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ON SEMIPRIME SEGMENTS OF RINGS

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Abstract

A semiprime segment of a ring R is a pair $P_2 \subset P_1$ of semiprime ideals of R such that $\bigcap I^n \subseteq P_2$ for all ideals I of R with $P_2 \subset I \subset P_1$. In this paper semiprime segments with P_1 a comparizer ideal are classified as either simple, exceptional, or archimedean, extending to several classes of rings a classification known for right chain rings. These three types of semiprime segments are also characterized in terms of the pseudo-radical.

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1. Introduction

A *right chain ring* is a ring whose right ideals are linearly ordered under inclusion. A *prime segment* of a right chain ring R is an interval in the lattice of ideals of R defined by a pair of neighbouring completely prime ideals.

In [5], it was proved that a prime segment $P_2 \subset P_1$ of a right chain ring R is either simple in which case there are no further ideals in this segment or it is exceptional in which case a prime ideal Q exists with $P_2 \subset Q \subset P_1$, or it is locally right invariant, that is, $P_1 a \subseteq a P_1$ for all $a \in P_1 \setminus P_2$. A similar result was obtained in [3] where prime segments of Dubrovin valuation rings were classified as either simple, exceptional, or archimedean (that is, $\bigcap_{n\in\mathbb{N}}(RaR)^n = P_2$ for every $a \in P_1 \setminus P_2$), and in [4] prime segments of skew fields were studied.

Recently, several papers showed that some features for right chain rings can be carried over to rings R containing a comparizer ideal, that is, an ideal P such that for any $a, b \in R$, either $aR \subseteq bR$ or $bP \subseteq aR$ (see for instance [8–10, 14, 15]). In this

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paper we extend the classification of prime segments to these rings, and moreover, we classify semiprime segments of arbitrary rings with identity (Theorem 3.2).

A *semiprime segment* of a ring R is a pair $P_2 \subset P_1$ of semiprime ideals of R such that $\bigcap_{n\in\mathbb{N}} I^n \subseteq P_2$ for all ideals I of R with $P_2 \subset I \subset P_1$. We show that every semiprime segment $P_2 \subset P_1$ with P_1 a comparizer ideal is either simple, exceptional, or archimedean (Theorem 4.1). We also characterize each of these three types of semiprime segments using prime right ideals within the segment, that is, in terms of the pseudo-radical (Corollary 4.4). To prove these results, in Section 2 and Section 3 we contribute some new insights to the general theory of ideals. As a consequence we obtain the aforementioned results of [3] and [5].

Throughout this paper, R denotes a ring with identity, and the Jacobson radical of R is denoted by $J(R)$. If I is an ideal (right ideal) of R, then we write $I \lhd R$ ($I <_r R$). The symbol $\subset$ stands for proper inclusion of sets.

2. Comparizer ideals

Let R be a ring and I a right ideal of R which is proper (that is, $I \neq R$). Then I is called *prime* if for any $a, b \in R$, $aRb \subseteq I$ implies either $a \in I$ or $b \in I$. If $ab \in I$ implies either $a \in I$ or $b \in I$, then I is called *completely prime*. Moreover, I is said to be *semiprime* if $aRa \subseteq I$ implies $a \in I$.

Recall that a proper right ideal I of a ring R is said to be a *waist* if I is comparable with each right ideal of R, that is, either $A \subseteq I$ or $I \subset A$ holds for each right ideal A of R (see [1]). Obviously, every waist I of R is contained in all maximal right ideals of R, and thus $I \subseteq J(R)$.

The following lemma is an immediate consequence of [9, Lemma 2.5] and the definitions above.

LEMMA 2.1. *If a completely prime right ideal P of a ring R is a waist, then P is an ideal of R and $aP = P$ holds for each $a \in R \setminus P$.*

A proper right ideal I of a ring R is called a *comparizer* (respectively, *strongly comparizer*) right ideal [15] if for any $a, b \in R$, either $aR \subseteq bR$ or $bI \subseteq aR$ (respectively, either $aR \subseteq bR$ or $bI \subseteq aI$). In the following proposition we list several properties of comparizer right ideals.

