 L_2\rangle$. □

We observe that $\langle aL\rangle \in \mathcal{L}(H)$ if $a \in H, L \in \mathcal{L}(H)$, but that $\langle LaH\rangle$ may not be an element in $\mathcal{L}(H)$. In Example 4.2 the sequence $\{x^n H \mid n \geq 0\} = L$ is in $\mathcal{L}(H)$, however $x^n H y H = yH$ for all n and $\langle\{LyH\}\rangle = \langle\{x^n H y H\}\rangle = \langle yH\rangle$ is not in $\mathcal{L}(H)$.

The above result suggests to define subrings R of $\hat{R}$ by applying suitable restrictions on the limit sets $\mathcal{L}(\alpha)$ for elements $\alpha \in R$.

If $\mathcal{B}$ is a subset of $\mathcal{L}(H)$, then we define $R_\mathcal{B} = \{\alpha \in \hat{R} \,|\, \mathcal{L}(\alpha) \subseteq \mathcal{B}\}$ as the set of elements in $\hat{R} = K\{\{H\}\}$ whose limit sets are in $\mathcal{B}$. Then the set $R_\mathcal{B}$ does not only contain the semigroup ring, but also the skew field $F_0 = K\{\{U(H)\}\}$.

Corollary 5.3. *Let H be a right invariant right cone in (G, Π) as in Proposition 4.1 and let $\mathcal{B} \subseteq \mathcal{L}(H)$. If*

(1) *$L_1, L_2 \in \mathcal{B}$ implies $\langle L_1 L_2\rangle \in \mathcal{B}$, and*
(2) *$L \in \mathcal{B}$ and $a \in H$ implies $\langle aL\rangle \in \mathcal{B}$ and $\langle LaH\rangle \in \mathcal{B}$ provided $\langle LaH\rangle \in \mathcal{L}(H)$, then $R_\mathcal{B}$ is a subring of $\hat{R}$.*

The ring $R_\mathcal{B}$, where $\mathcal{B}$ satisfies the conditions (1) and (2) of Corollary 5.3, will in general not be a right chain domain as the example $\Pi = H, \mathcal{B} = \emptyset$ with $R_\mathcal{B} = K[\Pi]$ shows. We want to find additional conditions on $\mathcal{B}$ so that H is associated with $R_\mathcal{B}$. The condition (b-i) in Lemma 4.3 is satisfied by the subring $R_\mathcal{B}$ of $\hat{R}$ if the following is true for $\mathcal{B}$:

(3) If $L \in \mathcal{B}$ and $gH \in \mathcal{L}$, then $g^{-1}L = \{hH \in W(H) \,|\, ghH \in L\} \in \mathcal{B}$.

To see this, let $\alpha = \sum h_i k_i \neq 0$ be an element in $R_\mathcal{B}$ with $h_0 = \min\{\mathrm{supp}(\alpha)\}, h_i \in H, k_i \in K$ and $\alpha = h_0 k_0 (1 + \sum_{i>0} h_0^{-1} h_i k_0^{-1} k_i) = h_0 k_0 (1 - m)$. Then, $h_0^{-1} h_i \in H$ for all i and $m = -\sum h_0^{-1} h_i k_0^{-1} k_i$ is an element in $R_\mathcal{B}$ by condition (3).

By Lemma 2.3 every right ideal I of H with $I \neq H$ and $I \neq cJ(H), c \in H$, defines a limit set $L_I = \{aH \in W(H) \,|\, aH \supset I\}$ of H and conversely $L = L_I$ for every $L \in \mathcal{L}(H)$ for $I = \cap_{aH \in L} aH$.

In Sec. 2, we associated with such a limit set $L = L_I \in \mathcal{L}(H)$ a completely prime ideal $P_r(I)$, and $P_r(I) = Q_r(I)$ since we assume here that H is right invariant. Further, we defined a right ideal I of H as distinguished if $I = cP$ for some $c \in H$, and P a completely prime ideal of H with $P \neq J(H)$. By [9, 2.8], $P_r(cP) = P$, and hence P is the associated prime ideal of L_{cP}, however L_{cP} is not a P-limit set, since $cH \in L_{cP}$, but $cpH \notin L_{cP}$ for all $p \in P$.

Let $\mathcal{D} = \{L_{cP} \,|\, c \in H, P \neq J(H), P$ completely prime$\} \subseteq \mathcal{L}(H)$ be the set of all distinguished limit sets of H, $\mathcal{D}$ includes $W(H) = L_\emptyset$. Then the condition (b-ii) of Lemma 4.3 is satisfied by $R_\mathcal{B}$ if $\mathcal{B}$ satisfies (1), (2) and $\mathcal{D} \subseteq \mathcal{B}$, that is:

(4) $\mathcal{B}$ contains $\mathcal{D}$, the set of all distinguished limit sets of H.

To prove this let $m = \sum h_i k_i \in R_\mathcal{B}$ with $h_i \in H, k_i \in K$, and $e < h_i < h_j$ for all $h_i, h_j \in \mathrm{supp}(m)$ and $i < j$. The inverse of $1 - m$ in $\hat{R}$ is then $\sum_{i=0}^{\infty} m^i = 1 + m + m^2 + \cdots$. We want to show that under the above conditions on H and $\mathcal{B}$ the element $(1-m)^{-1}$ is in $R_\mathcal{B}$, that is $\mathcal{L}[(1-m)^{-1}] \subseteq \mathcal{B}$ if $\mathcal{L}(m) \subseteq \mathcal{B}$.

The statement is true for $m = 0$. Otherwise there exists the element $h_1 = \min(\mathrm{supp}(m))$ with $e < h_1$ and $h_1 \in U(H)$ or there exists a prime segment $P'_{i_1} \supset P''_{i_1}$ of H with $h_1 \in P'_{i_1} \backslash P''_{i_1}$. Then $m = m_1 + m'_1$ with $m_1, m'_1 \in R_\mathcal{B}$ and $\mathrm{supp}(m_1)$

$\subseteq U(H), \mathrm{supp}(m_1') \subseteq J(H)$ in case $h_1 \in U(H)$, and $\mathrm{supp}(m_1) \subseteq S_{i_1} = P'_{i_1} \backslash P''_{i_1}$ with $\mathrm{supp}(m_1') \subseteq P''_{i_1}$ in case $h_1 \in S_{i_1}$.

It follows that $\alpha_1 = \sum_{i=0}^{\infty} m_1^i \in K\{\{U(H)\}\} \subseteq R_{\mathcal{B}}$ if $h_1 \in U(H)$, and we want to show that

$$\mathcal{L}(\alpha_1) \subseteq \bigcup_{t \in \mathbb{N}} \mathcal{L}(m_1^t) \cup L_{P''_{i_1}} \subseteq \mathcal{B} \quad \text{if } h_1 \in S_{i_1}.$$

The second containment is true by Lemma 5.2 and since we assume (1), (2), and $\mathcal{D} \subseteq \mathcal{B}$.

Since $\mathrm{supp}(m_1) \subseteq S_{i_1}$, it follows that $\mathrm{supp}(\alpha_1 - 1) \subseteq S_{i_1}$. Let $L \in \mathcal{L}(\alpha_1)$ and hence $P'_{i_1} \supset I_L \supseteq P''_{i_1}$. We can assume $I_L \neq P''_{i_1}$ since $\mathcal{D} \subseteq \mathcal{B}$.

Since the prime segment $P'_{i_1} \supset P''_{i_1}$ is right invariant, there exists a power h_1^n of h_1 with $P'_{i_1} \supset I_L \supset h_1^n H \supset P''_{i_1}$, see [9, Proposition 1.6(c)].

Since $h_1^n = \min(\mathrm{supp}(m_1^n))$, it follows that

$$L \in \mathcal{L}(1 + m_1 + m_1^2 + \cdots + m_1^{n-1}) \subseteq \bigcup_{t=1}^{n-1} \mathcal{L}(m_1^t) \subseteq \mathcal{B}.$$

Hence, $\alpha_1 \in R_{\mathcal{B}}$ and $(1-m)\alpha_1 = (1 - m_1 - m_1')\alpha_1 = (1 - m_1'\alpha_1) \in R_{\mathcal{B}}$ for $\alpha_1 = \beta_1 = 1 + M_1$ with $M_1 = \sum_{t=1}^{\infty} m_1^t$ in $R_{\mathcal{B}}$.

The above process is now repeated with $\tilde{m}_2 = m_1'\alpha_1 = m_2 + m_2'$ in place of m and with $h_2 = \min(\mathrm{supp}(m_1'\alpha_1)) \in P'_{i_2} \backslash P''_{i_2}$ for a prime segment $P''_{i_1} \supseteq P'_{i_2} \supsetneqq P''_{i_2}$ (or $J \supseteq P'_{i_2} \supsetneqq P''_{i_2}$) and $\mathrm{supp}(m_2) \subseteq S_{i_2} = P'_{i_2} \backslash P''_{i_2}$ and $\mathrm{supp}(m_2') \subseteq P''_{i_2}$.

