Quasilinear Maxwell Variational Inequalities in Ferromagnetic Shielding

joint work with Gabriele Caselli, Irwin Yousept

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Obstacle Problem in Ferromagnetic Shielding

Electromagnetic shielding

Effect of redirecting or blocking electromagnetic fields by barriers made of conductive or magnetic materials.

UDE

Electromagnetic shielding

Effect of redirecting or blocking electromagnetic fields by barriers made of conductive or magnetic materials.

Ferromagnetic shielding

Special case of Electromagnetic shielding: redirecting or blocking *magnetic fields* by ferromagnetic materials. Ferromagnetic materials are materials with high (relative) magnetic permeability, for example:

- Iron ($\mu/\mu_0 \approx 200.000$)
- Permalloy ($\mu/\mu_0 \approx 100.000$)
- Mu-metal ($\mu/\mu_0 \approx 50.000$)

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To model the ferromagnetic shielding effect, we combine a Maxwell-structured elliptic VI of the first kind with a nonlinearity $\nu = \mu^{-1}\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$, resulting in the problem

$$\text{(VI)} \qquad \begin{cases} \text{Find } (A, \phi) \in K \times H_0^1(\Omega), \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl}(v - A) + \int_\Omega \nabla\phi \cdot (v - A) \geq \int_\Omega J \cdot (v - A) \quad \forall v \in K \\ \displaystyle\int_\Omega A \cdot \nabla\psi = 0 \quad \forall \psi \in H_0^1(\Omega) \end{cases}$$

$$\& \qquad K := \{v \in H_0(\operatorname{curl})\colon |\operatorname{curl} v| \leq d(\cdot) \text{ a.e. on } \Omega\}.$$

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$$

- $\Omega \subseteq \mathbb{R}^3$ open, bounded, Lipschitz, simply connected
- $(J, d) \in L^2(\Omega) \times L^2(\Omega)$
- ν is ‘standard’, i.e. Carathéodory, strictly positive, bounded, strongly monotone and Lipschitz

The obstacle problem

We investigate:

- Is (VI) well-posed?
- How regular is its dual multiplier?
- Optimal control of (VI)

Main ingredient: A Yosida type penalization of (VI).

Optimal control of variational inequalities:

- V. Barbu
- F. Mignot and J.P. Puel
- M. Bergounioux
- ...
- K. Ito and K. Kunisch
- M. Hintermüller
- R. Herzog, C. Meyer and G. Wachsmuth

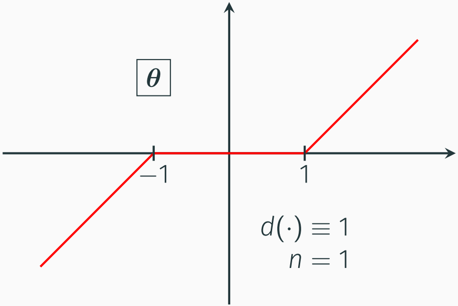
Optimal control of Maxwell-related PDEs:

- F. Tröltzsch
- A. Valli
- I. Yousept
- ...

Regularization of the Variational Inequality

We define the penalization term

$$\boldsymbol{\theta}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x, s) \mapsto \begin{cases} \max(|s| - d(x), 0)\dfrac{s}{|s|}, & s \neq 0 \\ 0, & s = 0. \end{cases}$$



