Primes and Completions of Right Chain Rings

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1 Introduction

A commutative valuation ring V which has no proper extension with the same value group and the same residue field as V is called maximal (KRULL [Kr32]). Such valuation rings can also be characterized by the fact that every pseudo convergent sequence in V has a limit in V, that is, V is maximally complete (KAPLANSKY [Ka42]).

A *right chain domain* R is a not necessarily commutative integral domain such that $aR \subseteq bR$ or $bR \subset aR$ for any elements $a, b \in R$. The concepts of maximal and maximally complete extend in a natural way to right chain domains and a maximally complete right chain domain is maximal but we will discuss an example of a maximal right chain domain that is not maximally complete; see section 4.

Essential for the understanding of the ideal theory of right chain rings R is the understanding of the ideals between two neighbouring completely prime ideals $P_1 \supset P_2$; such a pair (P_1, P_2) with no further completely prime between them is called a *prime segment* of R. Prime segments fall in exactly one of three classes: they are either invariant, simple, or exceptional and the last two types do not occur if the skew field of quotients of R is finite dimensional over its center.

On the other hand we are concerned with completions of a right chain ring R and say that R is I-complete (or I-compact) for a right ideal $I \neq R$ if the natural mapping of R into the inverse limit of R/I_λ is onto where I_λ, $\lambda \in \Lambda$, $(I_\alpha \supseteq I_\beta$ if $\alpha \leq \beta$ in $\Lambda)$ is any family of right ideals of R with $\bigcap_{\lambda\in\Lambda} I_\lambda = I$; this can also be expressed in terms of the existence of limits of pseudo convergent sequences and the definitions clearly can be extended to the right R-modules of the form R/I for a right ideal I of R.

If a right chain domain R is I-complete, then R is necessarily I'-complete for a certain set $C(I)$ of right ideals I' of R that contains all right ideals of the form aI for

*Supported in part by NSERC

$0 \neq a \in R$. More generally, if $s^{-1}B = t^{-1}A = \{r \in R \mid tr \in A\}$ for proper right ideals A and B of R and $s \notin B, t \notin A$ then A and B are called *related*, and it follows that R is A-complete if and only if R is B-complete. It is also proved that the union of right ideals related to I is equal to $P(I) = \{p \in R \mid Ip^{-1} \supset I\}$, the complete prime ideal of R associated with I.

As one of the consequences of this result it follows that $\mathcal{N}$, the family of right ideals I of R for which R/I is not maximally complete, is either empty or there exists a completely prime ideal P with $\mathcal{N} = \{I \mid I \subset P\}$ or $\mathcal{N} = \{I \mid I \subseteq P\}$. As another consequence one obtains the following extension of a theorem by (RIBENBOIM [Ri64]) for commutative valuation rings, see also (MAC LANE [McL38]). THEOREM: *Let R be a right chain domain with $P \neq P^2$ for all completely prime ideals $P \neq (0)$ of R. Then R is maximally complete if and only if R is P-complete for all completely prime ideals.*

Finally, we construct chain domains that are I-complete for a prescribed set of right ideals and are not B-complete for all right ideals outside this set by imposing certain conditions on the support of elements in a Malcev-Neumann power series ring.

2 Prime ideals

Let R be a right chain domain. We denote with $J = J(R)$ the unique *maximal right* and left *ideal* and it follows that $U = U(R) = R\backslash J$ is the *group of units* of R and R/J is the residue skew field. As usual, an ideal $P \neq R$ of R is called *prime* if $I_1I_2 \subseteq P$ for ideals I_1, I_2 of R implies $I_1 \subseteq P$ or $I_2 \subseteq P$. If the same property holds for elements of R instead of ideals, P is called *completely prime*.