Then $\alpha_2 = \sum_{t=0}^{\infty} m_2^t = 1 + M_2$ for $M_2 = \sum_{t=1}^{\infty} m_2^t$ satisfies $\mathrm{supp}(M_2) \subseteq S_{i_2}$ and $\mathcal{L}(\alpha_2) \subseteq \mathcal{B}$ with $(1-m)\alpha_1\alpha_2 = (1-m)(1+M_1)(1+M_2) = 1 - m_2'\alpha_2$, $\mathrm{supp}(m_2'\alpha_2) \subseteq P''_{i_2}$, and

$$\beta_2 = \alpha_1\alpha_2 = (1+M_1)(1+M_2) = (1+M_1) + \alpha_1 M_2 = \alpha_1 + \alpha_1 M_2 = \beta_1 + \beta_1 M_2 \in R_{\mathcal{B}}.$$

We prove by transfinite induction that there exists a well-ordered sequence

$$P'_{i_1} \supset P''_{i_1} \supseteq P'_{i_2} \supset P''_{i_2} \supseteq \cdots \supset P'_{i_\rho} \supset P''_{i_\rho} \supseteq P'_{i_{\rho+1}} \supset P''_{i_{\rho+1}} \supseteq \cdots$$

of prime segments of H with $S_{i_\rho} = P'_{i_\rho} \backslash P''_{i_\rho}$ (where $S_{i_1} = U(H) \backslash \{e\}$ is possible) and for all $\rho \geq 1$ an element

(∗) $\beta_\rho = 1 + \sum_{\lambda=1}^{\rho} (\sum_{h_i^{(\lambda)} \in A_\lambda} h_i^{(\lambda)} k_{\lambda,i})$ in $R_{\mathcal{B}}$ for well-ordered subset $A_\lambda \subseteq S_{i_\lambda}$ and elements $0 \neq k_{\lambda,i} \in K$ which do not depend on ρ, with $1 \leq \lambda < \rho$ in this sum for ρ a limit ordinal, so that

(∗∗) $(1-m)\beta_\rho = (1 - \tilde{m}_{\rho+1})$ for $\tilde{m}_{\rho+1} = 0$ or $\mathrm{supp}(\tilde{m}_{\rho+1}) \subseteq P''_{i_\rho}$, or $\mathrm{supp}(\tilde{m}_{\rho+1}) \subseteq P''_{i_\lambda}$ for all $\lambda < \rho$ if ρ is a limit ordinal.

The elements $\alpha_1 = \beta_1$ and $\alpha_1\alpha_2 = \beta_2$ with (∗) and (∗∗) were constructed above. If we assume that β_ρ exists and $\tilde{m}_{\rho+1} = 0$, the process stops. Otherwise let $\tilde{m}_{\rho+1} = m_{\rho+1} + m'_{\rho+1}$ with $h_{\rho+1} = \min \mathrm{supp}(\tilde{m}_{\rho+1}) \in P'_{i_{\rho+1}} \backslash P''_{i_{\rho+1}} = S_{i_{\rho+1}}$ for the prime segment $P'_{i_{\rho+1}} \supset P''_{i_{\rho+1}}$ and $\mathrm{supp}(m_{\rho+1}) \subseteq S_{i_{\rho+1}}, \mathrm{supp}(m'_{\rho+1}) \subseteq P''_{i_{\rho+1}}$, where $P''_{i_\rho} \supseteq P'_{i_{\rho+1}}$.

Then $\alpha_{\rho+1} = \sum_{t=0}^{\infty} m_{\rho+1}^t = 1 + M_{\rho+1} \in R_{\mathcal{B}}$, and

$$\begin{aligned}\beta_{\rho+1} &= \beta_\rho \alpha_{\rho+1} = \beta_\rho(1 + M_{\rho+1}) = \beta_\rho + \beta_\rho M_{\rho+1}\\ &= 1 + \sum_{\lambda=1}^{\rho+1}\left(\sum_{h_i^{(\lambda)} \in A_\lambda} h_i^{(\lambda)} k_{\lambda,i}\right) \in R_{\mathcal{B}}\end{aligned}$$

with $A_{\rho+1} = \mathrm{supp}(\beta_\rho M_{\rho+1}) \subseteq S_{i_{\rho+1}}$ satisfies $(*)$ for $\rho + 1$ instead of ρ.

Similarly, $(**)$ is also satisfied: $(1 - m)\beta_{\rho+1} = 1 - m'_{\rho+1}\alpha_{\rho+1} = 1 - \tilde{m}_{\rho+2}$ with $\tilde{m}_{\rho+2} = m'_{\rho+1}\alpha_{\rho+1} = 0$ or $\mathrm{supp}(\tilde{m}_{\rho+2}) \subseteq P''_{i_{\rho+1}}$.

To define β_σ in the case where σ is a limit ordinal we assume that for $\rho < \sigma$ the element $\beta_\rho \in R_{\mathcal{B}}$ has the form $(*)$ with property $(**)$.

The element β_σ is then defined as

$$\beta_\sigma = \alpha_1 + \sum_{1 \le \rho < \sigma} \beta_\rho M_{\rho+1} = 1 + \sum_{1 \le \lambda < \sigma}\left(\sum_{h_i^{(\lambda)} \in A_\lambda} h_i^{(\lambda)} k_{\lambda,i}\right).$$

This element is in $K\{\{H\}\} = \hat{R}$ since its support is the disjoint union of a well-ordered set of well-ordered sets A_λ of elements in H, and it is in $R_{\mathcal{B}}$ since $L \in \mathcal{L}(\beta_\sigma)$ implies either $P'_{i_\lambda} \supset I_L \supset P''_{i_\lambda}$ and then $L \in \mathcal{L}(\sum_{h_i^{(\lambda)} \in A_\lambda} h_i^{(\lambda)} k_{i,\lambda}) \subseteq \mathcal{B}$ for some $\lambda < \sigma$, or $I_L = P''_{i_\lambda}$ for some $\lambda < \sigma$, or $I_L = \bigcap P''_{i_\lambda}$ for some $\lambda < \sigma$ and $L \in \mathcal{D} \subseteq \mathcal{B}$ in these two cases.

The condition $(**)$ is satisfied for β_σ since for $\lambda < \sigma$ we have $\beta_\sigma = \beta_\lambda + \sum_{(\lambda+1) \le \tau < \sigma}(\sum_{h_i^{(\tau)} \in A_\tau} h_i^{(\tau)} k_{\tau,i})$ and

$$\begin{aligned}(1 - m)\beta_\sigma &= (1 - m)\beta_\lambda + (1 - m)(\beta_\sigma - \beta_\lambda)\\ &= 1 - \tilde{m}_{\lambda+1} + (1 - m)(\beta_\sigma - \beta_\lambda) = 1 - \tilde{m}_{\sigma+1}\end{aligned}$$

with $\tilde{m}_{\sigma+1} = \tilde{m}_{\lambda+1} - (1 - m)(\beta_\sigma - \beta_\lambda) \in P''_{i_\lambda}$ for all $\lambda < \sigma$.

It follows that there exists an ordinal number $\tilde{\rho}$ with $\tilde{m}_{\tilde{\rho}+1} = 0$ and $(1-m)\beta_{\tilde{\rho}} = 1$ for $\beta_{\tilde{\rho}} \in R_{\mathcal{B}}$. This proves part of the following main result in the right invariant case.

Theorem 5.4. *Let K be a skew field and let H be a right invariant right cone of an ordered group (G, Π) with $H \subseteq \Pi_P, U(H) = U(\Pi_P) \cap H$ for a prime ideal P of Π. Let $\mathcal{B}$ be a subset of $\mathcal{L}(H)$ that satisfies the following conditions*:

(1) $L_1, L_2 \in \mathcal{B}$ *implies* $L_1 L_2 \in \mathcal{B}$.
(2) $L \in \mathcal{B}, g \in H$ *implies* $\langle gL\rangle \in \mathcal{B}$ *and* $\langle LgH\rangle \in \mathcal{B}$ *if* $\langle LgH\rangle \in \mathcal{L}(H)$.
(3) *If* $L \in \mathcal{B}$ *and* $gH \in L$, *then* $g^{-1}L = \{hH \in W(H) \mid ghH \in L\} \in \mathcal{B}$.
(4) $\mathcal{D} \subseteq \mathcal{B}$.

Then:

(i) $R_{\mathcal{B}} = \{\alpha \in K\{\{H\}\} \mid \mathcal{L}(\alpha) \subseteq \mathcal{B}\}$ *is a right invariant right chain domain and* H *is associated with* $R_{\mathcal{B}}$.

(ii) *$R_{\mathcal{B}}$ is I-compact for every right ideal I with $L_{I\cap H} \in \mathcal{B}$ or of the form $I = cJ(R_{\mathcal{B}})$ with $0 \neq c \in R_{\mathcal{B}}$.*

(iii) *$R_{\mathcal{B}}$ is not I'-compact for any right ideal I' of $R_{\mathcal{B}}$ not listed in (ii) if every prime ideal P of H without a lower neighbor in the lattice of prime ideals satisfies $P = \bigcup_{i\in\mathbb{N}} P_i$ for $P_i \subset P$ completely prime in H.*

(iv) *$\hat{R} = K\{\{H\}\}$ is a maximal immediate extension of $R_{\mathcal{B}}$.*

Proof. (i) The conditions (1) and (2) imply by Corollary 5.3 that $R_{\mathcal{B}}$ is a subring of $\hat{R} = K\{\{H\}\}$. Since the conditions (3) and (4) on $\mathcal{B}$ imply conditions (b-i) and (b-ii) in Lemma 4.3 it follows that H is associated with $R_{\mathcal{B}}$; hence $R_{\mathcal{B}}$ is a right invariant right chain domain.

