Can. J. Math., Vol. XXX, No. 5, 1978, pp. 1079–1086

EMBEDDING RIGHT CHAIN RINGS IN CHAIN RINGS

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1. Introduction. The following problem was the starting point for this investigation: Can every desarguesian affine Hjelmslev plane be embedded into a desarguesian projective Hjelmslev plane [**8**]? An affine Hjelmslev plane is called *desarguesian* if it can be coordinatized by a right chain ring R with a maximal ideal $J(R)$ consisting of two-sided zero divisors. A projective Hjemslev plane is called desarguesian if the coordinate ring is in addition a left chain ring, i.e. a chain ring. This leads to the algebraic version of the above problem, namely the embedding of right chain rings into suitable chain rings. We can prove the following result.

Let R be a right chain ring of type (2) or (3) (the definitions are given in the next section) with finitely generated maximal ideal $J(R) = mR$. If we assume further that the characteristic of R is different from the characteristic of $R/J(R)$ then R is a chain ring. On the other hand, if we assume that there exists a ring monomorphism σ from R to R_2 with $rm = m\sigma(r)$, $\sigma(m) = m$, where R_2 is a subring of R, then R can be embedded into a chain ring whose lattice of right ideals is isomorphic to its lattice of left ideals and is isomorphic to the lattice of right ideals of R. This result is used to solve the above extension problem in case R contains a division ring of representatives of $R/J(R)$ and satisfies some additional condition.

2. Definitions and preliminaries. All rings considered in this paper have a unit element. A *right* (*left*) *chain ring* is a ring with a linearly ordered lattice of right (left) ideals. A ring which is a right and left chain ring is called a *chain ring*. If every element in $J = J(R)$, the maximal ideal of a (right) chain ring, is a two-sided zero divisor, R is called a (*affine*) *projective Hjelmslev ring*, for short (*AH*-) *PH*-ring. We write $U(R)$ for the group of units of R. A ring is said to be *right invariant* (*invariant*) if $Ra \leqq aR$ ($Ra = aR$) holds for all a in R. More details about the incidence structures mentioned in the introduction can be found in [**1**] and [**11**].

Our problem can be formulated as follows: Let R be a right chain ring. Does there exist a ring extension S of R which is a chain ring and satisfies the

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Received June 8, 1977 and in revised form April 13, 1978.
The first author was supported in part by the National Research Council of Canada under Grant No. A-7254. The second author wishes to thank the Deutsche Forschungsgemeinschaft and the University of Alberta for partial support.

following condition (1)?

(1) $U(S) \cap R = U(R)$; and for any a in S there exists an s in $U(S)$ with as in R.

This condition will guarantee that the lattices of right ideals of R and S respectively are isomorphic. If R is a right noetherian right chain ring with at least two nonzero prime ideals $R > xR > yR \neq (0)$ we have $xyR = yR$, but $Ry \not\supseteq Ryx \not\supseteq Ry$. This implies that for such a ring R no extension in the above sense exists (see [**4**]).

We therefore consider the following two types of right chain rings.

(2) $J(R)$ is the only prime ideal of R;

(3) $J(R)$ and (0) are the only prime ideals in R.

3. The case: char (*R*) $\neq$ char (*R*/*J*). We assume in this section that R is a right chain ring of type (2) or (3) satisfying

(4) char $(R) \neq$ char (R/J).

This property implies the existence of a central element $z \neq 0$ in R, contained in $J(R)$.

3.1. THEOREM. *Every right chain ring R with* (4) *of type* (2) *or* (3) *is right invariant.*

Proof. If R is of type (2) it follows that the elements in $J(R)$ are nilpotent and this together with the assumption that R is a right chain ring implies that R is right invariant. Now let (0) and J be the only prime ideals of R and let z be a nonzero element in J with $zR = Rz$. If $J = zR$ it follows that R, z^iR, $i = 1, \ldots$ and (0) are the only right ideals of R and R is right invariant. Otherwise we form the intersection L of all two-sided nonzero ideals of R. Two-sided ideals $Z \neq (0)$ lead to right chain rings R/Z of type (2) and this implies that every right ideal $I \supseteq L$ is a two-sided ideal in R. We are therefore left with the case $L \neq (0)$. It follows that L is not a prime ideal and elements a, b not in L exist with ab in L and $aRb \neq (0)$. We obtain $L = abR$, since abR is a two-sided ideal, and since abJ is a two-sided nonzero ideal as well, $L = abR = abJ$ follows. This implies $L = (0)$, a contradiction, and proves the theorem. (See [**5**] for related problems and results.)