PROPOSITION 2.2. *Let I be a comparizer right ideal of a ring R.*

(i) *If $I = I^2$, then I is a waist of R.*
(ii) *If I is a waist of R, then for each $a \in R$ the right ideal aI is a waist of R.*
(iii) *All prime right ideals of R properly contained in I are waists of R.*
(iv) *If I is a nonnilpotent ideal of R, then the ideal $\bigcap_{n\in\mathbb{N}} I^n$ is completely prime.*

(v) *For every ideal Q of R with $Q \subset I$, Q is semiprime if and only if Q is prime.*

(vi) *If $I = I^2$, then for each right ideal A of R with $A \subset I$ there exists $b \in I$ such that bI is a waist of R and $A \subseteq bI \subset I$.*

PROOF. Properties (i)–(v) follow from results of [15], Lemma 1.2 (i), Lemma 1.2 (ii), Proposition 1.9, Theorem 2.3 and Corollary 2.7, respectively.

(vi) Using (i) and (ii) we obtain that $I = \bigcup_{b \in I} bI$. Since $A \subset I$, there exists $b \in I$ with $A \subseteq bI$. If $bI = I$, then $b \in bI \subseteq bJ(R)$ and thus $b = 0$, leading to a contradiction $A \subset I = (0)$. Hence $bI \subset I$. □

Below we present several classes of rings with (strongly) comparizer ideals.

EXAMPLE 1. A ring R is said to be a *right chain* ring if the lattice of right ideals of R is linearly ordered by inclusion. Clearly, each proper right ideal of a right chain ring R is a comparizer waist, and each proper ideal of R is a strongly comparizer waist.

EXAMPLE 2. A ring R is called a *right pseudo-chain* ring [14] if and only if whenever $aR \not\subseteq bR$ for $a, b \in R$, then $bcR \subseteq aR$ for each non-unit $c \in R$. These rings are a common generalization of right chain rings and commutative pseudo-valuation rings [2]. It is proved in [14, Theorem 2.1] that a ring R is a right pseudo-chain ring if and only if R is a local ring and $J(R)$ is a comparizer ideal of R. Hence, in any right pseudo-chain ring R, $J(R)$ is a completely prime strongly comparizer waist and each proper right ideal of R is comparizer.

EXAMPLE 3. A ring R is called a *right distributive* ring if the lattice of right ideals of R is distributive. By [13, Lemma 3.1 (ii)] and [16, Proposition 2.1], each completely prime ideal $P \subseteq J(R)$ of a right distributive ring R is a strongly comparizer waist.

EXAMPLE 4. A *right Bezout* ring is a ring in which all finitely generated right ideals are principal. If R is a right Bezout ring, then all completely prime ideals of R contained in $J(R)$ are strongly comparizer waists ([15, Lemma 1.6]).

EXAMPLE 5. A ring R is called a ring *with comparability* [8] if for each completely prime ideal P of R with $P \subseteq J(R)$ and any $a, b \in R$ we have either $aR \subseteq bR$, or $bR \subseteq aR$, or $(aR)S^{-1} = (bR)S^{-1}$, where $(aR)S^{-1} = \{x \in R \mid xs \in aR$ for some $s \in R \setminus P\}$. Since the equality $(aR)S^{-1} = (bR)S^{-1}$ implies that $aP = bP$, it follows that all completely prime ideals contained in the Jacobson radical of a ring with comparability are strongly comparizer ([8, Remark 2.5]), and by [8, Lemma 1.3] they are also waists.

From the examples above further examples of rings with completely prime strongly comparizer waists can be constructed using the following.