(ii) We use condition (c) in Lemma 1.3. Let (a_ρ) be a pseudo-convergent sequence in $R_{\mathcal{B}}$ with breadth I and $L_{(I\cap H)} \in \mathcal{B}$. Then $v(a_\sigma - a_\rho) = h_\rho R_{\mathcal{B}}$ for $\sigma > \rho$ and $h_\rho \in H$ with $\langle\{h_\rho H\}\rangle = L \in \mathcal{B}$. It follows that the coefficients of h for $hH < h_\rho H$ in the power series representations of a_σ and a_ρ in $\hat{R}$ agree, but disagree for $h_\rho u$, for some $u \in U(H)$. We define an element $\alpha \in \hat{R}$ as $\alpha = \sum hk_h$ if k_h is the coefficient in K for h in the representation of a_ρ for some ρ and $hH \supset h_\rho H$. We claim that α is a limit of (a_ρ) in $R_{\mathcal{B}}$.

To show that α is in $\hat{R}$, it must be proved that $\mathrm{supp}(\alpha)$ is well-ordered in G if $\alpha \neq 0$. If $\emptyset \neq S \subseteq \mathrm{supp}(\alpha)$, then S contains an element $h \in H$ and $hH \supset h_\rho H$ for some ρ and $h \in \mathrm{supp}(a_\rho)$. Any element $g \in \mathrm{supp}(\alpha)$ with $g < h$ is also contained in the well-ordered set $\mathrm{supp}(\alpha_\rho)$ and S has therefore a smallest element, α is in $\hat{R}$.

To prove that α is in $R_{\mathcal{B}}$ we show that $\mathcal{L}(\alpha) \subseteq \bigcup \mathcal{L}(a_\rho) \cup \{L\}$. If $L' \in \mathcal{L}(\alpha)$, then either $L' = L \in \mathcal{B}$ and we are done, or $L' \subset L$, or $L \subset L'$. The last case cannot occur, since otherwise there exists $h'H \in L' \backslash L$ and $g \in \mathrm{supp}(\alpha)$ with $h_\rho H < h'H \leq gH$ for all ρ, a contradiction.

If $L' \subset L$, there exists $hH \in L\backslash L'$ and $h_\rho H \in L$ with $hH < h_\rho H$. It follows that $L' \in \mathcal{L}(a_\rho)$ and $\mathcal{L}(\alpha) \subseteq \mathcal{B}, \alpha \in R_{\mathcal{B}}$. The element α is a limit of the sequence (a_ρ) since $v(\alpha - a_\rho) = h_\rho R_{\mathcal{B}} = v(a_\sigma - a_\rho)$ for $\sigma > \rho$ and all ρ; the ring $R_{\mathcal{B}}$ is I-compact for the right ideal I of $R_{\mathcal{B}}$ with $L_{(I\cap H)} \in \mathcal{B}$.

(iii) Let $I' \neq R_{\mathcal{B}}$ be a right ideal of $R_{\mathcal{B}}$ not listed in (ii), and let $I_1 = I' \cap H$ with $P_r(I_1) = P_1$, and hence $P_r(I') = P_1 R_{\mathcal{B}}$ by Proposition 3.3. Hence, I_1 is not distinguished, and L_{I_1} is a P_1-limit set in $\mathcal{L}(H)$ by Theorem 2.6 and Corollary 3.6. We consider first the case where H contains a prime segment $P_1 \supset P_2$. Then, there exists a right ideal I_2 of H with $P_1 \supset I_2 \supset P_2$ and an element $t \in H$ with $t \cdot I_2 = I_1$ (see [9, 2.11]). The right ideal I_2 of H is not distinguished and $P_r(I_1) = P_r(I_2) = P_1$. By [9, Theorem 4.6], the P_1-limit set L_{I_2} is of the form $L_{I_2} = \{hH \mid \phi(hH_{P_1}\backslash P_2 H_{P_1}) < \rho\}$ for a real number $\rho > 0$ and the mapping ϕ from $W(H')$ to $(\mathbb{R}^+, +, \leq)$ with $H' = H_{P_1}\backslash P_2 H_{P_1}$, where H' is a rank-one right invariant right cone.

Hence, there exists a countable sequence of elements $h_i H \in L_{I_2}$ with $h_1 H < h_2 H < \cdots$ and $L_{I_2} = \langle\{h_i H \mid i \in \mathbb{N}\}\rangle$.

The elements $a_1 = h_1, a_2 = h_1 + h_2, \ldots, a_n = h_1 + \cdots + h_n$, form a pseudo-convergent sequence in $R_{\mathcal{B}}$ with breadth $I_2 R_{\mathcal{B}}$ and $L_{I_2} \in \mathcal{L}(\alpha)$ for any limit α of the sequence (a_n).

To see this, it is only necessary to observe that $v(\alpha - h_1) = h_2 R_{\mathcal{B}}$, hence $h_1 \in \text{supp}(\alpha)$; similarly $v(\alpha - (h_1 + h_2)) = h_3 R_{\mathcal{B}}$ and $h_1, h_2 \in \text{supp}(\alpha)$, more general, $h_n \in \text{supp}(\alpha)$ for all n follows: $L_{I_2} \in \mathcal{L}(\alpha)$. However, since $L_{I_1} \notin \mathcal{B}$, it follows that $L_{I_2} \notin \mathcal{B}, \alpha \notin R_{\mathcal{B}}$ and $R_{\mathcal{B}}$ is not I'-compact.

We use a similar argument in the case where $P_1 = \bigcup_{i \in \mathbb{N}} Q_i$ for completely prime ideals $Q_i \subset P_1, i = 1, 2, \ldots$. It then follows from [9, Corollary 3.5], that $I_1 = \bigcap_{i \in \mathbb{N}} h_i H, h_i H < h_j H$ for $i < j$, since $P_r(I_1) = P_1$.

Then $a_1 = h_1, a_2 = h_1 + h_2, \ldots, a_n = h_1 + h_2 + \cdots + h_n, \ldots$, forms a pseudo-convergent sequence in $R_{\mathcal{B}}$ so that any limit α of (a_n) in $R_{\mathcal{B}}$ must satisfy $\text{supp}(\alpha) \supseteq \{h_1, h_2, \ldots\}$. Hence, $L_{I_1} \in \mathcal{L}(\alpha)$ and $\alpha \notin R_{\mathcal{B}}$ since $L_{I_1} \notin \mathcal{B}$. This proves (iii).

(iv) We observe that $\hat{R} = K\{\{H\}\} = R_{\mathcal{L}(H)}$ and that $\hat{R}$ is therefore I-compact for all proper right ideals I of R and hence maximal (see [4, Theorem 4.12]). Since $\varphi(a + J(R_{\mathcal{B}})) = a + J(\hat{R}), a \in R_{\mathcal{B}}$, defines an isomorphism from $R_{\mathcal{B}}/J(R_{\mathcal{B}})$ onto $\hat{R}/J(\hat{R}) \cong K\{\{U(H)\}\}$, and since H is associated with both $R_{\mathcal{B}}$ and $\hat{R}$, it follows that $\hat{R}$ is an immediate extension of $R_{\mathcal{B}}$. □

We prove in the next result that the assumptions in Theorem 5.4 are satisfied for certain particular subset $\mathcal{B}$ of $\mathcal{L}(H)$.

Proposition 5.5. *Let H be a right invariant right cone in a group G and let $\mathcal{D} = \{hL_P \mid h \in H, P \neq J(H)$ completely prime ideal of H or $P = \emptyset\}$ be the set of distinguished limit sets of H. Then $\mathcal{D}$ satisfies the conditions* (1)–(4) *in Theorem* 5.4.

Proof. We first prove that $\mathcal{D}$ satisfies condition (2) and $\langle tL \rangle \in \mathcal{D}$ for $t \in H, L \in \mathcal{D}$ follows from the definition. It can happen that for $L = L_P$ and $t \in H$ the set $LtH = \{atH \mid aH \supset P\}$ has a last element dH and LtH is not a limit set of H.

Otherwise we claim that $\langle L_P tH \rangle = \langle t \cdot L_{P'} \rangle$ for $L' = \langle \{a'H \mid at = ta'$ for $aH \supset P\} \rangle = L_{P'}$ for $P' = \emptyset$ or P' a completely prime ideal of H. This follows since for $cH, dH \in L'$ there exist $aH \supset P, bH \supset P$ with $cH \supseteq a'H, dH \supseteq b'H$ and $cdH \supseteq a'b'H$ where $at = ta', bt = tb'$. Hence $cH \cdot dH \in L'$ and $L' = L_{P'}$ for $P' = \emptyset$ or some completely prime ideal P' of H. This proves that condition (2) is satisfied by $\mathcal{D}$.