3.2. COROLLARY. *A prime right chain ring satisfying* (3) *and* (4) *has no zero divisors.*

We need the result that the semigroup H of principal right ideals of a right chain ring R of type (2) is commutative (see also [**3**; **7**]). H is a linearly ordered semigroup and $h_1 \leqq h_2$ holds for elements $h_1 = aR$, $h_2 = bR$ if and only if $aR \geqq bR$. H has a unit element $e = R$ and a largest element $0 = (0)$.

It follows that $h_1 \leqq h_2$ holds if and only if there exists an element h_3 in H with $h_1h_3 = h_2$. In addition, the cancellation laws hold in the following form:

(i) $h_1h_2 = h_1h_3 \neq 0$ implies $h_2 = h_3$, and
(ii) $h_2h_1 = h_3h_1 \neq 0$ implies $h_2 = h_3$.

To prove this let $h_1 = aR$, $h_2 = bR$ and $h_3 = cR$ and assume $b = cd$ for d in $J(R)$. This leads to the contradiction $acdR = acR$ in the case (i). In the case (ii) one obtains $cdaR = caR$ and using (i), $daR = aR$ follows. The ideal $I = \{r \in R; raR \subsetneqq aR\}$ is a prime ideal different from $J(R)$ and (0) and this contradiction proves (ii).

3.3. LEMMA. *The semigroup H of a right chain ring of type* (3) *is commutative.*

Proof. The result is obvious if $H' = H\backslash\{e\}$ contains a least element. We can assume that H' does not have a smallest element. Let h_1, h_2 be two elements with $h_1h_2 \neq h_2h_1$. If we assume $h_1h_2 < h_2h_1 \neq 0$ we proceed as follows: $h_2h_1 = h_1h_2c$; c in H'. There exists an element z in H' with $z^2 \leqq c$, $z \leqq h_1$, $z \leqq h_2$ and integers m, n with $z^m \leqq h_1 < z^{m+1}$ and $z^n \leqq h_2 < z^{n+1}$. We obtain $h_2h_1 = h_1h_2c \geqq z^{m+n+2} > h_2h_1$, a contradiction. If we assume $h_1h_2 < h_2h_1 = 0$ we consider first the case $h_2 < h_1$. Then there exists $k \geqq 1$ with $h_2{}^k < h_1 \leqq h_2{}^{k+1}$ and $h_1 = h_2{}^kh$ for some h in H'. We get $h \leqq h_2$, and $h_2h \leqq h_1 < 0$ and $hh_2 \leqq h_1h_2 < 0$ follows. Application of the first part shows that h and h_2 and therefore h_2 and h_1 commute. It remains to consider the case $h_1h_2 < h_2h_1 = 0$ and $h_1 < h_2$. As before we obtain an integer $k \geqq 1$ and an element h in H' with $h_1{}^k < h_2 \leqq h_1{}^{k+1}$ and $h_2 = h_1{}^kh$, and as before, $h \leqq h_1$. We see that $h_1h \neq 0$ and if $hh_1 = 0$ we apply the previous argument with $h < h_1$ to prove that h_1 and h commute.

We can now prove the main result of this section.

3.4. THEOREM. *Let R be a right chain ring of type* (2) *or* (3) *with finitely generated maximal ideal $J = mR$ and* char $(R) \neq$ char (R/J). *Then R is a chain ring.*

It is sufficient to prove this result for chain rings of type (2). The next lemma leads immediately to the proof of the theorem and can actually be used to prove the above result for a larger class of right chain rings (see Remark 3.6).

3.5. LEMMA, *Let R be a chain ring of type* (2). *We assume further that there exists an element m in J with $0 \neq Rm^k = m^kR$ for some $k \geqq 1$ and that*

(5) $\quad (m^k)^r = \{a \in R; m^ka = 0\} \leqq mR.$

Then $mR = Rm$.

Proof. We define a sequence of subrings R_i of R in the following fashion:

$$R = R_1, \quad R_{i+1} = \{b \text{ in } R; \exists\, a \text{ in } R_i \text{ with } am = mb\}.$$

It follows that the R_i form a descending chain:

$$R \geqq R_1 \geqq R_2 \geqq \ldots \geqq R_i \geqq R_{i+1} \geqq \ldots$$

The associated semigroup H of all the principal right ideals of R is commutative. This implies that elements a, b in R with $ab = 0$ commute. In particular $am = 0$ implies $ma = 0$ and a is contained in $\bigcap R_i$ together with the element m. Let n be the nilpotency index of m, i.e. $m^n = 0$, $m^{n-1} \neq 0$. Using the above notation we have $Rm^k = m^k R_{k+1} = m^k R$. For each a in R exists therefore an element b in R_{k+1} with $m^k a = m^k b$ and $a - b$ in $(m^k)^r \leqq mR$ follows.