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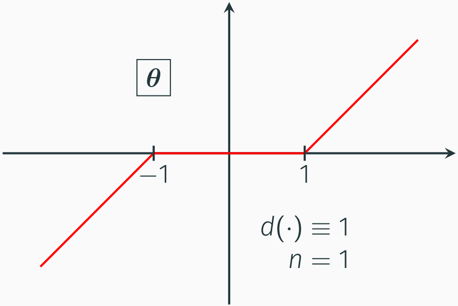
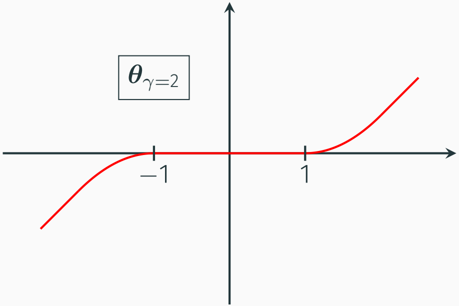
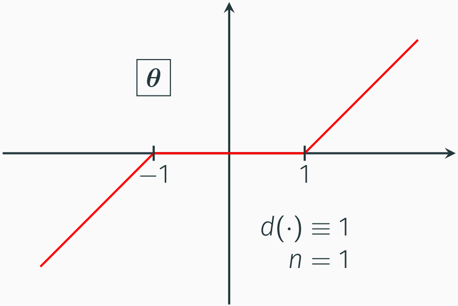


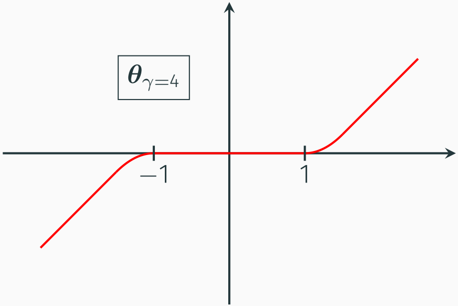


Vector-valued penalization and smoothing

We define the penalization term

$$\boldsymbol{\theta}_\gamma : \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x, s) \mapsto \begin{cases} \max_\gamma(|s| - d(x), 0)\,\dfrac{s}{|s|}, & s \neq 0 \\ 0, & s = 0. \end{cases}$$





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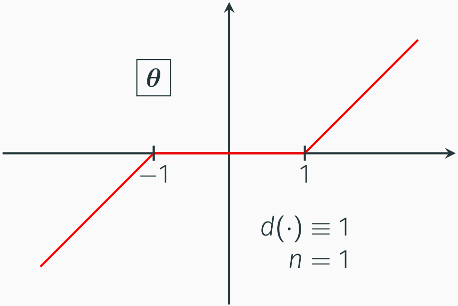

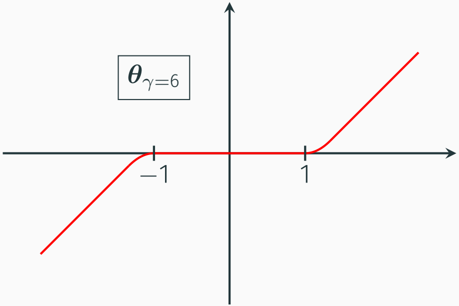

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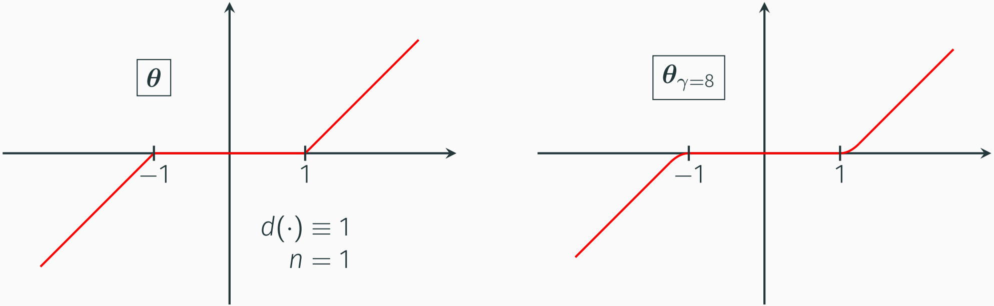