PROPOSITION 2.3. *If P is a completely prime strongly comparizer waist of a ring T, and R is a subring of T containing P, then P is a completely prime strongly comparizer waist of R.*

PROOF. From Lemma 2.1 it follows that P is a completely prime waist of R. To show that P is strongly comparizer in R, we consider any elements $a, b \in R$ with $bP \not\subseteq aP$. Since P is strongly comparizer in T, $a = bt$ for some $t \in T$. If $t \notin P$, then Lemma 2.1 implies $bP = btP = aP$, a contradiction. Hence $t \in P \subset R$ and thus $aR \subseteq bR$, which ends the proof. □

3. Classification of semiprime segments

Let $P_2 \subset P_1$ be semiprime ideals of a ring R satisfying $\bigcap_{n\in\mathbb{N}} I^n \subseteq P_2$ for every ideal I of R with $P_2 \subset I \subset P_1$. The interval $P_2 \subset P_1$ is then called a *semiprime segment* of R. Later on we will need the following

LEMMA 3.1. *If $P_2 \subset P_1$ is a semiprime segment of a ring R, then exactly one of the following possibilities occurs.*

(1) *There are no further ideals of R between P_2 and P_1, and P_2 is comparable with each ideal of R contained in P_1.*

(2) *There exists a semiprime ideal Q of R such that $P_2 \subset Q \subset P_1$ and Q is comparable with each ideal of R contained in P_1.*

(3) *$P_1aR + RaP_1 \subset RaR$ for all $a \in P_1 \setminus P_2$.*

(4) *$P_1 = Ra_1R + \cdots + Ra_nR + P_2$ for some $a_1, \ldots, a_n \in R$ with $Ra_iR \subset P_1$, and $P_1aR + RaP_1 = RaR$ for some $a \in P_1 \setminus P_2$.*

PROOF. Observe that if $a \in R$ and I is an ideal of R such that $a \in IaR + RaI$, then for any $n \in \mathbb{N}$, $a \in I^naR + RaI^n$, and so $a \in \bigcap_{n\in\mathbb{N}} I^n$. In particular, if $P_1^2 + P_2 \subset P_1$, then the possibility (c) occurs. Therefore, to finish the proof we assume that $P_1^2 + P_2 = P_1$. Let M be the sum of all ideals I of R with $I \subset P_1$. Then $P_2 \subseteq M \subseteq P_1$. If $M = P_2$ holds, then clearly the alternative (a) occurs.

Next assume that $P_2 \subset M \subset P_1$, and let A be any ideal of R with $A^2 \subseteq M$. Then $A^2 \subseteq P_1$, and since P_1 is semiprime, we conclude that $A \subseteq P_1$. The case $A = P_1$ leads to a contradiction, since then $P_1 = P_1^2 + P_2 = A^2 + P_2 \subseteq M$. Thus $A \subset P_1$ and $A \subseteq M$ follows. Hence M is a semiprime ideal of R and (b) occurs.

We are finally left with the case $M = P_1$, and we assume that (c) does not occur. We want to show that (d) holds. Since $M = P_1$, for every $p \in P_1$ there exist $a_1, \ldots, a_n \in R$ such that $p = a_1 + \cdots + a_n$ and $Ra_iR \subset P_1$ for each i. Thus, if $RpR = P_1$ for some $p \in P_1$, then (d) occurs. Assume that $RpR \subset P_1$ for every

$p \in P_1$. Since (c) does not occur, $a \in P_1aR + RaP_1$ for some $a \in P_1 \setminus P_2$ and we can write $a = \sum_i p_i a r_i + \sum_j s_j a q_j$ for some $p_i, q_j \in P_1$ and $r_i, s_j \in R$. Set $I = \sum_i Rp_iR + \sum_j Rq_jR$. Then $a \in IaR + RaI$. If $I + P_2 \subset P_1$, then by the remark at the beginning of the proof, we obtain $a \in P_2$, a contradiction. Thus $I + P_2 = P_1$ and (d) occurs.

It is easy to see that the alternatives (a)–(d) are mutually exclusive. □

In the following theorem we classify semiprime segments of arbitrary rings.