We prove, using induction on j, that $m^j R \leqq R_{k+1}$ holds for $j = n - 1, \ldots, 1, 0$.

The containment $m^{n-1}R \leqq R_{k+1}$ is trivial. We assume $m^{j+1}R \leqq R_{k+1}$. Let $r = m^j a$ be an element in $m^j R$. Then there exists an element b in R_{k+1} with $a - b$ in mR, say $a - b = mc$ for some c in R. This leads to $r = m^j b + m^{j+1}c$ which is an element in R_{k+1} using induction. We conclude that $R = R_{k+1} = R_2$ and $Rm = mR$ follows.

3.6. *Remark.* The statement in Theorem 3.4 remains true for right chain rings of type (2) or (3) satisfying (4) as long as the associated semigroup of principal right ideals is isomorphic to one of the following semigroups:

(i) $(Q, +)$; (ii) $(Q, +) \cap [0, 1]$; (iii) $(Q, +) \cap [0, 1]) \cup \{\infty\}$.

In addition, it must be assumed that the principal ideal generated by the central element (whose existence is guaranteed by (4)) is not the upper neighbour of the zero ideal. Condition (5) in Lemma 3.4 will then be satisfied and an abritrary principal right ideal aR can be obtained from mR by either "taking roots" $((aR)^n = bR)$ or by using powers of certain right ideals. One obtains $aR = Ra$ for arbitrary a in R.

4. Two embedding theorems. We can now restrict ourselves to the case in which

(6) $\quad \operatorname{char}(R) = \operatorname{char}(R/J)$

is satisfied.

We begin with the solution of our problem for right invariant right chain rings of type (3). We need the result, that the semigroup of nonzero principal right ideals of such a ring R is commutative (Lemma 3.3 and [**3**]). It is obvious that R is an integral domain and embeddable in a division ring of quotients $Q(R)$.

4.1. THEOREM. *Let R be a right invariant right chain ring of type* (3). *Then $S = \bigcup R_a$, $0 \neq a$ in $J(R)$, $R_a = aRa^{-1}$, is a chain ring extension of R and the lattices of right ideals in R and S respectively are isomorphic.*

Proof. Since R is right invariant, $Ra \leqq aR$ follows for every element a in R. But, the multiplication of principal right ideals is commutative which implies that $J(R)a \leqq aJ(R)$. From this we conclude that $U(S) \cap R = U(R)$ holds; otherwise there exist nonzero elements x, a in $J(R)$ with x^{-1} in R_a. This leads to $x^{-1} = ara^{-1}$ and $a = xar = ax'r$ for some x' in $J(R)$, r in R, and the contradiction $a = 0$, To prove the second part of condition (1) (Section 2) for S let $y = bxb^{-1}$ be an element in S for some x in R, b in $J(R)$. Then there exist a unit t in R with $xb = bxt$ and $y(btb^{-1}) = x$ for the unit btb^{-1} in S. One checks, by computing it directly, that $Sy \leqq yS \leqq Sy$ for all y in S, and it follows that S is an invariant chain ring, satisfying condition (1).

The first example of an AH-ring which is not a PH-ring was probably given by Baer in [**2**] using an idea of Ore in [**9**]:

Let F be a commutative field with a monomorphism σ which is not an isomorphism. The vector space $F \oplus F$ can be made into a ring E using the multiplication $(a, b)(a', b') = (aa', a^\sigma b' + ba')$. The right ideals of E are $E \supset I = \{(0, b); b \in F\} \supset (0)$ and E is a right chain ring, but $E(0, b) \not\supseteq E(0, b^\sigma) \not\supseteq E(0, b)$ for b in $F \backslash F^\sigma$.

The next result gives a solution to our problem for right chain rings of type (2) (or (3)) if the maximal ideal is a principal right ideal and an additional condition is satisfied:

4.2. THEOREM. *Let R be a right chain ring of type* (2) (*or* (3)) *with finitely generated maximal right ideal $J = mR$. We assume further:*

(7) *There exists a monomorphism σ from R into R with $\sigma(m) = m$ and $rm = m\sigma(r)$.*

Then there exists a chain ring S satisyfing condition (1) *and solving our embedding problem.*