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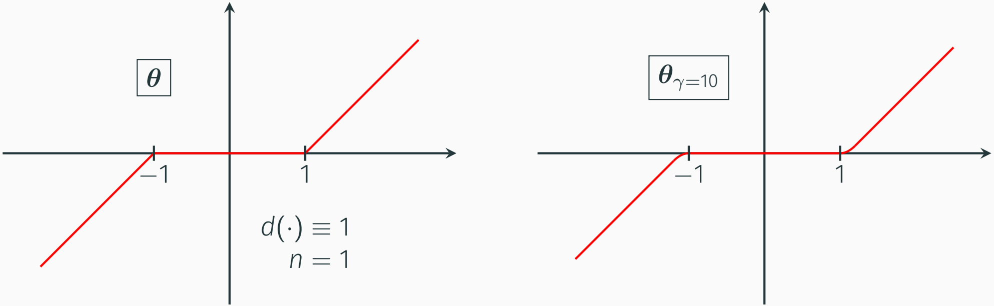

The regularized VI

For $\gamma > 0$, we consider the regularized (unconstrained) problem

$$
(\mathrm{VE}_{\gamma}^{\mathrm{sol}}) \qquad \begin{cases} \text{Find } \boldsymbol{A}_{\gamma} \in \boldsymbol{X}_{N,0} := \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0), \text{ s.t.} \\ \displaystyle\int_{\Omega} \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_{\gamma}|)\,\mathbf{curl}\,\boldsymbol{A}_{\gamma} \cdot \mathbf{curl}\,\boldsymbol{v} + \gamma \int_{\Omega} \boldsymbol{\theta}_{\gamma}(\cdot, \mathbf{curl}\,\boldsymbol{A}_{\gamma}) \cdot \mathbf{curl}\,\boldsymbol{v} \quad = \int_{\Omega} \boldsymbol{J}_{\mathrm{sol}} \cdot \boldsymbol{v} \\ \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \end{cases}
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Lemma

For every $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$, the regularized problem $(\mathrm{VE}_\gamma^{\mathrm{sol}})$ admits a unique solution $\boldsymbol{A}_\gamma$.

Left-hand side induces a monotone and coercive operator $\boldsymbol{X}_{N,0} \to \boldsymbol{X}_{N,0}^*$.

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Theorem

For $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$*, the unique solution* $\boldsymbol{A}_\gamma$ *of* $(\mathrm{VE}_\gamma^{\mathrm{sol}})$ *converges strongly in* $\boldsymbol{X}_{N,0}$ *to the unique solution of the problem*

$$(\mathrm{VI}_{\mathrm{sol}}) \qquad \begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0), \ \text{s.t.} \\ \int_\Omega \nu(\cdot, |\operatorname{curl} \boldsymbol{A}|) \operatorname{curl} \boldsymbol{A} \cdot \operatorname{curl}(\boldsymbol{v} - \boldsymbol{A}) \geq \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{v} - \boldsymbol{A}) \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0). \end{cases}$$

Well-Posedness and Regularity

Well-posedness and dual formulation

Corollary

For every $J \in L^2(\Omega)$*, there exists a unique solution* $(A, \phi) \in K \times H_0^1(\Omega)$ *to* (VI)*. Moreover, there exists a unique multiplier* $m \in X_{N,0}$ *such that the solution* (A, ϕ) *is characterized by the dual formulation*

$$\begin{cases} \int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl} v + \nabla\phi \cdot v + \operatorname{curl} m \cdot \operatorname{curl} v = \int_\Omega J \cdot v \quad \forall v \in H_0(\operatorname{curl}) \\ \int_\Omega A \cdot \nabla\psi = 0 \quad \forall \psi \in H_0^1(\Omega) \\ \int_\Omega \operatorname{curl} m \cdot \operatorname{curl}(v - A) \leq 0 \quad \forall v \in K. \end{cases}$$

Well-posedness and dual formulation

Corollary

For every $J \in L^2(\Omega)$, *there exists a unique solution* $(A, \phi) \in K \times H_0^1(\Omega)$ *to* (VI). *Moreover, there exists a unique multiplier* $\boldsymbol{m} \in X_{N,0}$ *such that the solution* (A, ϕ) *is characterized by the dual formulation*

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How regular are the appearing multipliers?