If $\langle t_1 L_{P_1} \rangle, \langle t_2 L_{P_2} \rangle$ are elements in $\mathcal{D}$ and $L_{P_1} t_2 H$ has a last element dH, then

$$\langle t_1 L_{P_1} \rangle \langle t_2 L_{P_2} \rangle = \langle t_1 d L_{P_2} \rangle$$

is in $\mathcal{D}$.

Otherwise it was shown above that $\langle L_{P_1} t_2 \rangle = \langle t_2 L_{P_1'} \rangle$ for some completely prime ideal P_1' of H and

$$L_{P_1'} L_{P_2} = \begin{cases} L_{P_2} & \text{if } P_1' \supseteq P_2, \\ L_{P_1'} & \text{if } P_1' \subset P_2. \end{cases}$$

Hence $\langle t_1 L_{P_1}\rangle\langle t_2 L_{P_2}\rangle \in \mathcal{D}$ in all cases which proves (1) for $\mathcal{D}$.

To prove that condition (3) holds for $\mathcal{D}$ we have to consider $\langle a^{-1}\langle tL_P\rangle\rangle$ for $tL_P \in \mathcal{D}$ and $a \in H$ with $aH \in tL_P$. If $t = ad$, then $\langle a^{-1}\langle tL_P\rangle\rangle = \langle dL_P\rangle \in \mathcal{D}$, and $a = tb$ for $b \in H$ otherwise. Then $\langle a^{-1}tL_P\rangle = \langle b^{-1}L_P\rangle$ which equals $L_P \in \mathcal{D}$ since $b \notin P$. This shows that $\mathcal{D}$ satisfies condition (3) and proves the proposition. □

For the next result we recall the notation given after Definition 4.5 in [9].

Let H be a right invariant right cone in a group G with $H \neq G$. Let $\mathrm{spec}(H) = \{P \mid P$ completely prime ideal in H or $P = \emptyset\}$ be the chain of completely prime ideals of H, including the empty set $\emptyset$, and let $\mathrm{spec}_0(H)$ be the subset of $\mathrm{spec}(H)$ consisting of those prime ideals P of H which have a lower neighbor P'' in $\mathrm{spec}(H)$; hence, $P \in \mathrm{spec}_0(H)$ if there exists $P'' \in \mathrm{spec}(H)$ with $P = P' \supset P''$ a prime segment of H. In that case it follows that $H_P^0 = H_P \backslash P'' H_P$ is a right invariant right cone of rank-one and a monomorphism ϕ_P exists from $\overline{H_P^0} = (\{aH_P^0 \mid a \in H_P^0\}, \cdot)$ into $(\mathbb{R}^+, +)$, the semigroup of non-negative real numbers under addition. We write $L_{0,P} = L_P$ for $J \neq P \in \mathrm{spec}(H)$ and $L_{\rho,P} = \{aH \in W(H) \mid \phi_P(aH_P^0) < \rho\}$ for $P \in \mathrm{spec}_0(H)$ with $P = P^2$ and $0 < \rho \in \mathbb{R}$.

Proposition 5.6. *Let $H \neq G$ be a right invariant right cone in G. Then*

$$\mathcal{B} = \mathcal{D} \cup \{\langle tL_{\rho,P}\rangle \mid t \in H, P \in \mathrm{spec}_0(H), P = P^2, 0 < \rho \in \phi_P(\overline{H_P^0})\}$$

satisfies the conditions (1)–(4).

Proof. We note that $\mathcal{D} = \{\langle tL_{0,P}\rangle \mid t \in H, J \neq P \in \mathrm{spec}(H)\}$ and show first that condition (2) holds for $\mathcal{B}$.

If $L \in \mathcal{B}$ and $t' \in H$, then $\langle t'L\rangle \in \mathcal{B}$ by definition of $\mathcal{B}$. By [9, Proposition 5.6] it follows that $\langle Lt'H\rangle$ is either equal to $\langle t'H\rangle$ and not a limit set or $\langle t'H\rangle = \langle t'L'\rangle$ for $L' = L_{0,P'} \in \mathcal{D}$ if $L \in \mathcal{D}$, and $L' = L_{\rho',P'} \in \{\langle tL_{\rho,P}\rangle \mid t \in H, P \in \mathrm{spec}_0(H), P = P^2, 0 < \rho \in \phi_P(\overline{H_P^0})\}$ if L is contained in this set. This proves condition (2) for $\mathcal{B}$.

That condition (1) holds for $\mathcal{B}$ follows from Proposition 5.6 combined with Proposition 4.7 in [9] where it is shown that

$$L_{\rho_1,P_1} L_{\rho_2,P_2} = \begin{cases} L_{\rho_1+\rho_2,P} & \text{for } P_1 = P_2, \\ L_{\rho_1,P_1} & \text{for } P_1 \subset P_2, \\ L_{\rho_2,P_2} & \text{for } P_1 \supset P_2, \end{cases}$$

with $J \neq P_i \in \mathrm{spec}(H)$ if $\rho_i = 0$ and $P_i^2 = P_i \in \mathrm{spec}_0(H)$ if $0 < \rho_i \in \mathbb{R}$.

Since condition (4) is satisfied for $\mathcal{B}$ by definition, it remains to show that condition (3) holds. If $L \in \mathcal{D}$ we saw in Proposition 5.5 above that $\langle a^{-1}L\rangle \in \mathcal{D}$ for $aH \in L$. If $L = \langle tL_{\rho,P}\rangle \in \mathcal{B}$ with $P = P^2 \in \mathrm{spec}_0(H), 0 < \rho \in \phi_P(\overline{H_p^0})$, and $aH \in L$, then either $t = ad$ or $a = tb$ for some $d, b \in H$. In the first case we have $\langle a^{-1}L\rangle = \langle dL_{\rho,P}\rangle \in \mathcal{B}$ and we are done.

In the second case $a = tb$ it follows that $bH \in L_{\rho,P}$, hence $\phi_P(bH_p^0) < \rho$ and $\langle a^{-1}L\rangle = L_{\rho-\phi_P(bH_p^0),P}$. However, $\rho = \phi_P(hH_P^0)$ for some $h \in H$, since $L \in \mathcal{B}$, then

$bH \supset hH, h = bd$ for some $d \in H$ and $\langle a^{-1}L \rangle = L_{\phi_P(dH_P^0),P} = L_{\rho',P} \in \mathcal{B}$. This shows that condition (3) holds for $\mathcal{B}$ and proves the proposition. □

6. Examples

Example 6.1. Let $H = (\mathbb{R}^+, +)$ be the semigroup of non-negative real numbers under addition and let $\hat{R} = K\{\{H\}\}$ be the generalized power series ring of H over a commutative field K. Then $\mathcal{D} = \{W(H)\} = \{L_\phi\}$ with $W(H) = \{a + H \,|\, a \in H\}$ and, applying Proposition 5.5, $R_1 = R_{\mathcal{D}} = \{\alpha \in \hat{R} \,|\, \mathcal{L}(a) \subseteq \{L_\phi\}\}$ is a rank-one valuation ring that is (0)-compact, but not compact for any other ideal $\neq rJ(R_1)$ for $0 \neq r \in R_1$.

The ideals for any valuation ring $R \subseteq \hat{R}$ associated with H are $R, (0)$, $\{cJ(R) \,|\, c \in H\}$ and $\{I_c = \bigcap_{c > a \in H} aR = \bigcap_{a+H \in L_c} aR = cR \,|\, 0 < c \in H = \mathbb{R}^+\}$ where $L_c = \{a + H \,|\, \phi(a + H) = a < c\} \in L(H)$.

If $\mathcal{D} \subset \mathcal{B} \subseteq \mathcal{L}(H)$, then $\mathcal{B}$ contains a limit set L_c for some $0 < c \in \mathbb{R}$, and hence $L_{c'} \in B$ for all $0 < c' \in \mathbb{R}$, since either $c' = c + c''$ for $c'' \in H$ or $c = c' + \tilde{c}$ for $\tilde{c} \in H$, and $L_{c'} \in \mathcal{B}$ follows by conditions (2) and (3) that hold for $\mathcal{B}$.

It follows that for $H = (\mathbb{R}^+, +)$ and $R \subseteq \hat{R}$ associated with H, there are exactly two possibilities for $\mathcal{B}$: Either $\mathcal{B} = \mathcal{B}_1 = \{L_\phi\}$ or $\mathcal{B} = \mathcal{B}_2 = \{L_\phi\} \cup \{L_c \,|\, 0 < c \in \mathbb{R}\} = \mathcal{L}(H)$. Then $R_2 = R_{\mathcal{B}_2} = \hat{R}$ is I-compact for all its proper ideals.