THEOREM 3.2. *If $P_2 \subset P_1$ is a semiprime segment of a ring R, then exactly one of the following possibilities occurs.*

(i) *The semiprime segment $P_2 \subset P_1$ is simple; that is, there are no further ideals of R between P_2 and P_1, and P_2 is comparable with each ideal of R contained in P_1.*

(ii) *The semiprime segment $P_2 \subset P_1$ is exceptional; that is, there exists a semiprime ideal Q of R such that $P_2 \subset Q \subset P_1$ and Q is comparable with each ideal of R contained in P_1.*

(iii) *The semiprime segment $P_2 \subset P_1$ is archimedean; that is, for every $a \in P_1 \setminus P_2$ there exists an ideal I of R with $a \in I \subseteq P_1$ and $\bigcap_{n\in\mathbb{N}} I^n \subseteq P_2$.*

(iv) *The semiprime segment $P_2 \subset P_1$ is decomposable; that is, the semiprime segment $P_2 \subset P_1$ is not archimedean and $P_1 = A + B$ for some ideals A, B of R properly contained in P_1.*

PROOF. All possible cases are listed in Lemma 3.1. Clearly, in the case (a) the semiprime segment $P_2 \subset P_1$ is simple, in the case (b) it is exceptional, and in the case (d) it is either archimedean or decomposable.

Assume that the possibility (c) occurs and the semiprime segment $P_2 \subset P_1$ is not archimedean. If for all $a \in P_1 \setminus P_2$, we have $(RaR)^2 + P_2 \subset P_1$, then

$$\bigcap_{n\in\mathbb{N}} (RaR)^n \subseteq \bigcap_{n\in\mathbb{N}} ((RaR)^2 + P_2)^n \subseteq P_2,$$

so the semiprime segment $P_2 \subset P_1$ is archimedean, a contradiction. Hence for some $a \in P_1 \setminus P_2$, $(RaR)^2 + P_2 = P_1$. Since by (c), $(RaR)^2 \subseteq P_1aR \subset RaR \subseteq P_1$, the semiprime segment $P_2 \subset P_1$ is decomposable.

It is easy to verify that the alternatives (i)–(iv) are mutually exclusive. □

Each of the types of semiprime segments listed in Theorem 3.2 is possible. Indeed, in the ring $\mathbb{Z}_{30}$ of integers modulo 30, the semiprime segment $(0) \subset 6\mathbb{Z}_{30}$ is simple, and the semiprime segment $6\mathbb{Z}_{30} \subset 2\mathbb{Z}_{30}$ is decomposable. Moreover, if $\mathbb{Z}_{(2)}$ is the localization of the ring of integers $\mathbb{Z}$ at $2\mathbb{Z}$, then the set $\mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}$ with componentwise addition and multiplication defined by

$$(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1 + x_2y_2)$$

is a ring in which the semiprime segment $(0) \subset (0) \times \mathbb{Z}_{(2)}$ is exceptional, and the semiprime segment $(0) \subset (0) \times 2\mathbb{Z}_{(2)}$ is archimedean.

Next we apply Theorem 3.2 to the class of Dubrovin valuation rings. Let R be a Dubrovin valuation ring; that is, R is a Bezout order in a simple artinian ring A and $R/J(R)$ is simple artinian. A prime ideal P of R is called a *Goldie prime* if the factor ring R/P is Goldie. A prime segment of R is defined as a pair of Goldie primes $P_2 \subset P_1$ of R such that no further Goldie prime exists between P_2 and P_1 (see [3]). Since the ideals of a Dubrovin valuation ring R are linearly ordered by inclusion, and for every proper ideal I of R, the ideal $\bigcap_{n\in\mathbb{N}} I^n$ is Goldie prime (see [3, Theorem 5]), it follows that all prime segments of R are also semiprime segments. Thus as a special case of Theorem 3.2 we obtain the following well known classification.

COROLLARY 3.3 ([3, Theorem 6]). *For a prime segment $P_2 \subset P_1$ of a Dubrovin valuation ring R, exactly one of the following possibilities occurs.*

(i) *The prime segment $P_2 \subset P_1$ is simple.*

(ii) *The prime segment $P_2 \subset P_1$ is exceptional. In this case there exists a prime ideal Q of R with $P_2 \subset Q \subset P_1$ that is not a Goldie prime.*

(iii) *The prime segment $P_2 \subset P_1$ is archimedean.*

Examples can be found in [3] showing that each of the alternatives listed in Corollary 3.3 is possible.

