Quasilinear Maxwell Variational Inequalities in Ferromagnetic Shielding

joint work with Gabriele Caselli, Irwin Yousept

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Computational Methods in Applied Mathematics
29.08 - 02.09, 2022

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Variational Inequality in Ferromagnetic Shielding

Electromagnetic shielding
Effect of redirecting or blocking electromagnetic fields by barriers made of conductive or magnetic materials.

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Ferromagnetic shielding

Special case of Electromagnetic shielding: redirecting or blocking *magnetic fields* by ferromagnetic materials. Ferromagnetic materials are materials with high (relative) magnetic permeability, for example:

- Iron ($\mu/\mu_0 \approx 200.000$)
- Permalloy ($\mu/\mu_0 \approx 100.000$)
- Mu-metal ($\mu/\mu_0 \approx 50.000$)

To model the ferromagnetic shielding effect, we combine a Maxwell-structured elliptic VI of the first kind with a nonlinearity $\nu = \mu^{-1} \colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$, resulting in the problem

$$
\text{(VI)} \qquad \begin{cases} \text{Find } (A, \phi) \in K \times H_0^1(\Omega), \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl}(v - A) + \int_\Omega \nabla\phi \cdot (v - A) \geq \int_\Omega J \cdot (v - A) \quad \forall v \in K \\ \displaystyle\int_\Omega A \cdot \nabla\psi = 0 \quad \forall \psi \in H_0^1(\Omega) \end{cases}
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\& \qquad K := \{v \in H_0(\operatorname{curl}) \colon |\operatorname{curl} v| \leq d(\cdot) \text{ a.e. on } \Omega\}.
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- $\Omega \subseteq \mathbb{R}^3$ open, bounded, Lipschitz, simply connected
- $(J, d) \in L^2(\Omega) \times L^2(\Omega)$
- ν is ‘standard’, i.e. Carathéodory, strictly positive, bounded, strongly monotone and Lipschitz

We investigate:

- Is (VI) well-posed?
- How regular is its dual multiplier?
- Optimal control of (VI)

Main ingredient: A Yosida type penalization of (VI).

Optimal control of variational inequalities:

- V. Barbu
- F. Mignot and J.P. Puel
- M. Bergounioux
- ...
- K. Ito and K. Kunisch
- M. Hintermüller
- R. Herzog, C. Meyer and G. Wachsmuth

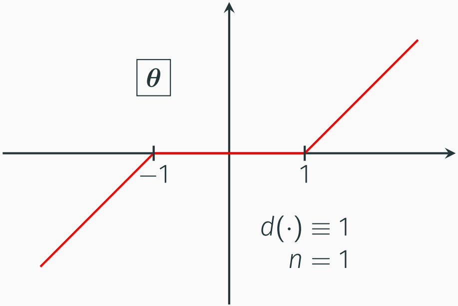
Optimal control of Maxwell-related PDEs:

- F. Tröltzsch
- A. Valli
- I. Yousept
- ...

Regularization of the Variational Inequality

We define the penalization term

$$\boldsymbol{\theta}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x,s) \mapsto \begin{cases} \max(|s| - d(x), 0)\dfrac{s}{|s|}, & s \neq 0 \\ 0, & s = 0. \end{cases}$$



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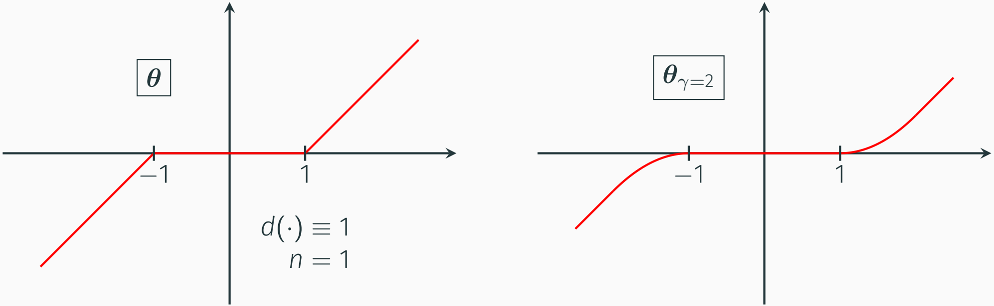
$$\boldsymbol{\theta}_\gamma : \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x, s) \mapsto \begin{cases} \max_\gamma(|s| - d(x), 0)\dfrac{s}{|s|}, & s \neq 0 \\ 0, & s = 0. \end{cases}$$