Theorem

Let $\partial\Omega$ be connected. For $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, let $(\boldsymbol{A}, \phi, \boldsymbol{m}) \in \boldsymbol{X}_{N,0} \times H_0^1(\Omega) \times \boldsymbol{X}_{N,0}$ denote the unique solution to the previous dual formulation. Then, the following multiplier regularity results hold true:

$$p \in [2,3],\ \boldsymbol{J} \in \boldsymbol{L}^p(\Omega),\ d \in L^p(\Omega) \Rightarrow \phi \in W_0^{1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

$$p \in [2,6],\ \boldsymbol{J} \in \boldsymbol{L}^p(\Omega),\ d \in L^p(\Omega),\ \Omega \text{ of class } \mathcal{C}^{1,1} \Rightarrow \phi \in W_0^{1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

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$$\boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl}),\ d \in L^\infty(\Omega),\ \nu(\cdot, |\mathbf{curl}\,\boldsymbol{A}|) \in \mathcal{C}^{0,1}(\overline{\Omega}),\ \Omega \text{ of class } \mathcal{C}^{2,1} \Rightarrow \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^\infty(\Omega)$$

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The proof is mainly based on an L^p-Helmholz-decomposition and elliptic regularity theory.

Optimal Control

$$
(\mathrm{P}) \qquad \begin{cases}
\displaystyle \min_{(J,A)\in L^2(\Omega)\times X_{N,0}} \frac{1}{2}\|\operatorname{curl} A - B_{\mathrm{d}}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|J\|^2_{L^2(\Omega)} \\
\text{subject to} \\
\displaystyle \int_\Omega \nu(\cdot, |\operatorname{curl} A|)\operatorname{curl} A \cdot \operatorname{curl}(v - A) + \int_\Omega \nabla\phi \cdot (v - A) \geq \int_\Omega J \cdot (v - A) \quad \forall v \in K \\
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The optimal control problem

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$$

Analysis of (P)

Theorem

There exists an optimal solution $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ *to the problem* (P).

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The solution mapping

$$G\colon H(\mathrm{div}{=}0) \to X_{N,0}, \quad J \mapsto A$$

is weak-strong continuous.

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Task: Find optimality conditions for optimal controls $J^\star$.
Problem: The mapping G is not directionally differentiable.

The regularized optimal control problem

$$
(\mathrm{P}_\gamma) \qquad
\begin{cases}
\displaystyle\min_{(J_\gamma, A_\gamma)\in H(\operatorname{div}=0)\times X_{N,0}} \frac{1}{2}\|\operatorname{curl} A_\gamma - B_\mathrm{d}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|J_\gamma\|^2_{L^2(\Omega)} + \frac{\lambda}{4}\|J_\gamma - J^\star\|^2_{L^2(\Omega)} \\
\text{subject to} \\
\displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \theta_\gamma(\cdot, \operatorname{curl} A_\gamma) \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \\
\forall v \in X_{N,0}.
\end{cases}
$$

$$
(\mathrm{P}_\gamma) \qquad \begin{cases} \min\limits_{(J_\gamma, A_\gamma) \in H(\mathrm{div}=0) \times X_{N,0}} \dfrac{1}{2} \| \operatorname{curl} A_\gamma - B_\mathrm{d} \|^2_{L^2(\Omega)} + \dfrac{\lambda}{2} \|J_\gamma\|^2_{L^2(\Omega)} + \dfrac{\lambda}{4} \|J_\gamma - J^\star\|^2_{L^2(\Omega)} \\ \text{subject to} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \theta_\gamma(\cdot, \operatorname{curl} A_\gamma) \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \\ \forall v \in X_{N,0}. \end{cases}
$$

The solution mapping

$$
G_\gamma : H(\mathrm{div}=0) \to X_{N,0}, \quad J_\gamma \mapsto A_\gamma
$$

is weak-strong continuous, i.e. there exists a minimizer $(J_\gamma, A_\gamma) \in H(\mathrm{div}=0) \times X_{N,0}$ for (P_γ). Especially, as a result of our smoothing process, G_γ is weakly Gâteaux differentiable.