LEMMA 2.1 *Let R be a right chain domain. Then:*

(i) *An ideal $I \neq R$ is completely prime if $x^2 \in I$ implies $x \in I$ for $x \in R$.*

(ii) *Let $I \neq R$ be an ideal in R. Then $P = \bigcap_n I^n$ is a completely prime ideal.*

PROOF: (i) Assume that the condition holds for I and that $xy \in I$ for $x, y \in R$. If $x = yz$ for $z \in R$ then $x^2 = xyz \in I$ and $x \in I$. If $y = xz$ then $(yx)^2 \in I$ implies $yx \in I$ and with the previous argument $y \in I$.

(ii) If $I = I^2$, then $P = I$ and we assume $x^2 \in I$, $x \notin I$ for an element $x \in R$. Then $I \subseteq xJ$ and $I = I^2 \subseteq xI \subseteq x^2J \subset x^2R \subseteq I$, a contradiction that shows that $I = P$ is completely prime in this case. If $P = I^n$ for some $n \geq 2$, then $P = I^n = (I^n)^2$ is completely prime by the previous argument. Finally, if $P \subset I^n$ for all n, and $x \notin P$ then $I^n \subset xR$ for some n and $P \subset I^{2n} \subseteq xI^n \subseteq x^2R$ and $x^2 \notin P$ follows. ∎

We consider prime segments (P_1, P_2) of R, i.e. $P_1 \supset P_2$ are completely prime ideals of R with no further completely prime ideal of R between P_1 and P_2. By considering $L = \bigcup I$, the union of ideals of R properly contained in P_1 it can be proved that the following three possibilities arise (see [BBT90]):

THEOREM 2.2 *Let (P_1, P_2) be a prime segment of a right chain domain. Then one of the following alternatives occurs:*

(a) *$L = P_1$ if and only if $P_1a \subseteq aP_1$ for all $a \in P_1 \setminus P_2$; the prime segment is called* invariant *in this case.*

(ii) *Let A be a proper right ideal in R. Then $P(A)$ is equal to the union of the right ideals of R related to A.*

PROOF: (i) The first part is proved in Lemma 3.1 in [BT92], the second in Theorem 4.15 in (see [BT89]).

(ii) If B is related to A then $t^{-1}B = s^{-1}A$ for some $t \notin B$, $s \notin A$. For $b \in B$ there exists $w \in t^{-1}B$ with $tw = b$ and $sw \in A$. Hence, $w \in P(A)$ and $b \in P(A)$, since $P(A)$ is an ideal. Conversely, let $z \in P(A)$; hence there exists $t \notin A$ with $tz \in A$ and $z \in t^{-1}A$ follows. This proves (ii). ■

We can apply these results to *discrete right chain domains* R. This is a right chain domain such that $P^2 \neq P$ for every completely prime ideal $P \neq (0)$ of R.

THEOREM 3.2 *Let R be a discrete right chain domain, then R is maximally complete if and only if R is P-complete for every completely prime ideal P of R.*

PROOF: Let A be any right ideal of R and we want to show that R is A-complete. This is clear if $A = (0)$. Otherwise let $P = P(A)$ be the associated prime ideal of A and $P \neq P^2$ by assumption. It follows from Lemma 2.1(ii) that $P' = \bigcap_n P^n$ is a completely prime ideal and that (P, P') is an invariant segment using Theorem 2.2 and the fact that $P_1 = P_1^2$ if (P_1, P_2) is a prime segment of type (b) or (c). We are done by Proposition 3.1(i) if we can show that P is actually related to A. Since $P \supset P^2$ there exists by Proposition 3.1(ii) a right ideal B related to A with $P \supseteq B \supset P^2$ and $P(B) = P$. If $P \supset B \supset P^2$ then there exists $x \in P \setminus B$ and $b \in B \setminus P^2$ and $xt = b$ for some $t \in R \setminus P$ (because $b \notin P^2$). Since $b \in B$ and $x \in R \setminus B$, $xt = b$ implies $t \in P(B) = P$ by definition of $P(B)$. But we know that $t \notin P$. This is a contradiction that proves $B = P$, and R is A-complete since A is related to P. ■

This result was proved in the commutative case by MAC LANE [McL38]. Valuation rings that are P-complete for every prime ideal P are closely related to the perfect valuation rings considered by KRULL [Kr32] and those that are 'complet par etages' as defined by RIBENBOIM [Ri64].