Proof. If $R = R^\sigma$ we can take $S = R$. Otherwise we consider a set S_1 which is the disjoint union of the set R and a set T with $T = \{t_k; k \in R \backslash R^\sigma\}$. We can extend the mapping σ to a one to one and onto mapping σ_1 from S_1 to R by mapping t_k in S_1 to k in R. This mapping can be used to define a ring structure on S_1 and σ_1 is then an isomorphism between S_1 and R, and S_1 contains R as a subring. This process is repeated and we obtain a sequence of rings

$$R = S_0 \subset S_1 \subset S_2 \subset \ldots$$

with isomorphisms σ_i from S_i to S_{i-1} with $\sigma_0 = \sigma$ and σ_{i+1} is an extension of σ_i. The lattice of right ideals in S_i is still of the form

$$S_i \supset mS_i \supset \ldots \supset m^n S_i \supset m^{n+1} S_i \supset \ldots \supset (0) \quad \text{and}$$
$$rm = m\sigma_i(r) \text{ holds for } r \text{ in } S_i.$$

This last statement is proved by induction on i:

$$rm = \sigma_i^{-1}(\sigma_i(r)m) = \sigma_i^{-1}(m\sigma_{i-1}(\sigma_i(r)) = m\sigma_i(r).$$

We form the ring $S = \bigcup S_n \supset R$. This ring is a local ring with maximal ideal mS. For an element r in S there exists an index i and an element q in S_{i+1} with

$\sigma_{i+1}(q) = r$ in S_i. This implies $Sm = mS$ and it follows that S is a chain ring containing R and satisfying condition (1).

This result can be applied immediately to the following situation:

4.3. COROLLARY. *Assume R is a right chain ring of type (2) with finitely generated maximal ideal $J = mR$. If R contains a division ring D of representatives of R/J and $dm = m\sigma(d)$ holds for any element d in D, with $\sigma(d)$ in D also, then R is embeddable in a chain ring S satisfying* (1).

Let R be a right chain ring as in Corollary 4.3 without the special condition that $\sigma(d)$ is again in D. We will then have the more general equation $dm = m(d_1 + md_2 + \ldots + m^{n-2}d_{n-1})$ with d_j in D for $j = 1, \ldots, n-1$ where n is the nilpotency index of m. This case will be treated in the next section.

5. *AH*-rings as skew polynomial rings. In this section the following assumptions are made: R is an AH-ring of type (2), R contains a skew field D of representatives of R/J and $J = mR$ is finitely generated as a right ideal. Finally, let n be the nilpotency index of m.

The multiplication in R is determined by

(8) $$dm = md^{\delta_1} + m^2d^{\delta_2} + \ldots + m^{n-1}d^{\delta_{n-1}}$$

where the δ_i are mappings from D into D. Since R is a right vector space with basis $\{1, m, \ldots, m^{n-1}\}$, it is obvious that the δ_i's are endomorphisms of the additive group of D; δ_1 is a monomorphism from D into D. We will use the notation and some arguments from [**10**]. If we put $a^{\delta_i} = a_i$ and

$$a_{(k,t)} = \sum_{\substack{j_1+\ldots+j_k=t \\ j_i=1,\ldots,n-1}} a_{j_1 j_2 \ldots j_k} \quad (a_{(k,t)} = 0 \text{ for } k > t)$$

one obtains

(9) $$(ab)_i = \sum_{k=1}^{i} a_{(k,i)}b_k \quad \text{for } i = 1, \ldots, n-1,\ a, b \text{ in } D.$$

The following identity, needed later, can be easily checked:

(10) $$\sum_{i=0}^{n-1} a_{i+1(w,n-1-i)} = a_{(w+1,n)} \quad \text{for } w = 1, 2, \ldots.$$

We would like to apply Theorem 4.2 to solve our embedding problem for a ring R satisfying the assumptions listed at the beginning of this section.

This means that a monomorphism σ from R into a subring S_1 of R must be found with $\sigma(m) = m$ and $rm = m\sigma(r)$ for r in R.

This we could do under an additional assumption on the mappings δ_i:

(11) $$D^{\delta_2} \subseteq D^{\delta_1} \quad \text{and} \quad \delta_{i+1} = \delta^i\delta_1 \quad \text{with } \delta = \delta_2\delta_1^{-1}, \quad i = 1, \ldots, n-2.$$

Assuming (11) one checks that the following identity is true:

(12) $$a_{1(k,t-1)} + a_{(k+1,t-1)} = a_{(k+1,t)}, \quad a \text{ in } D.$$

Given an element A in R. Then A can be written as

$$A = a(0) + ma(1) + \ldots + m^{n-1}a(n-1) \quad \text{with } a(i) \text{ in } D.$$

We have $Am = mB$ for some element B in R and $B = \sum_{i=0}^{n-1} m^i b(i)$, but only the $b(i)$'s for $i = 0, \ldots, n-2$ are uniquely determined by A:

$$(13) \quad b(i) = a(i)_1 + a(i-1)_2 + \ldots + a(0)_{i+1} \quad (0 \leqq i \leqq n-2).$$

In order to make (13) a valid equation for $n-1$ as well, we define a mapping δ_n from D into D by

$$\delta_n = \delta^{n-1}\delta_1.$$

It is now possible to prove (9) for $i = n$ and a monomorphism σ from R to R with $rm = m\sigma(r)$ can be given.