Vector-valued penalization and smoothing

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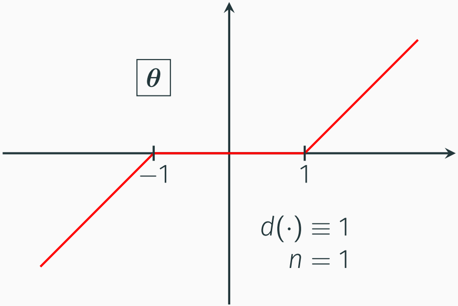

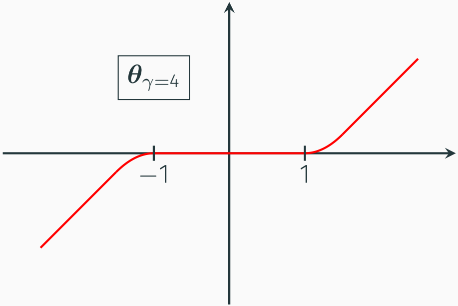

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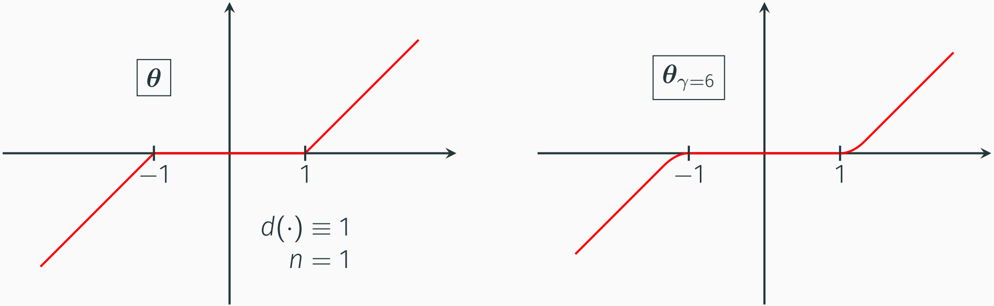

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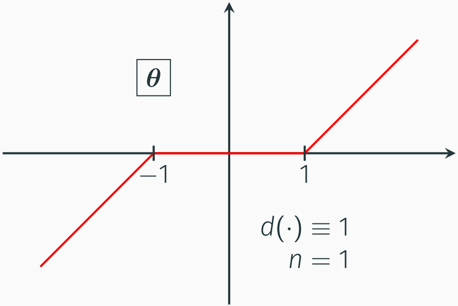

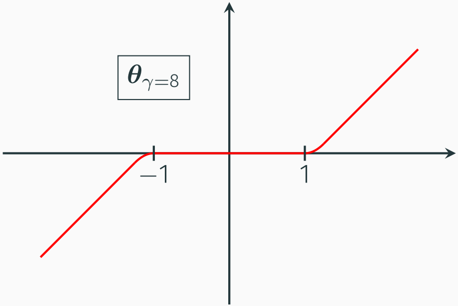

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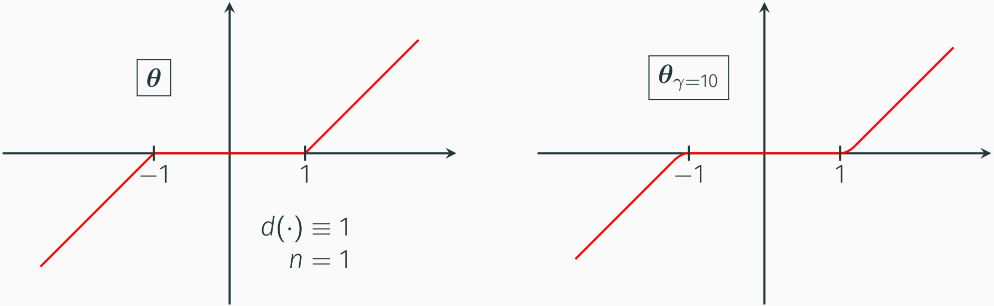

For $\gamma > 0$, we consider the regularized (unconstrained) problem

$$(\mathrm{VE}_\gamma^{\mathrm{sol}}) \qquad \begin{cases} \text{Find } \boldsymbol{A}_\gamma \in X_{N,0} := \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0), \text{ s.t.} \\ \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{v} + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v} \quad = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot \boldsymbol{v} \\ \forall \boldsymbol{v} \in X_{N,0}. \end{cases}$$

The regularized VI

For $\gamma > 0$, we consider the regularized (unconstrained) problem

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\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{v} + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v} \quad = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot \boldsymbol{v} \\
\forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}.
\end{cases}
$$

UDE

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Lemma

For every $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$, the regularized problem $(\mathrm{VE}_\gamma^{\mathrm{sol}})$ admits a unique solution $\boldsymbol{A}_\gamma$.