If R is a right chain ring and I a right ideal, then R/I is a right R-module whose lattice of submodules is totally ordered. The earlier introduced notion of maximally complete right chain domain carries over to right R-modules with the just mentioned property. One obtains the following result:

THEOREM 3.3 *Let $\mathcal{N}$ be the family of right ideals I of a right chain ring R such that R/I is not maximally complete. Then $\mathcal{N} = \emptyset$ or there exists a completely prime ideal P of R with $\mathcal{N} = \{I \mid I \subset P\}$ or $\mathcal{N} = \{I \mid I \subseteq P\}$.*

PROOF: Let I be a right ideal of R such that R/I is not maximally complete. Then R is not L-complete for some right ideal L of R with $I \leq L$. We consider the set $\mathcal{N}$ of right ideals L' of R with R not L'-complete. It follows by the above observation that the union V of right ideals $L' \in \mathcal{N}$ is equal to the union of right ideals I of R with R/I not maximally complete.

From Proposition 3.1 we conclude that the set $\mathcal{N}$ consists of classes of related right ideals and the union of elements in a class is a completely prime ideal. Hence, V itself is the union of completely prime ideals which implies $V = P$ for a completely prime

ideal if $\mathcal{N}$ is not empty. We therefore have the three alternatives listed in the theorem: The ring R and hence R/I is maximally complete for every I, i.e. $\mathcal{N} = \emptyset$. The ring R/P is not maximally complete and $\mathcal{N} = \{I \mid I \subseteq P\}$. Finally, $P \notin \mathcal{N} = \{I \mid I \subset P\}$.
■

We close this section by showing that an almost maximally complete right chain ring with zero-divisors is actually complete. As was observed above the notion of I-complete and maximally complete extend immediately to right chain rings which are not necessarily domains. A right chain ring R is called *almost maximally complete* if it is I-complete for all right ideals I with the possible exception of $I = (0)$.

If $a \neq 0 \neq b$ in a right chain ring R with $ab = 0$ then $I = a^{-1}(0) \ni b$ is a nonzero right ideal of R related to (0). By Proposition 3.1(i) it follows that R is (0)-complete if it is I-complete. In the commutative case this was proved by GILL (see [Gi71]).

4 Examples and applications

We use the results in the previous sections to discuss the example of a maximal right chain domain R that is not maximally complete, and examples of right chain domains that are I-complete exactly for the right ideals in a predetermined set.

EXAMPLE 4.1 Let $\mathbb{Q}(t_1, t_2, \ldots)$ be the function field over the field $\mathbb{Q}$ of rational numbers in the variables $t_i, i = 1, 2, \ldots$ and $\mathbb{Q}(t_1, t_2, \ldots)[x]_{(x)}$ is the localization at the prime ideal (x) of the polynomial ring $\mathbb{Q}(t_1, t_2, \ldots)[x]$; it is a discrete valuation ring with a monomorphism σ defined by $\sigma(x) = t_1,\ \sigma(t_i) = t_{i+1}$.
Finally,

$$R = \mathbb{Q}(t_1, t_2, \ldots)\,[x]_{(x)}\,[[y, \sigma]] = \{\sum_{n=0}^{\infty} y^n f_n\,(t_i, x) \mid f_n\,(t_i, x) \in Q(t_1, t_2, \ldots)\,[x]_{(x)}\}$$

with $f_n(t_i, x)y = y\sigma(f_n(t_i, x))$, the skew power series ring in the variable y, is a discrete right invariant right chain domain with $\{y^n x^m R \mid n, m \geq 0\}$ the set of right ideals $\neq (0)$. This ring has rank 2 and $R \supset xR \supset yR \supset (0)$ is its chain of completely prime ideals; in addition, R is xR as well as (0)-complete. By Theorem 3.2 it follows that R is maximally complete if and only if R is (y)-complete. However, $R/yR \cong Q(t_1, t_2, \ldots)\,[x]_{(x)}$ is not complete and hence, R is not (y)-complete and not maximally complete. Before we indicate why R is a maximal right chain domain we recall the definition of an immediate extension.