Theorem

$J_\gamma \in H(\mathrm{div}{=}0)$ *optimal control for* (P_γ). *Then, there exists* $(A_\gamma, Q_\gamma) \in X_{N,0} \times X_{N,0}$, *s.t.*

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \theta_\gamma(\cdot, \operatorname{curl} A_\gamma) \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \quad \forall v \in X_{N,0}$$

$$\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\, [\operatorname{curl} A_\gamma])^\top \operatorname{curl} Q_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \mathrm{D}_s \theta_\gamma(\cdot, \operatorname{curl} A_\gamma) \operatorname{curl} Q_\gamma \cdot \operatorname{curl} v$$

$$= \int_\Omega (\operatorname{curl} A_\gamma - B_\mathrm{d}) \cdot \operatorname{curl} v \quad \forall v \in X_{N,0}$$

$$J_\gamma = -\frac{2}{3}\lambda^{-1} Q_\gamma + \frac{1}{3} J^\star.$$

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Theorem

$J_\gamma \in H(\mathrm{div}{=}0)$ optimal control for (P_γ). Then, there exists $(A_\gamma, Q_\gamma) \in X_{N,0} \times X_{N,0}$, s.t.

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \int_\Omega \underbrace{\gamma \boldsymbol{\theta}_\gamma(\cdot, \operatorname{curl} A_\gamma)}_{:=\boldsymbol{\xi}_\gamma} \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \quad \forall v \in X_{N,0}$$

$$\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\,[\operatorname{curl} A_\gamma])^\top \operatorname{curl} Q_\gamma \cdot \operatorname{curl} v + \int_\Omega \underbrace{\gamma \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \operatorname{curl} A_\gamma) \operatorname{curl} Q_\gamma}_{:=\boldsymbol{\lambda}_\gamma} \cdot \operatorname{curl} v$$

$$= \int_\Omega (\operatorname{curl} A_\gamma - B_\mathrm{d}) \cdot \operatorname{curl} v \quad \forall v \in X_{N,0}$$

$$J_\gamma = -\frac{2}{3}\lambda^{-1} Q_\gamma + \frac{1}{3} J^\star.$$

UDE

Given an optimal control $J^\star \in H(\mathrm{div}{=}0)$ of (P), we obtain

- a sequence $\{J_\gamma^\star\}_{\gamma>0} \subseteq H(\mathrm{div}{=}0)$ of minimizers to (P_γ) satisfying

$$J_\gamma^\star \to J^\star \quad \text{strongly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

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$$J_\gamma^\star \to J^\star \quad \text{strongly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

- a sequence

$$\left\{\left(A_\gamma^\star, Q_\gamma^\star, \xi_\gamma^\star, \lambda_\gamma^\star\right)\right\}_{\gamma>0} \subseteq X_{N,0} \times X_{N,0} \times L^2(\Omega) \times L^2(\Omega)$$

  of states and multipliers as well as limiting fields, s.t.

$$\begin{array}{llll} A_\gamma^\star \to A^\star & \text{strongly} & \text{in } X_{N,0} & \text{as } \gamma \to \infty \\ Q_\gamma^\star \rightharpoonup Q^\star & \text{weakly} & \text{in } X_{N,0} & \text{as } \gamma \to \infty \\ \left(\mathbb{P}_{\mathrm{curl}\, X_{N,0}} \xi_\gamma^\star, \mathbb{P}_{\mathrm{curl}\, X_{N,0}} \lambda_\gamma^\star\right) \rightharpoonup (\mathrm{curl}\, m^\star, \mathrm{curl}\, n^\star) & \text{weakly} & \text{in } L^2(\Omega) \times L^2(\Omega) & \text{as } \gamma \to \infty. \end{array}$$