Example 6.2. We now consider the case where H is non-discrete, rank-one with $H \subset \mathbb{R}^+$ and want to prove that there are infinitely many distinct subsets $\mathcal{B} \subseteq \mathcal{L}(H)$ satisfying the defining conditions (1)–(4) in Theorem 5.4. Hence, there are infinitely many distinct valuation rings $R_{\mathcal{B}} \subseteq \hat{R} = K\{\{H\}\}$ associated with H and I-compact exactly for $I = (0)$ and the set $\{I \,|\, L_{I \cap H} = L_\rho \in B\}$ of ideals of $R_{\mathcal{B}}$ besides the trivially I-compact ideals $aJ(R_{\mathcal{B}}), a \in H$; $\hat{R}$ is defined as in Example 6.1. It follows from [9, Theorem 4.6] that $\mathcal{L}(H) = \{W(H) = L_\phi \cup L_\rho \,|\, \rho \in \mathbb{R}\}$ where $L_\rho = \{a + H \,|\, \phi(a + H) = a < \rho$ for $a \in H$ and with $0 < \rho \in \mathbb{R}\}$.

For any subset $\mathcal{B} \subseteq L(H)$ that satisfies the conditions (1)–(4) in Theorem 5.4 we define the subset $T(\mathcal{B}) = \{\rho \in \mathbb{R} \,|\, L_\rho \in \mathcal{B}\}$ of $\mathbb{R}^+ \backslash \{0\}$ which satisfies the following two conditions:

(a) $\rho_1, \rho_2 \in T(\mathcal{B})$ implies $\rho_1 + \rho_2 \in T(\mathcal{B})$.
(b) $\rho \in T(\mathcal{B}), q \in H$ implies $\rho + q \in T(\mathcal{B})$, and $\rho - q \in T(\mathcal{B})$ if in addition $\rho > q$.

Conversely, any subset $T \subseteq \mathbb{R}^+ \backslash \{0\}$ that satisfies the conditions (a) and (b) defines a subset $\mathcal{B}$ of $L(H)$ that satisfies conditions (1)–(4) if $\mathcal{B} = \{L_\phi\} \cup \{L_\rho \,|\, \rho \in T\}$ (see [9, Proposition 4.7].

We want to prove that there are infinitely many such subsets T of $\mathbb{R}^+ \backslash \{0\}$ satisfying conditions (a) and (b).

Let $\tilde{H} = H \cup -H$ be the subgroup of $(\mathbb{R}, +)$ generated by H. For each $0 < \rho \in \mathbb{R}$ we consider the coset $\rho + \tilde{H}$ of $\tilde{H}$ in $\mathbb{R}$ and define $D_\rho = (\rho + \tilde{H}) \cap (\mathbb{R}^+ \backslash \{0\}) = \{\rho + q \,|\, q \in H\} \cup \{\rho - q \,|\, \rho > q \in H\}$.

The element $\rho + \tilde{H}$ has either infinite order in $\mathbb{R}/\tilde{H}$, in which case all the sets $D_\rho, D_{2\rho}, \ldots, D_{n\rho}, \ldots, n \in \mathbb{N}$, are mutually disjoint, or $\rho + \tilde{H}$ has finite order n_ρ in $R/\tilde{H}$ in which case the sets $D_\rho, D_{2\rho}, \ldots, D_{n_\rho \rho} = H\backslash\{0\}$ are mutually disjoint.

Any subset T of $\mathbb{R}^+\backslash\{0\}$ that contains $0 < \rho \in \mathbb{R}$ and satisfies condition (b) also contains D_ρ. If T also satisfies condition (a) and contains D_{ρ_1} and D_{ρ_2}, then T contains $D_{\rho_1+\rho_2}$ for $0 < \rho_1, \rho_2 \in \mathbb{R}$.

If $H \subset \mathbb{R}^+$ and $\rho + \tilde{H}$ has infinite order in $\mathbb{R}/\tilde{H}$ for $0 < \rho \in \mathbb{R}$, then $T_p = \bigcup_{n\in\mathbb{N}} D_{np\rho}$ satisfies conditions (a) and (b) for a prime $p \in \mathbb{N}$. The sets T_p will be distinct for distinct primes and have empty intersection with H.

They define infinitely many distinct sets $\mathcal{B}_p = \{L_\phi\} \cup \{L_\sigma \mid \sigma \in T_p\} \subseteq \mathcal{L}(H)$ and hence infinitely many distinct rank-one valuation rings $R_p = R_{\mathcal{B}_p} \subseteq \hat{R}$, associated with H and I-compact for $I = (0)$ and the ideals in the set $\{I \mid L_{I\cap H} = L_\sigma$ for some $\sigma \in T_p\}$; these rings will not be I-compact for any principal ideal $I \neq (0)$.

If the order of every element $\rho + \tilde{H}$ in $\mathbb{R}/\tilde{H}$ is finite, say n_ρ, then the set of these orders cannot be finite. Otherwise, there is a finite set $n_{\rho_1}, n_{\rho_2}, \ldots, n_{\rho_t} \in \mathbb{N}$ so that $n(\rho + \tilde{H}) = n\rho + \tilde{H} = \tilde{H}$ for $n = n_{\rho_1} \cdots n_{\rho_t}$ and every $0 < \rho$ in $\mathbb{R}$. The contradiction $n\mathbb{R} = \mathbb{R} \subseteq \tilde{H} \subset \mathbb{R}$ shows that there is an infinite list $\rho_1, \ldots, \rho_k, \ldots$ of elements in $\mathbb{R}^+\backslash\{0\}$ so that $\rho_k + \tilde{H}$ in $\mathbb{R}/\tilde{H}$ has order n_k and $n_1 < n_2 < n_3 < \cdots$. It follows that the sets ${}_kT = \bigcup_{j=1}^{n_k} D_{j\rho_k}$ are distinct subsets in $\mathbb{R}^+\backslash\{0\}$ that contain $H\backslash\{0\}$ and satisfy conditions (a) and (b) for all $k \in \mathbb{N}$.

The corresponding rank-one valuation rings ${}_kR = R_{{}_k\mathcal{B}} \subseteq \hat{R}$ with ${}_k\mathcal{B} = \{L_\phi\} \cup \{L_\rho \mid \rho \in {}_kT\}$ will be I-compact for all principal ideals of ${}_kR$ and in addition for some non-principal ideals $I \neq cJ({}_kR), c \in H$, if ${}_kT \supset H\backslash\{0\}$.

Let H be a right cone in a group G as in Proposition 4.1 of rank-one. Then there exists a monomorphism ϕ from the semigroup $\bar{H} = (\{aH \mid a \in H\}, \circ)$ with $aH \circ bH = abH$ as operation, into $(\mathbb{R}^+, +)$ and the arguments above can be used to prove the following result.

Lemma 6.3. *Let H be a right cone in the group G of rank-one as in Proposition 4.1. If $\phi(\bar{H}) \subseteq \mathbb{R}^+$ has a smallest positive element, then $\mathcal{B} = \{\{aH \mid a \in H\}\} = \{W(H)\} = \mathcal{L}(H)$ is the only possible set $\mathcal{B}$. If $\phi(\bar{H}) = \mathbb{R}^+$, then there exist exactly two such sets $\mathcal{B}$, satisfying conditions* (1)–(4)*: $\mathcal{B}_1 = \mathcal{D} = \{W(H)\}$ and $\mathcal{B}_2 = \mathcal{L}(H)$. If H is not discrete and $\phi(\bar{H}) \subset \mathbb{R}^+$, then there exist infinitely many such subsets $\mathcal{B}$ in $\mathcal{L}(H)$.*

By Theorem 5.4 there exists for each $\mathcal{B}$ in Lemma 6.3 the right invariant rank-one right chain domain $R_\mathcal{B}$ which is I-compact exactly for the ideals I with $L_{I\cap H} \in \mathcal{B}$ besides the ideals of the form $cJ(R_\mathcal{B})$ for $0 \neq c \in R_\mathcal{B}$.

Example 6.4. For $H = \mathbb{Q}^+ \subset \mathbb{R}^+$ we choose $\rho = \sqrt{2}$ and obtain infinitely many examples for $\mathcal{B}$ in $\mathcal{L}(H)$ with $\mathcal{B}_p = \{W(H)\} \cup \{L_\sigma \mid \sigma \in T_p\}$ and $T_p = \{q + pn\sqrt{2} \mid q + pn\sqrt{2} > 0, q \in \mathbb{Q}, n \in \mathbb{N}\}$ and p a prime in $\mathbb{N}$.

Example 6.5. For $H = \{\frac{a}{2^n} \mid 0 \leq a \in \mathbb{Z}, n \in \mathbb{N}\}$ and $\rho = \frac{1}{3}$ we have $n_{\frac{1}{3}} = 3$ and the sets $D_{\frac{1}{3}}, D_{\frac{2}{3}}, D_1 = H\backslash\{0\}$.

In the next example we consider right cones H as in Proposition 4.1, but of arbitrary rank.