Let R be a right chain domain. We say that a ring extension $R_1 \supseteq R$ is an *immediate extension* of R if the follwing conditions hold:

(i) Every element $a_1 \in R_1$ has the form $a_1 = a \cdot u_1$ for $a \in R$ and a unit $u_1 \in U(R_1)$.

(ii) $\varphi(r + J(R)) = r + J(R_1)$ defines an isomorphism between the skew fields $R/J(R)$ and $R_1/J(R_1)$.

It follows that for any right chain domain R there exist maximal immediate extensions and that every element $z \in R_1$ where R_1 is an immediate extension of R, is the limit of a pseudo convergent sequence in R that has no limit in R.

We return to our example R above and we know that R is not maximally complete. We want to show that on the other hand R has no proper immediate extension $R_1 \supset R$. Otherwise such a ring R_1 is again a right invariant right chain domain of rank two with $\{y^n x^m R \mid n, m \geq 0\}$ as its set of non-zero right ideals; however, $R_1/yR_1 \supset R/yR$. The existence of an element $f \in R_1/yR_1$ which is not in R/yR leads to the contradiction that x^k is a unit in R for some $k > 0$. Hence, no such proper immediate extension R_1 of R exists, i.e. R is maximal even though it is not maximally complete.
We don't know the answer to the following question.

PROBLEM 4.2 *Do there exist right and left chain domains which are maximal but not maximally complete?*

If R is a right chain domain with maximal ideal $J = J(R) \neq (0)$ and $0 \neq c \in R$, then $\bigcap_\lambda I_\lambda = cJ$ for a family of right ideals $I_\lambda \subset R$ implies $I_\lambda = cJ$ for an index λ since otherwise $I_\lambda \supseteq cR$ for all λ and $\bigcap_\lambda I_\lambda \supseteq cR \supset cJ$.The ring R is therefore trivially I-complete for every right ideal I of the form $cJ \neq (0)$.

A right ideal $I \neq R$ of R is either of the form $cJ \neq (0)$ or the set $L_I = \{aR \mid aR \supset I\}$ has no last element, i.e. L_I is a limit point of $W(R)$, the totally ordered set of principal ideals of R with $aR \leq bR$ if and only if $bR \subseteq aR$. Here we say that a convex subset L of a totally ordered set $(W, \geq)$ with first element $e \in W$ is a *limit point* of W if $e \in L$ and L has no last element.

The set $W(R)$ of a right invariant rank one right chain domain forms a semigroup with ideal multiplication as operation that can be identified with a subsemigroup D of $(\mathbb{R}^+, +)$ the non-negative real numbers under addition. With D' we denote the closure of D in $\mathbb{R}^+$; it follows that $D' = \mathbb{R}^+$ or $D' = \mathbb{N}_0$ in the discrete case. A limit point L of D is therefore given by a real number $0 \neq \rho \in D'$ with $L = \{b \in D \mid b < \rho\}$ or $L = D$.

We now consider a right ideal $I \neq cJ$ in a right chain ring R. Let $P_1 = P(I)$ be the prime ideal associated with I. We assume further that P_1 has a lower neighbour P_2 in the chain of completely prime ideals and that the rank one valuation ring $\overline{R}(P_1, P_2) = R_{P_1}/P_2R_{P_1}$ associated with the prime segment (P_1, P_2) is right invariant with value semigroup D and completion $D' \subseteq \mathbb{R}^+$. Then either $I = dP_1$ for some non-zero $d \in R$ or there exists a right ideal I_1 of R related to I with $P_1 \supset I_1 \supset P_2$ and $\overline{I}_1 = I_1\overline{R}(P_1, P_2)$ defines a limit point of D which in turn is defined by a real number $\rho \in D'$.