We have:

$$\begin{aligned}
(14) \quad (ab)_n &= (ab)^{\delta\delta_{n-1}} = (ab)^{\delta_2\delta_1^{-1}\delta_{n-1}} \\
&= (a_2b_1 + a_{11}b_2)^{\delta_1^{-1}\delta_{n-1}} \quad \text{(using (9))} \\
&= (a^\delta b + a_1 b^\delta)^{\delta_{n-1}} \quad \text{(using (11))} \\
&= \sum_{k=1}^{n-1} a^\delta_{(k,n-1)} b_k + \sum_{k=1}^{n-1} a_{1(k,n-1)} b_k^\delta \quad \text{(using (9))} \\
&= \sum_{k=1}^{n} a_{(k,n)} b_k \quad \text{(using (11) and (12))}.
\end{aligned}$$

We claim that

$$(15) \quad \sigma(m^i a) = m^i a_1 + m^{i+1}a_2 + \ldots + m^{n-1}a_{n-i}; \quad i = 0, \ldots, n-1$$

defines a monomorphism from R to R with $\sigma(m) = m$, $rm = m\sigma(r)$.

Let a, b be elements in D. We will show that $\sigma(ab) = \sigma(a)\sigma(b)$. It is enough to prove that $(ab)_n$ equals the coefficient of m^{n-1} in $\sigma(a)\sigma(b)$.

Let

$$\sigma(a) = \sum_{v=0}^{n-1} m^v a_{v+1} \quad \text{and} \quad \sigma(b) = \sum_{w=0}^{n-1} m^w b_{w+1}.$$

Using (9) we obtain

$$\sigma(a)\sigma(b) = \sum_{v=0}^{n-1} m^v \sum_{w=0}^{n-1} \sum_{s=0}^{n-1} m^s a_{v+1(w,s)} b_{w+1}.$$

The coefficient of m^{n-1} is therefore equal to

$$\sum_{w=0}^{n-1} \sum_{v=0}^{n-1} a_{v+1(w,n-1-v)} b_{w+1},$$

since $v + s = n + 1$. If we apply (10) we see that this expression is equal to

$$\sum_{w=0}^{n-1} a_{(w+1,n)} b_{w+1} = \sum_{w=1}^{n} a_{(w,n)} b_w = (ab)_n.$$

The mapping σ defined by (15) is therefore a homomorphism from R into R with $\sigma(m) = m$, $rm = m\sigma(r)$. That σ is also a one-to-one mapping is obvious. We obtain therefore the following result:

5.1. THEOREM. *Let R be an AH-ring of type* (2) *containing a skew field D of representatives of R/J, where $J = mR$ is the maximal ideal of R. Let*

$$am = ma^{\delta_1} + m^2a^{\delta_2} + \ldots + m^{n-1}a^{\delta_{n-1}} \quad \textit{for } a \textit{ in } D,$$

and assume that $D^{\delta_2} \subseteq D^{\delta_1}$ and $\delta_i = \delta^{i-1}\delta_1$ holds for $\delta = \delta_2\delta_1^{-1}$ and $i = 1, \ldots, n-1$. Then R can be embedded into a chain ring S satisfying (1).

5.2. *Remark.* The assumption $\delta_i = \delta^{i-1}\delta_1$ is always true if the nilpotency index n of m is equal to 3.

5.3. *Remark.* Let R be given as in the beginning of this section. Then $R/m^{n-1}R$ is embeddable in a chain ring.

Example [**6**; p. 38]. Let $K[y; \alpha, \delta]$ be a skew polynomial ring over a (skew) field K with monomorphism α and an α-derivation δ. Let $ya = a^\alpha y + a^\delta$. In the quotient field $K(y; \alpha, \delta)$ consider the subring generated by K and y^{-1}. We obtain with $x = y^{-1}$ the following:

$$ax = xa^\alpha + xa^\delta x = xa^\alpha + x^2s^{\delta\alpha} + x^2a^{\delta^2}x = \ldots$$

We see that $K[x]/(x^n) = R$ provides us with an example of a ring R satisfying the conditions of Theorem 5.1.

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