Example 6.6. Let H be a right cone in the ordered group G as in Proposition 4.1 with H right invariant and $\mathcal{B}$ be a subset of $\mathcal{L}(H)$ that satisfies the conditions (1)–(4) in Theorem 5.4. Then, for $P \in \operatorname{spec}_0(H)$, the set

$$T_P(\mathcal{B}) = \{0 < \rho \in \mathbb{R} \mid L_{\rho,P} \in \mathcal{B}\} \subseteq \mathbb{R}^+ \backslash \{0\}$$

satifies the following conditions:

(a) $\rho_1, \rho_2 \in T_P(\mathcal{B})$ implies $\rho_1 + \rho_2 \in T_P(\mathcal{B})$.
(b) $\rho \in T_P(\mathcal{B}), q \in \phi_P(\bar{H}_P^0)$ implies $\rho + q \in T_P(\mathcal{B})$, and $\rho - q \in T_P(\mathcal{B})$ if $\rho > q$.
(c) $L_{\hat{\rho},\hat{P}} \in \mathcal{B}, t \in H$ with $\langle L_{\hat{\rho},\hat{P}} tH\rangle = \langle tL_{\rho,P}\rangle$ implies $\rho \in T_P(\mathcal{B})$.

The notation and the definition of $L_{\rho,P}, 0 < \rho \in \mathbb{R}, P = P^2 \in \operatorname{spec}_0(H)$ were recalled before Proposition 5.6; we also use Theorem 4.6, Propositions 4.7 and 5.6 in [9].

Lemma 6.7. *Let H be as above. Assume that for every completely prime ideal $P \in \operatorname{spec}_0(H)$ with $P = P^2$ a subset $T_P(\mathcal{B})$ of $\mathbb{R}^+ \backslash \{0\}$ is given so that the conditions* (a), (b), *and* (c) *above are satisfied. Then*

$$\mathcal{B} = \mathcal{D} \cup \bigcup_{P=P^2 \in \operatorname{spec}_0(H)} \{\langle tL_{\rho,P}\rangle \mid t \in H, \rho \in T_P(\mathcal{B})\}$$

satisfies the conditions (1)–(4) *in Theorem* 5.4.

Proof. Condition (4), $\mathcal{D} \subseteq \mathcal{B}$, is satisfied by definition of $\mathcal{B}$. To prove that condition (1) holds for $\mathcal{B}$, let $L_1 = \langle t_1 L_{\rho_1,P_1}\rangle$ and $L_2 = \langle t_2 L_{\rho_2,P_2}\rangle \in \mathcal{B}$. Here, we assume either $\rho_1 \in T_{P_1}(\mathcal{B}), \rho_2 \in T_{P_2}(\mathcal{B})$ or $\rho_1 = 0$ or $\rho_2 = 0$ with $L_{0,P} = L_P = \{aH \mid a \in H, aH \supset P\}$ for a completely prime ideal $P \neq J(H)$ of H, with $P = \emptyset$ and $L_\emptyset = W(H)$ possible.

Then $\langle t_1 L_{\rho_1,P_1}\rangle\langle t_2 L_{\rho_2,P_2}\rangle = \langle t_1 t_2 L_{\hat{\rho}_1,\hat{P}_1} L_{\rho_2,P_2}\rangle$ with $\hat{\rho}_1 = 0$ if $\rho_1 = 0$, and $\hat{\rho}_1 \in T_{\hat{P}_1}(\mathcal{B})$ if $\rho_1 > 0$ by condition (c) above, unless $\langle L_{\rho_1,P_1} t_2 H\rangle = \{t_2 H\}$ is not a limit set; see [9, Proposition 5.6]. In this last case we have $L_1 \cdot L_2 = t_1 t_2 L_{\rho_2,P_2} \in \mathcal{B}$. In the first case we have $L_{\hat{\rho}_1,\hat{P}_1} L_{\rho_2,P_2} \in \mathcal{B}$ as in the proof of Proposition 5.6 since in case $\hat{P}_1 = P_2$ we have $\hat{\rho}_1 + \rho_2 \in T_{P_2}(\mathcal{B})$ by conditions (a) and (c). This shows that condition (1) holds for $\mathcal{B}$.

The first part of condition (2), $L \in \mathcal{B}, g \in H$ implies $\langle gL\rangle \in \mathcal{B}$, is true by definition of $\mathcal{B}$, and the second part, $L \in \mathcal{B}, g \in H$, then $\langle LgH\rangle \in \mathcal{L}(H)$ implies $\langle LgH\rangle \in \mathcal{B}$ was proved in the proof of condition (1).

To prove that condition (3) holds for $\mathcal{B}$, we observe that $L \in \mathcal{D}, gH \in L$ implies $\langle g^{-1}L\rangle \in \mathcal{D}$ as it was shown in the proof of Proposition 5.5.

Let $L = \langle tL_{\rho,P}\rangle \in \mathcal{B}$ for $P = P^2 \in \operatorname{spec}_0, \rho \in T_P(\mathcal{B})$ and $t \in H$. If $gH \in L, g \in H$, then either $t = gt'$ for some $t' \in H$ and $\langle g^{-1}L\rangle = \langle t' L_{\rho,P}\rangle \in \mathcal{B}$, or $g = tg'$ for some $g' \in H$. In this last case we have $\langle g^{-1}L\rangle = \langle g'^{-1} L_{\rho,P}\rangle = L_{\rho-\phi_P(g' H_P^0),P} \in \mathcal{B}$, since $\rho \in T_P(\mathcal{B})$ and $\rho > q = \phi_P(g'H_P^0)$, using condition (b). This proves Lemma 6.7. □

If we choose $T_P(\mathcal{B})$ to be the empty set for every $P = P^2 \in \operatorname{spec}_0(H)$, then we obtain $\mathcal{B} = \mathcal{D}$ in Lemma 6.7, the case considered in Proposition 5.5. It follows from Proposition 5.6 that the sets $T_P(\mathcal{B}) = \Phi_P(\overline{H_P^0}\backslash\{0\}) \subseteq \mathbb{R}\backslash\{0\}$ also satisfy the conditions (a), (b), and (c). If the maximum condition holds for the prime ideals of H, then $T_P(\mathcal{B}) = \mathbb{R}^+\backslash\{0\}$ for all $P = P^2 \in \operatorname{spec}_0(H)$ leads to $\mathcal{B} = \mathcal{L}(H)$. If H is commutative then condition (c) is satisfied by any subset of $\mathbb{R}^+$.

We saw in Example 6.2 that in the case where H has rank-one with $P = J(H) = P^2$ the conditions (a) and (b) are sufficient to define $T_P(\mathcal{B})$ and the corresponding $\mathcal{B} = \{W(H)\} \cup \{L_{\rho,P} \mid \rho \in T_P(\mathcal{B})\}$. We show in the next example that this is no longer true in the rank two case.

Example 6.8. Let $H = \{(k,a) \mid a, 1 \leq k \in \mathbb{Q}$ with $a \geq 0$ if $k = 1\}$ be the cone of the ordered group $G = \{(k,a) \mid a, 0 < k \in \mathbb{Q}\}$ with $(k,a)(k',a') = (kk', ka' + a)$ defining the operation. Then H has rank 2 with $P_1 = J(H) = H\backslash\{(1,0)\}$ and $P_2 = \{(k,a) \in H \mid k > 1\}$ the completely prime ideals of H. We have $H = H_{P_1}$ and $H_{P_1}^0 = H_{P_1}\backslash P_2 = \{(1,a) \mid 0 \leq a \in \mathbb{Q}\}$ with $\Phi_{P_1}((1,a)H_{P_1}^0) = a \in \mathbb{Q}^+$.

Since $(k,b)^{-1} = (k^{-1}, -k^{-1}b)$ for $(k,b) \in H$ and $(k^{-1}, -k^{-1}b)(1,a)(k,b) = (1, k^{-1}a)$, it follows that $\langle L_{\rho,P_1}(k,b)H\rangle = \langle (k,b)L_{k^{-1}\rho,P_1}\rangle$ for $0 < \rho \in \mathbb{R}$ and $L_{\rho,P_1} \in \mathcal{L}(H)$.

If, for example, we consider the smallest subset $T_{P_1}(\mathcal{B})$ of $\mathbb{R}^+\backslash\{0\}$ that contains $\sqrt{2}$ and satisfies conditions (a), (b), and (c) we obtain

$$T_{P_1}(\mathcal{B}) = \{q + q'\sqrt{2} \mid q, 0 < q' \in \mathbb{Q}, q + q'\sqrt{2} > 0\}.$$

We see that the subset $\{q + n\sqrt{2} \mid q \in \mathbb{Q}, n \in \mathbb{N}, q + n\sqrt{2} > 0\}$ satisfies conditions (a) and (b) as in Example 6.2, but not condition (c); it is not a suitable candidate for a set $T_{P_1}(\mathcal{B})$.

The localization $H_{P_2} = H \cup \{(1,a) \mid 0 \leq a \in \mathbb{Q}\}^{-1}$ is equal to $H_{P_2}^0$ and $\overline{H_{P_2}^0} = \{(k,a)H_{P_2} = (k,0)H_{P_2} \mid 1 \leq k \in \mathbb{Q}\}$ with $\Phi_{P_2}[(k,0)H_{P_2}] = \ln k \in \mathbb{R}^+$.