Let R be a right chain ring. A prime segment of R is defined as a pair $P_2 \subset P_1$ of completely prime ideals of R such that no further completely prime ideal exists between P_2 and P_1 (see [5]). Since for every nonnilpotent proper ideal I of R, the ideal $\bigcap_{n\in\mathbb{N}} I^n$ is completely prime (see Proposition 2.2 (iv)), it follows that every prime segment of R is a semiprime segment. Thus we can apply Theorem 3.2 to obtain the following well known classification of prime segments of right chain rings.

COROLLARY 3.4 ([5, Theorem 1.14]). *For a prime segment $P_2 \subset P_1$ of a right chain ring R, exactly one of the following possibilities occurs.*

(i) *The prime segment $P_2 \subset P_1$ is simple.*

(ii) *The prime segment $P_2 \subset P_1$ is exceptional. In this case there exists a prime ideal Q of R with $P_2 \subset Q \subset P_1$ that is not completely prime.*

(iii) *The prime segment $P_2 \subset P_1$ is locally right invariant; that is, $P_1 a \subseteq a P_1$ for every $a \in P_1 \setminus P_2$.*

PROOF. By Theorem 3.2, we only need to show that if the semiprime segment $P_2 \subset P_1$ is archimedean, then it is locally right invariant. Let $a \in P_1 \setminus P_2$ and $p \in P_1$. If $aP_1 \subseteq paP_1$, then since the semiprime segment $P_2 \subset P_1$ is archimedean, we obtain

$$aP_1 \subseteq \bigcap_{n\in\mathbb{N}} p^n a P_1 \subseteq \bigcap_{n\in\mathbb{N}} (RpR)^n \subseteq P_2,$$

a contradiction. Hence $paR \subset aR$ and $P_1a \subset aR$ follows. Let $r \in R$ be such that $pa = ar$. If $r \notin P_1$, then applying Lemma 2.1 we obtain $aP_1 = arP_1 = paP_1$, which is impossible, as we noted earlier. Thus $P_1a \subseteq aP_1$. □

Clearly, in each rank one commutative valuation domain R with maximal ideal M, the prime segment $(0) \subset M$ is locally right invariant and thus archimedean. Examples of right chain rings with simple segments were given by Mathiak [12] and Dubrovin [6], and a right chain domain with an exceptional segment was constructed by Dubrovin in [7]. Thus all types of prime segments described in Corollary 3.4 are possible.

4. Classification of comparizer semiprime segments

As we have seen in Corollary 3.4, prime segments of a right chain ring are classified as either simple, exceptional, or locally right invariant. In this section we extend the classification to semiprime segments $P_2 \subset P_1$ of any ring with P_1 a completely prime strongly comparizer waist (examples of such rings were given in Section 2).

In the following theorem we classify semiprime segments $P_2 \subset P_1$ with P_1 a comparizer ideal. We recall that any neighbouring completely prime ideals $P_2 \subset P_1$ with P_1 a comparizer ideal form a semiprime segment (Proposition 2.2 (iii)–(iv)).

THEOREM 4.1. *Let $P_2 \subset P_1$ be a semiprime segment of a ring R. If P_1 is a comparizer ideal of R, then exactly one of the following possibilities occurs.*

(i) *The semiprime segment $P_2 \subset P_1$ is simple. In this case P_1 is a completely prime waist of R.*

(ii) *The semiprime segment $P_2 \subset P_1$ is exceptional. In this case there exists a prime waist $Q \lhd R$ with $P_2 \subset Q \subset P_1$ such that there are no further ideals of R between Q and P_1. Moreover, P_1 and P_2 are completely prime waists of R.*