Left-hand side induces a monotone and coercive operator $X_{N,0} \to X_{N,0}^*$.

Convergence property of the regularization

Theorem

For $J_{\text{sol}} \in \boldsymbol{H}(\text{div}{=}0)$*, the unique solution* $\boldsymbol{A}_\gamma$ *of* $(\text{VE}_\gamma^{\text{sol}})$ *converges strongly in* $\boldsymbol{X}_{N,0}$ *to the unique solution of the problem*

$$(\text{VI}_{\text{sol}}) \qquad \begin{cases} \textit{Find } \boldsymbol{A} \in K \cap \boldsymbol{H}(\text{div}{=}0), \ \textit{s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl}(v - A) \geq \int_\Omega J_{\text{sol}} \cdot (v - A) \quad \forall v \in K \cap \boldsymbol{H}(\text{div}{=}0). \end{cases}$$

Well-Posedness and Regularity

Corollary

For every $J \in L^2(\Omega)$*, there exists a unique solution* $(A, \phi) \in K \times H_0^1(\Omega)$ *to* (VI)*. Moreover, there exists a unique multiplier* $m \in X_{N,0}$ *such that the solution* (A, ϕ) *is characterized by the dual formulation*

$$\begin{cases} \int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl} v + \nabla\phi \cdot v + \operatorname{curl} m \cdot \operatorname{curl} v = \int_\Omega J \cdot v \quad \forall v \in H_0(\operatorname{curl}) \\ \int_\Omega A \cdot \nabla\psi = 0 \quad \forall \psi \in H_0^1(\Omega) \\ \int_\Omega \operatorname{curl} m \cdot \operatorname{curl}(v - A) \leq 0 \quad \forall v \in K. \end{cases}$$

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How regular are the appearing multipliers?

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Theorem

Let $\partial\Omega$ be connected. For $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, let $(\boldsymbol{A}, \phi, \boldsymbol{m}) \in \boldsymbol{X}_{N,0} \times H_0^1(\Omega) \times \boldsymbol{X}_{N,0}$ denote the unique solution to the previous dual formulation. Then, the following multiplier regularity results hold true:

$$p \in [2,3],\ \boldsymbol{J} \in \boldsymbol{L}^p(\Omega),\ d \in L^p(\Omega) \Rightarrow \phi \in W_0^{1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

$$p \in [2,6],\ \boldsymbol{J} \in \boldsymbol{L}^p(\Omega),\ d \in L^p(\Omega),\ \Omega \text{ of class } \mathcal{C}^{1,1} \Rightarrow \phi \in W_0^{1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

$$p \in [2,\infty),\ \boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl}),\ d \in L^p(\Omega),\ \Omega \text{ of class } \mathcal{C}^{2,1} \Rightarrow \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

$$\boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl}),\ d \in L^\infty(\Omega),\ \nu(\cdot, |\mathbf{curl}\,\boldsymbol{A}|) \in \mathcal{C}^{0,1}(\overline{\Omega}),\ \Omega \text{ of class } \mathcal{C}^{2,1} \Rightarrow \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^\infty(\Omega)$$

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\begin{aligned}
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\end{aligned}
$$

The proof is mainly based on an L^p-Helmholz-decomposition and elliptic regularity theory.