We have $(k,0)H_{P_2}(k_1,a_1) = (k_1,a_1)(k,0)H_{P_2}$ for $(k,0),(k_1,a_1) \in H$ and hence $\langle L_{\rho,P_2}(k_1,a_1)H\rangle = \langle (k_1,a_1)L_{\rho,P_2}\rangle$ for $0 < \rho \in \mathbb{R}^+$. The set $T_{P_2}(\mathcal{B})$ therefore only has to satisfy conditions (a) and (b) for $\mathcal{B}$ to satisfy conditions (1)–(4).

There are infinitely many choices for $T_{P_1}(\mathcal{B})$ as well as for $T_{P_2}(\mathcal{B})$ which can be paired arbitrarily to obtain infinitely many choices for subsets $\mathcal{B} \subseteq \mathcal{L}(H)$ that satisfy conditions (1)–(4) in Theorem 5.4.

In Example 6.9 we consider a class of non-commutative examples of right invariant right cones where the otherwise troublesome condition (c) is satisfied by any subset of $\mathbb{R}^+\backslash\{0\}$.

Example 6.9. Let H be any right invariant right cone in a group G. The semigroup $\overline{H} = \{aH \mid a \in H\}$ with $aHbH = abH$ defining the operation is embeddable into a group if $aU \subseteq Ua$ for $U = U(H)$, the group of units of H, and $a \in H$. Since

$Ua \subseteq aU$ by the right invariance of H, we then have $aU = Ua$ and U is a normal subgroup of G since every $g \in G$ has the form $g = ab^{-1}$, for some $a, b \in H$. In this case $\bar{H} \cong \{aU \mid a \in H\}$ is a right cone in $\bar{G} = G/U$. This condition $aU \subseteq Ua$ is certainly satisfied if $U = \{e\}$ or if H is also left invariant.

The left cancellation law for $\bar{H}$, that is $aHbH = aHcH$ for $a, b, c \in H$ implies $bH = cH$, holds for any right invariant, right cone H in a group G, as well as the Ore condition, $aH = bHcH$ or $bH = aHcH$ for any $a, b \in H$ and a suitable $c \in H$.

However, $\bar{H}$ is not embeddable into a group if the right cancellation law does not hold. In that case we have $bHaH = cHaH$ and, say $bH \supset cH$ for some $a, b, c \in H$. Then $c = bd, d \in H\backslash U(H)$ and $aH = daH, da = ak$ for $k \in U(H), d, a \in H\backslash U(H)$.

In Example 4.2 we constructed a right invariant right cone H of rank 2 in an ordered group G as in Proposition 4.1 that contains elements $x, y \in H\backslash U$ with $xy = yu$ for some $u \in U(H)$, exactly the condition given above that indicates that $\bar{H}$ is not embeddable into a group.

We will modify this example to consider right invariant right cones H as in Proposition 4.1 for which the condition (c) in Lemma 6.7 for $T_P(\mathcal{B})$ is automatically satisfied since $a \cdot b \in bU(H)$ for $a \in P_1\backslash P_2$ for a prime segment $P_1 \supset P_2$ of H, and $b \in P_2$.

Let $G = G_1 \wr G_2$ be the wreath product of the ordered groups $G_1 = G_2 = (\mathbb{Q}, +)$ with base groups

$$B = \sum_{r' \in \mathbb{Q}} (G_1)_{r'}.$$

The elements $g \in G$ have the form $g = rb$ with $r \in G_2 = \mathbb{Q}$ and $b = (b_{r'}) \in B$. The operation is defined by $br = r\hat{b}$ for $b \in B, r \in G_2 = \mathbb{Q}$ and $\hat{b} = (\hat{b}_{r'}) \in B$ with $\hat{b}_{r'} = b_{r'-r}$ if $b = (b_{r'}) \in B$.

Then $H = \{rb \mid 0 \le r \in \mathbb{Q}; b = (b_{r'}) \in B \text{ with } b_{r'} = 0 \text{ for } r' < 0 \text{ and } b_0 \ge 0\}$ is a right invariant right cone of rank two in the ordered group G as in Proposition 4.1.

The maximal ideal $J(H) = P_1 = \{rb \in H \mid 0 \le r \in \mathbb{Q}, b = (b_{r'}) \in B$ with $b_{r'} = 0$ for $r' < 0$ and $b_0 > 0$ if $r = 0\}$ is generated as a right ideal by $\{b = (b_{r'}) \in B \subset G \mid b_{r'} = 0$ for $r' < 0$ and $b_0 > 0\}$.

The set $P_2 = \{rb \in H \mid r > 0\}$ is the only other prime ideal. We have

$$\Phi_{P_1}(\overline{H^0_{P_1}}) = \Phi_{P_1}(\overline{H\backslash P_2}) = \mathbb{Q}^+ \quad \text{with } \Phi_{P_1}[(b_{r'})H^0_{P_1}] = b_0 \in \mathbb{Q}^+$$

and $\Phi_{P_2}(\overline{H^0_{P_2}}) = \Phi_{P_2}(\bar{H}_{P_2}) = \mathbb{Q}^+$ with $\Phi_{P_2}(rbH_{P_2}) = r \in \mathbb{Q}^+$.

In order to construct subsets $\mathcal{B}$ of $\mathcal{L}(H)$ that satisfy the conditions (1)–(4) in Theorem 5.4 it is sufficient by Lemma 6.7 to prescribe subsets $T_{P_1}(\mathcal{B})$ and $T_{P_2}(\mathcal{B})$ of $\mathbb{R}^+\backslash\{0\}$ each satisfying the conditions (a), (b) and (c) in this lemma.

However, condition (c) will be automatically satisfied, since $\langle L_{\hat{\rho},\hat{P}}tH\rangle$, for $\rho \in T_{\hat{P}}(\mathcal{B})$ and $t \in H$, is equal to $\langle tL_{\hat{\rho},\hat{P}}\rangle$ if either $\hat{P} = P_1, \rho \in T_{P_1}(\mathcal{B})$ and $t \in H\backslash P_2$, or if $\hat{P} = P_2$ and $\hat{\rho} \in T_{P_2}(\mathcal{B})$ which follows from [9, 4.7 and 5.6] or can be checked directly.

In the remaining case $\hat{P} = P_1, \hat{\rho} \in T_{P_1}(\mathcal{B})$ and $t \in P_2$ it follows that $\langle L_{\hat{\rho},\hat{P}} tH\rangle = \langle \{tH\}\rangle$ is not a limit set, since by construction of H we have $h_1 p_2 = p_2 \cdot u$ for $h \in H\backslash P_2, p_2 \in P_2$ and a unit $u \in U(H)$. We can therefore choose any subset $T_{P_1}(\mathcal{B})$ of $\mathbb{R}^+\backslash\{0\}$, say

$$T_{P_1}(\mathcal{B}) = \{q + n\sqrt{2} + m\sqrt{3} \,|\, q \in \mathbb{Q}, (0,0) \neq (n,m) \in \mathbb{N} \times \mathbb{N}, q + n\sqrt{2} + m\sqrt{3} > 0\}$$

as in Example 6.2, and similarly a subset $T_{P_2}(\mathcal{B})$ of $\mathbb{R}^+\backslash\{0\}$, say

$$T_{P_2}(\mathcal{B}) = \{q + 5 \cdot k\pi \,|\, q \in \mathbb{Q}, k \in \mathbb{N}, q + 5k\pi > 0\}$$

that just satisfy conditions (a) and (b) and obtain a subset

$$\mathcal{B} = \mathcal{D} \cup \{\langle tL_{\rho_1,P_1}\rangle \,|\, t \in H, \rho_1 \in T_{P_1}(\mathcal{B})\} \cup \{\langle tL_{\rho_2,P_2}\rangle \,|\, t \in H, \rho_2 \in T_{P_2}(\mathcal{B})\}$$

and a right invariant right chain domain $R_{\mathcal{B}}$ that is I-compact for ideals I with $L_{I\cap H} \in \mathcal{B}$.

The right invariant right cone H' in the wreath product $(G_1 \wr G_2) \wr G_3 = G'$ with $G_1 = G_2 = G_3 = (\mathbb{Q}, +)$ and

$$H' = \left\{ qb \in G' \,|\, 0 \leq q \in \mathbb{Q}, b \in \sum_{q' \in G_3} (G_1 \wr G_2)_{q'} \text{ with } b_{q'} = 0 \text{ for } q' < 0 \text{ and } b_0 \in H \right\}$$

has rank 3, and the sets $T_P(\mathcal{B}), P \in \operatorname{spec}_0(H')$, only need to satisfy conditions (a) and (b).

So far we have considered subsets $\mathcal{B}$ of $\mathcal{L}(H)$ for a right invariant right cone H that were defined by prime ideals of H and ideals I with $P_r(I) = P = P^2 \in \operatorname{spec}_0(H)$, that is, P has a lower neighbor in the lattice of prime ideals. In the next and final Example 6.10 we consider subsets $\mathcal{B}$ so that the corresponding rings $R_{\mathcal{B}}$ are I-compact for ideals I with $P_r(I) = P = \bigcup P_i$ for a limit prime ideal P with prime ideals $P_i \subset P$.