(iii) *The semiprime segment $P_2 \subset P_1$ is archimedean. In this case $P_1aP_1 \subset aR$ for every $a \in P_1 \setminus P_2$, and P_2 is a completely prime waist of R. If furthermore P_1 is a waist of R, then $P_1aP_1 \subseteq aP_1$ for every $a \in P_1 \setminus P_2$.*

PROOF. From (v) and (iii) of Proposition 2.2 it follows that P_2 is a waist of R. If for some $a \in P_1 \setminus P_2$ we have $a \in P_1aR + RaP_1$, then $a \in \bigcap_{n\in\mathbb{N}} P_1^n$ and thus necessarily $P_1 = P_1^2$. This observation and Proposition 2.2 (vi) imply that one of the alternatives (a), (b), or (c) listed in Lemma 3.1 occurs. Clearly, if (a) occurs, then applying (i) and (iv) of Proposition 2.2 we obtain the possibility (i). If (b) occurs, then there exists a semiprime ideal Q with $P_2 \subset Q \subset P_1$, and Q is a prime ideal and a waist by Proposition 2.2 (v) and (iii). Moreover, by Proposition 2.2, P_1 and P_2 are completely prime waists of R. Finally, if (c) occurs, then for every

$a \in P_1 \setminus P_2$, $aR \not\subseteq P_1aR$, and since P_1 is comparizer, $P_1aP_1 \subset aR$ follows. In particular, $\bigcap_{n\in\mathbb{N}}(RaR)^n \subseteq P_2$, which implies that the semiprime segment $P_2 \subset P_1$ is archimedean with P_2 a completely prime waist. Assume that P_1 is a waist and suppose that $paP_1 \not\subseteq aP_1$ for some $p \in P_1$. Then by Proposition 2.2 (ii), $aP_1 \subseteq paP_1$ and we obtain $aP_1 \subseteq \bigcap_{n\in\mathbb{N}} p^naP_1 \subseteq \bigcap_{n\in\mathbb{N}}(RpR)^n \subseteq P_2$, a contradiction. □

To prove the following corollary it is enough to apply again the arguments of Corollary 3.4.

COROLLARY 4.2. *Let $P_2 \subset P_1$ be an archimedean semiprime segment of a ring R. If P_1 is a strongly comparizer ideal of R, then $P_1a \subset aR$ for every $a \in P_1 \setminus P_2$. If furthermore P_1 is a completely prime waist of R, then the semiprime segment $P_2 \subset P_1$ is locally right invariant; that is, $P_1a \subseteq aP_1$ for every $a \in P_1 \setminus P_2$.*

Let R be a ring and I an ideal of R. The *pseudo-radical* $\overline{\mathrm{ps}}(I)$ determined by the ideal I is defined as the intersection of all prime right ideals of R properly containing I; that is, $\overline{\mathrm{ps}}(I) = \bigcap\{P <_r R \mid P \text{ is prime and } I \subset P\}$ (see [11]). We will apply the operator '(P_2, P_1)-closure' described below to determine the pseudo-radical $\overline{\mathrm{ps}}(P_2)$ in the case of a simple semiprime segment $P_2 \subset P_1$.

Let P_2, P_1 be ideals of a ring R and I a right ideal of R. We define the *(P_2, P_1)-closure* of I as $\overline{I} = \{a \in R \mid aP_1 \subseteq IP_1 + P_2\}$. It is easy to see that $I \subseteq \overline{I}$ and $\overline{I}$ is the largest right ideal of R such that $\overline{I}P_1 + P_2 = IP_1 + P_2$. Hence $\overline{\overline{I}} = \overline{I}$, which justifies the name '(P_2, P_1)-closure'.