Optimal Control

The optimal control problem

$$
(\mathrm{P}) \quad \begin{cases}
\min\limits_{(J,A)\in L^2(\Omega)\times X_{N,0}} \dfrac{1}{2}\|\operatorname{curl} A - B_{\mathrm{d}}\|^2_{L^2(\Omega)} + \dfrac{\lambda}{2}\|J\|^2_{L^2(\Omega)} \\
\text{subject to} \\
\displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl}(v - A) + \int_\Omega \nabla\phi \cdot (v - A) \geq \int_\Omega J \cdot (v - A) \quad \forall v \in K \\
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(\mathrm{P}) \qquad \begin{cases} \min\limits_{(J,A)\in H(\mathrm{div}=0)\times X_{N,0}} \dfrac{1}{2}\|\operatorname{curl} A - B_{\mathrm{d}}\|^2_{L^2(\Omega)} + \dfrac{\lambda}{2}\|J\|^2_{L^2(\Omega)} \\ \text{subject to} \\ \displaystyle\int_\Omega \nu(\cdot,|\operatorname{curl} A|)\operatorname{curl} A\cdot\operatorname{curl}(v-A) + \int_\Omega \nabla\phi\cdot(v-A) \geq \int_\Omega J\cdot(v-A) \quad \forall v\in K \\ \displaystyle\int_\Omega A\cdot\nabla\psi = 0 \quad \forall\psi\in H^1_0(\Omega). \end{cases}
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Theorem

There exists an optimal solution $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ *to the problem* (P).

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The solution mapping

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is weak-strong continuous.

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Task: Find optimality conditions for optimal controls $J^\star$.
Problem: The mapping G is (likely) not directionally differentiable.

The regularized optimal control problem

$$
(\mathrm{P}_\gamma) \qquad \begin{cases} \min\limits_{(J_\gamma, A_\gamma) \in H(\mathrm{div}=0) \times X_{N,0}} \dfrac{1}{2} \| \operatorname{curl} A_\gamma - B_\mathrm{d} \|^2_{L^2(\Omega)} + \dfrac{\lambda}{2} \|J_\gamma\|^2_{L^2(\Omega)} + \dfrac{\lambda}{4} \|J_\gamma - J^\star\|^2_{L^2(\Omega)} \\ \text{subject to} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \theta_\gamma(\cdot, \operatorname{curl} A_\gamma) \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \\ \forall v \in X_{N,0}. \end{cases}
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$$

The solution mapping

$$
G_\gamma : H(\mathrm{div}{=}0) \to X_{N,0}, \quad J_\gamma \mapsto A_\gamma
$$

is weak-strong continuous, i.e. there exists a minimizer $(J_\gamma, A_\gamma) \in H(\mathrm{div}{=}0) \times X_{N,0}$ for (P_γ). Especially, as a result of our smoothing process, G_γ is weakly Gâteaux differentiable.

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Theorem

$J_\gamma \in H(\mathrm{div}{=}0)$ *optimal control for* (P_γ). *Then, there exists* $(\boldsymbol{A}_\gamma, \boldsymbol{Q}_\gamma) \in X_{N,0} \times X_{N,0}$, *s.t.*

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \operatorname{curl} A_\gamma) \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \quad \forall v \in X_{N,0}$$

$$\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\,[\operatorname{curl} A_\gamma])^\top \operatorname{curl} Q_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \operatorname{curl} A_\gamma) \operatorname{curl} Q_\gamma \cdot \operatorname{curl} v$$

$$= \int_\Omega (\operatorname{curl} A_\gamma - B_\mathrm{d}) \cdot \operatorname{curl} v \quad \forall v \in X_{N,0}$$

$$J_\gamma = -\frac{2}{3}\lambda^{-1} Q_\gamma + \frac{1}{3} J^\star.$$

UDE

Theorem

$J_\gamma \in H(\mathrm{div}{=}0)$ optimal control for (P_γ). Then, there exists $(A_\gamma, Q_\gamma) \in X_{N,0} \times X_{N,0}$, s.t.

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \int_\Omega \underbrace{\gamma \boldsymbol{\theta}_\gamma(\cdot, \operatorname{curl} A_\gamma)}_{:= \boldsymbol{\xi}_\gamma} \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \quad \forall v \in X_{N,0}$$

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Limiting Analysis of (P_γ)

Given an optimal control $J^\star \in H(\mathrm{div}{=}0)$ of (P), we obtain

- a sequence $\{J_\gamma^\star\}_{\gamma>0} \subseteq H(\mathrm{div}{=}0)$ of minimizers to (P_γ) satisfying

$$J_\gamma^\star \to J^\star \quad \text{strongly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

Limiting Analysis of (P_γ)