Example 6.10. Let H be a right invariant right cone in a group G as in Proposition 4.1. It follows from Lemma 6.7 that the set

$$\mathcal{B} = \mathcal{D} \cup \bigcup_{P = P^2 \in \operatorname{spec}_0(H)} \{\langle tL_{\rho,P}\rangle \,|\, t \in H, 0 < \rho \in \mathbb{R}\}$$

satisfies conditions (1)–(4) in Theorem 5.4 and hence a right chain domain $R_{\mathcal{B}}$ exists that is I-compact for all distinguished ideals I of $R_{\mathcal{B}}$ and all ideals I with $P_r(I) = P = P^2 \in \operatorname{spec}_0(H)$. If H satisfies the maximum condition for prime ideals, then $R_{\mathcal{B}} = \hat{R}$. If, however, there exist limit primes $P = \bigcup P_i, P \supset P_i$ completely prime, then $R_{\mathcal{B}}$ is not I-compact for any non-distinguished ideal I with $P_r(I) = PR_{\mathcal{B}} = Q_r(I)$, where the last equation follows from [9, Corollary 2.4].

To illustrate this situation we consider the following example:

Let $G = \sum_{i\in\mathbb{N}} \mathbb{Z}_i$ be the direct sum of $\mathbb{Z}_i = (\mathbb{Z}, +)$ for $i = 1, 2, 3, \ldots$. Then G is ordered with cone $H = \{(a_i) \in G \,|\, (a_i) = (0) \text{ or } a_{i_0} > 0 \text{ for } i_0 = \operatorname{lind}(a_i)\}$, where

$\text{lind}(a_i) = \min\{i \mid a_i \neq 0\}$. In this alphabetical order we have

$$(1, 0, 0, \ldots) > (0, 1, 0, \ldots) > (0, 0, 1, 0, \ldots) \cdots .$$

The prime ideals of H are $P_n = \{(a_i) \in H \mid \text{lind}(a_i) \leq n\} \subset P_{n+1}$ for $n = 1, 2, \ldots$ and the limit prime

$$P = J(H) = H \backslash \{(0)\} = \bigcup_{n \in \mathbb{N}} P_n.$$

It follows that $\text{spec}_0(H) = \{P_n \mid n \in \mathbb{N}\}$, however, $P_n \supset P_n^2$ for all n.

Let $\hat{G} = \prod_{\in \mathbb{N}} \mathbb{Z}_i$ be the direct product of the $\mathbb{Z}_i = (\mathbb{Z}, +)$ equipped with the alphabetical order that is the extension of the order defined for $G \subset \hat{G}$, and let $\hat{H} \supset H$ be the corresponding cone in $\hat{G}$. It follows from [9, Example 3.8] that the non-distinguished ideals I of H with $P_r(I) = P = H \backslash \{(0)\}$ are given by $I = I_{\hat{g}} = \{(a_i) \in H \mid (a_i) \geq \hat{g} \text{ in } \hat{H}\}$ for $\hat{g} \in \hat{H} \backslash \{(0)\}$.

We can see directly that $Q_r(I_{\hat{g}}) = P$ holds for any $(0) < \hat{g} \in \hat{H}$ as follows: If p is any element in P then there exists $s < \hat{g}$ in H with $s + p \geq \hat{g}$. If $p \geq \hat{g}$, $s = (0)$ will do, otherwise there exists $s \in H$ with $\hat{g} > s > \hat{g} - p$ by the density of H in $\hat{H}$.

It follows that $p \in Q_r(I)$ and $Q_r(I) = P_r(I) = P$, using Corollary 2.4 in [9]. To show that $I_{\hat{g}}$ is not distinguished, we assume by contradiction that $I_{\hat{g}} = c + P$ for some $c \in H$. It then follows that $c \notin c + P$, hence $c < \hat{g}$ and h' exists in h with $c < h' < \hat{g}$. Therefore $h' = c + p \in c + P$ for some $p \in P$, but $h' \notin I_{\hat{g}}$, a contradiction that shows that $I_{\hat{g}}$ is not distinguished.

Assume that $\mathcal{B} \subseteq \mathcal{L}(H)$ satisfies the conditions (1)–(4) in Theorem 5.4, and let

$$P = H \backslash \{(0)\} = \bigcup_{n \in \mathbb{N}} P_n$$

be the limit prime of H. Then $T_P(\mathcal{B}) = \{(0) < \hat{g} \in \hat{H} \mid L_{\hat{g}} \in \mathcal{B} \subseteq \hat{H} \backslash \{(0)\}\}$, with $L_{\hat{g}} = \{h + H \mid h \in H, h < \hat{g} \text{ in } \hat{H}\}$, satisfies the following conditions:

(a) $\hat{g}_1, \hat{g}_2 \in T_P(\mathcal{B})$ implies $\hat{g}_1 + \hat{g}_2 \in T_P(\mathcal{B})$.
(b) $\hat{g} \in T_P(\mathcal{B}), h \in H$, implies $\hat{g} + h \in T_P(\mathcal{B})$, and $\hat{g} - h \in T_P(\mathcal{B})$ if $\hat{g} > h$.

The condition (a) follows since

(i) $L_{\hat{g}_1} \cdot L_{\hat{g}_2} = L_{\hat{g}_1 + \hat{g}_2}$

is in $\mathcal{B}$ for $L_{\hat{g}_1}, L_{\hat{g}_2} \in \mathcal{B}$. Condition (b) follows for $T_P(\mathcal{B})$ since conditions (2) and (3) in Theorem 5.4 hold for $\mathcal{B}$.

We also have

(ii) $$L_{P_n} L_{P_m} = \begin{cases} L_{P_n} & \text{for } n \leq m, \\ L_{P_m} & \text{for } m < n, \end{cases}$$

by the proof of Proposition 5.5.

To compute $L_{P_n} \cdot L_{\hat{g}}$ for $\hat{g} = (a_1, \dots, a_n, \dots, a_r, a_{r+1}, \dots) \in \hat{H} \backslash \{(0)\}$, we pick $r \in \mathbb{N}$ with $r \geq \max\{n, \operatorname{lind} \hat{g}\}$ and $t \in H$ with $t = (a_1, \dots, a_n, \dots, a_r, t_{r+1}, 0, 0, \dots)$ and $t_{r+1} < a_{r+1} \in \mathbb{Z}$. Then $\hat{g} = t + (0, 0, \dots, 0, a_{r+1} - t_{r+1}, a_{r+2}, \dots)$ with $\operatorname{lind}(\hat{g} - t) = r + 1$ and $L_{\hat{g}} = t + L_{\hat{g}-t}$ with $L_{\hat{g}-t} \subseteq L_{P_n}$.

We therefore obtain:

(iii) $L_{P_n} \cdot L_{\hat{g}} = L_{P_n} \cdot [t + L_{\hat{g}-t}] = t + L_{P_n}$.

It follows from (i), (ii), and (iii) that, given a subset $T_P(\mathcal{B})$ of $\hat{H} \backslash \{(0)\}$ that satisfies the conditions (a) and (b) above, then

$$\mathcal{B} = \mathcal{D} \cup \{L_{\hat{g}} \mid \hat{g} \in T_P(\mathcal{B})\} \subseteq \mathcal{L}(H)$$

satisfies the conditions (1)–(4) in Theorem 5.4 and a chain domain $R_{\mathcal{B}}$ exists that is I-compact exactly for all distinguished ideals I and for those ideals I with $L_{I \cap H} = L_{\hat{g}}$ for $\hat{g} \in T_P(\mathcal{B})$.

As an example we choose $\mathcal{B}_1 = \mathcal{D} \cup \{L_{\hat{g}} \mid \hat{g} \in T_P(\mathcal{B}_1)\}$ with

$$\begin{aligned} T_P(\mathcal{B}_1) = {} & \{nq(1, 1, 1, \dots) + h \mid h \in H, n \in \mathbb{N}\} \\ & \cup \{nq(1, 1, 1, 0, \dots) - h > (0) \mid h \in H, n \in \mathbb{N}\} \end{aligned}$$

for q a prime in $\mathbb{N}$. Then $T_P(\mathcal{B})$ satisfies the conditions (a) and (b), hence $\mathcal{B}$, satisfies the conditions (1)–(4) in Proposition 4.1 and the chain domain $R_1 = R_{\mathcal{B}_1}$ associated with H exists. It follows that this ring R_1 is I-compact for all distinguished ideals I and for those ideals I with $L_{I \cap H} = L_{\hat{g}}$ for some $\hat{g} \in T_P(\mathcal{B}_1)$. Since $T_P(\mathcal{B}_1) \cap H = \emptyset$, it follows that R_1 is not I-compact for any principal ideal $I \neq (0)$ in R_1.

This example shows also that infinitely many distinct subsets $T_P(\mathcal{B}) \subseteq \hat{H} \backslash \{(0)\}$ exist that satisfy conditions (a) and (b).

As a second example we can choose $\mathcal{B}_2 = \mathcal{D} \cup \{L_{\hat{g}} \mid \hat{g} \in H \backslash \{(0)\}\}$. Then $R_2 = R_{\mathcal{B}_2}$ is I-compact for all distinguished ideals I and all principal ideals.

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