Optimality system for (P)

Theorem

The limiting fields $(A^\star, Q^\star, \operatorname{curl} m^\star, \operatorname{curl} n^\star) \in X_{N,0} \times X_{N,0} \times \operatorname{curl} X_{N,0} \times \operatorname{curl} X_{N,0}$ *satisfy*

$$
\begin{aligned}
&\int_\Omega \nu(\cdot, |\operatorname{curl} A^\star|) \operatorname{curl} A^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl} v = \int_\Omega J^\star \cdot v \quad \forall v \in X_N^0 \\
&\int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl}(v - A^\star) \leq 0 \quad \forall v \in K \\
&\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\,[\operatorname{curl} A^\star])^\top \operatorname{curl} Q^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} n^\star \cdot \operatorname{curl} v \\
&= \int_\Omega (\operatorname{curl} A^\star - B_\mathrm{d}) \cdot \operatorname{curl} v \quad \forall v \in X_N^0 \\
&J^\star = -\lambda^{-1} Q^\star.
\end{aligned}
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UDE

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$$J^\star = -\lambda^{-1} Q^\star.$$

Further characterization of the adjoint multiplier curl $n^\star$

In the scalar H^1-setting (without an additional quasilinearity) with an obstacle set

$$K = \{v \in H_0^1(\Omega) \colon v \geq 0 \text{ a.e. on } \Omega\}$$

it is known that the adjoint multiplier is characterized[1] by

$$\int_\Omega (\text{adjoint multiplier}) \cdot (\text{state}) = 0$$
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As a reminder, we have

$$K = \{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \colon |\,\mathbf{curl}\,\mathbf{v}| \leq d(\cdot) \ \text{ a.e. on } \Omega\}.$$

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$$\int_\Omega \operatorname{curl} n^\star \cdot \left(d \frac{\operatorname{curl} A^\star}{|\operatorname{curl} A^\star|} - \operatorname{curl} A^\star \right) = 0 \qquad \boxed{?}$$

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We recall that

$$\mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}^\star_\gamma \rightharpoonup \operatorname{curl} n^\star \quad \text{weakly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

UDE

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In particular, there exist $\boldsymbol{\sigma}_{d_+}^\star, \boldsymbol{\sigma}_{d_-}^\star \in \boldsymbol{L}^2(\Omega)$, s.t.

$$\chi_{\{|\operatorname{curl} A_\gamma^\star| > d\}} \mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}_\gamma^\star \rightharpoonup \boldsymbol{\sigma}_{d_+}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty$$
$$\chi_{\{|\operatorname{curl} A_\gamma^\star| \le d\}} \mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}_\gamma^\star \rightharpoonup \boldsymbol{\sigma}_{d_-}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty$$

and

$$\operatorname{curl} n^\star = \boldsymbol{\sigma}_{d_+}^\star + \boldsymbol{\sigma}_{d_-}^\star .$$

Further characterization of the adjoint multiplier $\operatorname{curl} n^\star$

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d \frac{\operatorname{curl} A^\star}{|\operatorname{curl} A^\star|} - \operatorname{curl} A^\star \right) = 0 \quad \boxed{?}$$

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and

$$\operatorname{curl} n^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}.$$

Theorem

The adjoint multiplier $\operatorname{curl} n^\star \in L^2(\Omega)$ *is additionally characterized by*

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d \frac{\operatorname{curl} A^\star}{|\operatorname{curl} A^\star|} - \operatorname{curl} A^\star \right) = 0$$

$$\operatorname{curl} n^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}$$

$$\int_\Omega \operatorname{curl} n^\star \cdot \operatorname{curl} Q^\star \geq 0.$$

Thank you for your attention!