LEMMA 4.3. *Let $P_2 \subset P_1$ be ideals of a ring R such that $P_1 \neq R$ and there are no further ideals of R between P_2 and P_1.*

(i) *If P_2 is prime and I is a right ideal of R with $P_2 \subset I \subset P_1$, then $\overline{I}$ is a prime right ideal of R.*

(ii) *If P_2 is semiprime, then $\overline{\mathrm{ps}}(P_2) = P_2$.*

PROOF. (i) We noted earlier that $\overline{I}$ is a right ideal of R. If we have $1 \in \overline{I}$, then $P_1 \subseteq IP_1 + P_2 \subseteq I \subset P_1$, a contradiction. Hence $\overline{I} \neq R$. To finish the proof we assume that $xRy \subseteq \overline{I}$ and $y \notin \overline{I}$. Then $y \notin P_2$ and since the ideal P_2 is prime, we deduce $P_2 \subset RyP_1 + P_2 \subseteq P_1$. Thus $RyP_1 + P_2 = P_1$, and $x \in \overline{I}$ follows.

(ii) The ideal P_2, being semiprime, is the intersection of prime ideals containing P_2, and thus $\overline{\mathrm{ps}}(P_2) = P_2$ follows for the case when P_2 is not prime. Hence we assume that P_2 is prime and we set $A = \overline{\mathrm{ps}}(P_2)$. Without loss of generality we can assume that $P_2 = (0)$ and then we have to show that $A = (0)$. If $P_1 \not\subseteq J(R)$, then $P_1J(R) \subseteq P_1 \cap J(R) = (0)$ and consequently $J(R) = (0)$. Since all maximal right ideals of a ring are prime, it follows that $A = (0)$. Next we consider the case when $P_1 \subseteq J(R)$. Suppose that $A \neq (0)$ and let $a \in A \setminus (0)$. Since R is prime, there

exists an element $b \in aP_1 \setminus (0)$. If $b \in bP_1$, then $b \in bJ(R)$ and we obtain $b = 0$, a contradiction. Thus $(0) \subset bP_1 \subset P_1$, and (i) implies that $a \in A \subseteq \overline{bP_1}$. Hence $b \in aP_1 \subseteq bP_1$, which is impossible as is shown above. Therefore, $A = (0)$. □

It is easy to see that under assumptions of the above lemma, if furthermore P_2 is prime, then for a right ideal I of R with $P_2 \subset I \subset P_1$ we have $\overline{I} = I$ if and only if I is prime. Hence in this case the operator '(P_2, P_1)-closure' not only generates but also identifies prime right ideals lying between P_2 and P_1.

We conclude this paper with the following result which shows that semiprime segments can also be classified by the use of the pseudo-radical.

COROLLARY 4.4. *Let $P_2 \subset P_1$ be a semiprime segment of a ring R. If P_1 is a comparizer ideal of R, then*

(i) *The semiprime segment $P_2 \subset P_1$ is simple if and only if $\overline{\mathrm{ps}}(P_2) = P_2$.*
(ii) *The semiprime segment $P_2 \subset P_1$ is exceptional if and only if $P_2 \subset \overline{\mathrm{ps}}(P_2) \subset P_1$.*
(iii) *The semiprime segment $P_2 \subset P_1$ is archimedean if and only if $\overline{\mathrm{ps}}(P_2) = P_1$.*

PROOF. By Theorem 4.1, it is enough to prove necessity in each case (i)–(iii).

(i) We apply Lemma 4.3 (ii).

(ii) Let Q be the ideal of R described in Theorem 4.1 (ii). Then Q is a waist and by Proposition 2.2 (iii) there are no further prime right ideals of R between P_2 and Q. Hence $\overline{\mathrm{ps}}(P_2) = Q$.

(iii) Since P_1 is semiprime, P_1 coincides with the intersection of all prime ideals of R containing P_1, and thus $\overline{\mathrm{ps}}(P_2) \subseteq P_1$. Suppose that $\overline{\mathrm{ps}}(P_2) \subset P_1$. Then there exists a prime right ideal I of R such that $P_2 \subset I$ and $P_1 \not\subseteq I$. Hence by [15, Proposition 1.9], $P_2 \subset I \subset P_1$ and I is a waist of R. Let $a \in P_1 \setminus I$, then $I \subset RaR$ and $I \subseteq \bigcap_{n\in\mathbb{N}}(RaR)^n$ follows. Since the segment is archimedean, the last containment implies $I \subseteq P_2$, a contradiction. □

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