Given an optimal control $J^\star \in H(\mathrm{div}{=}0)$ of (P), we obtain

- a sequence $\{J^\star_\gamma\}_{\gamma>0} \subseteq H(\mathrm{div}{=}0)$ of minimizers to (P_γ) satisfying

$$J^\star_\gamma \to J^\star \quad \text{strongly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

- a sequence

$$\left\{ \left(A^\star_\gamma, Q^\star_\gamma, \xi^\star_\gamma, \lambda^\star_\gamma \right) \right\}_{\gamma>0} \subseteq X_{N,0} \times X_{N,0} \times L^2(\Omega) \times L^2(\Omega)$$

of states and multipliers as well as limiting fields, s.t.

$$\begin{aligned}
A^\star_\gamma &\to A^\star && \text{strongly} && \text{in } X_{N,0} && \text{as } \gamma \to \infty \\
Q^\star_\gamma &\rightharpoonup Q^\star && \text{weakly} && \text{in } X_{N,0} && \text{as } \gamma \to \infty \\
\left(\mathbb{P}_{\mathrm{curl}\, X_{N,0}} \xi^\star_\gamma, \mathbb{P}_{\mathrm{curl}\, X_{N,0}} \lambda^\star_\gamma \right) &\rightharpoonup (\mathrm{curl}\, m^\star, \mathrm{curl}\, n^\star) && \text{weakly} && \text{in } L^2(\Omega) \times L^2(\Omega) && \text{as } \gamma \to \infty.
\end{aligned}$$

Theorem

The limiting fields $(A^\star, Q^\star, \operatorname{curl} m^\star, \operatorname{curl} n^\star) \in X_{N,0} \times X_{N,0} \times \operatorname{curl} X_{N,0} \times \operatorname{curl} X_{N,0}$ *satisfy*

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A^\star|) \operatorname{curl} A^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl} v = \int_\Omega J^\star \cdot v \quad \forall v \in X_N^0$$

$$\int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl}(v - A^\star) \le 0 \quad \forall v \in K$$

$$\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\,[\operatorname{curl} A^\star])^\top \operatorname{curl} Q^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} n^\star \cdot \operatorname{curl} v$$

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UDE

Theorem

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In the scalar H^1-setting (without an additional quasilinearity) with an obstacle set

$$K = \{v \in H_0^1(\Omega) \colon v \geq 0 \ \text{ a.e. on } \Omega\}$$

it is known that the adjoint multiplier is characterized[1] by

$$\int_\Omega (\text{adjoint multiplier}) \cdot (\text{state}) = 0$$

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Further characterization of the adjoint multiplier $\operatorname{curl} n^\star$

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We recall that

$$\mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}^\star_\gamma \rightharpoonup \operatorname{curl} n^\star \quad \text{weakly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

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In particular, there exist $\boldsymbol{\sigma}_{d_+}^\star, \boldsymbol{\sigma}_{d_-}^\star \in \boldsymbol{L}^2(\Omega)$, s.t.

$$\chi_{\{|\operatorname{curl} A_\gamma^\star| > d\}} \mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}_\gamma^\star \rightharpoonup \boldsymbol{\sigma}_{d_+}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty$$
$$\chi_{\{|\operatorname{curl} A_\gamma^\star| \leq d\}} \mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}_\gamma^\star \rightharpoonup \boldsymbol{\sigma}_{d_-}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty$$

and

$$\operatorname{curl} n^\star = \boldsymbol{\sigma}_{d_+}^\star + \boldsymbol{\sigma}_{d_-}^\star .$$

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d \frac{\operatorname{curl} A^\star}{|\operatorname{curl} A^\star|} - \operatorname{curl} A^\star \right) = 0 \quad \boxed{?}$$

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Further characterization of the adjoint multiplier $\operatorname{curl} n^\star$

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$$\operatorname{curl} n^\star = \boldsymbol{\sigma}_{d_+}^\star + \boldsymbol{\sigma}_{d_-}^\star.$$

Theorem

The adjoint multiplier $\operatorname{curl} n^\star \in L^2(\Omega)$ *is additionally characterized by*

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d \frac{\operatorname{curl} A^\star}{|\operatorname{curl} A^\star|} - \operatorname{curl} A^\star \right) = 0$$

$$\operatorname{curl} n^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}$$

$$\int_\Omega \operatorname{curl} n^\star \cdot \operatorname{curl} Q^\star \geq 0.$$

Thank you for your attention!