We summarize these results:

LEMMA 4.3 *Let R be a right chain domain, (P_1, P_2) a prime segment of R and I a right ideal of R with $P(I) = P_1$. If the associated rank one chain domain ring $\overline{R}(P_1, P_2)$ is right invariant, then either $I = dP_1$ for some $0 \neq d \in R$ or there exists a real number ρ and a right ideal I_1 of R related to I with $P_1 \supset I_1 \supset P_2$ such that $L_{I_1} = \{aR \mid a\overline{R}(P_1, P_2) < \rho\}$.*

This lemma provides a satisfactory description of the right ideals of a right chain domain with maximum condition for completely prime ideals and for which all the rank one chain domains corresponding to prime segments of R are right invariant, we say that R is locally invariant in this case.

We don't have similar results if completely prime ideals exist that are the union of completely prime ideals, or nearly simple or exceptional rank one chain domains correspond to some prime segments.

We return to the question about the description of the set of right ideals I' for which the right chain domain R must be I'-complete if it is known that R is I-complete for all right ideals in a set $\mathcal{Z}$. We restrict ourselves to locally invariant right chain domains R with maximum condition on completely prime ideals. Each right ideal I is then either of the form $cJ \neq (0)$ and R is trivially I-complete, or of the distinguished form dP for a completely prime ideal P for d in R, or is related to a right ideal I_1 that corresponds to a real number ρ in one of these 'local maps' $D' \subseteq \mathbb{R}^+$ that describe the limit points of the corresponding right invariant rank one right chain domains. It is possible to obtain some results on the closure properties of the set of right ideals I for which R must be I-complete under these circumstances. Finally, chain domains can be constructed that are I-complete exactly for right ideals I in a predesigned set $\mathcal{V}$. We illustrate this in the next example.

EXAMPLE 4.4 Let $D = \mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0\} \subset D' = \mathbb{R}^+$. Let T be a subset of D' with the following properties: T does not contain 0, but $s \in D$ and $r \in T$ implies $s + r \in T$; if $s + \mu = \rho \in T$ for $s \in D$ and $\mu \in D' \backslash \{0\}$, then $\mu \in T$. We then consider $\hat{R} = \mathbb{Q}\{\{D\}\}$, the Malcev-Neumann ring of all power series $\alpha = \sum a_i d_i$ with the coefficients a_i in $\mathbb{Q}$ and d_i in D such that $support(\alpha) = \{d_i \mid a_i \neq 0\}$ is a well ordered subset of D. We define R_T as the subset of $\hat{R}$ consisting of those elements α in $\hat{R}$ such that L a limit point of D and $L \cap support(\alpha)$ cofinal in L implies $L = D$ or $L = \{d \in D \mid d < \rho\}$ for some $\rho \in T$. By computing the set of limit points for $\alpha + \beta, \alpha\beta$ and $\sum \gamma^n$ for $\alpha, \beta \in \hat{R}, \gamma \in J(\hat{R})$ with given sets of limit points, one can prove that R_T is a chain domain of rank one with D as associated semigroup of values and that R_T is I-complete exactly for $I = (0)$, corresponding to $L = D$, for $I = I\rho = \cap zR_T, z < \rho$ for $\rho \in T$, and trivially for $cJ(R_T)$ for all $0 \neq c \in R_T$ and no other right ideals. We give a concrete example for T in the present situation:

$$T_1 = \{r = q + n\sqrt{2} \in \mathbb{R}^+ \mid r > 0, q \in \mathbb{Q}, n \in \mathbb{N}\}$$

satisfies the above condition as does

$$T_2 = \{r = q + n_1\sqrt{2} + n_2\sqrt{3} \in \mathbb{R}^+ \mid r > 0, q \in \mathbb{Q}, n_i \in \mathbb{N}\}.$$

Results similar to those illustrated by this example are obtained in a forthcoming paper in the case where D is replaced by certain semigroups, socalled right invariant right cancellative right holoids of finite rank n. The set $\mathcal{V}$ is then essentially determined by a collection of n subsets $T_i \subseteq \mathbb{R}^+$ satisfying appropriate properties.

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