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QUASILINEAR VARIATIONAL INEQUALITIES IN FERROMAGNETIC SHIELDING: WELL-POSEDNESS, REGULARITY, AND OPTIMAL CONTROL*

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Abstract. This paper examines the analysis and optimal control of an H(curl)-quasilinear first kind variational inequality with a bilateral vector curl-constraint, stemming from the ferromagnetic shielding phenomenon. We propose a tailored regularization approach based on the Helmholtz decomposition and a reduction of the first-order constraint to the zeroth-order one in combination with a smoothed Yosida penalization. In this way, a suitable family of approximating quasilinear variational equalities is obtained. The corresponding limiting analysis not only leads to a well-posedness result for the variational inequality but also reveals its dual formulation. Thereafter, as a second novelty, we prove a regularity result for the dual multiplier by means of the L^p-Helmholtz decomposition in a careful combination with elliptic regularity results for Dirichlet and Neumann problems. The last part of this paper is devoted to the analysis of the corresponding optimal control, which is mainly complicated by the involving H(curl)-quasilinearity, the bilateral vector curl-constraint, and the nonsmoothness. On the basis of the proposed regularization, as the final novelty, we derive necessary optimality conditions, including a characterization of the limiting dual multiplier through curl-projection and cut-off type arguments.

Key words. H(curl)-quasilinear variational inequalities of the first kind, bilateral vector curl-constraints, ferromagnetic shielding, well-posedness, regularity, optimal control system

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1. Introduction.

Electromagnetic (EM) shielding is a physical process of redirecting or reducing electromagnetic fields by conductive or magnetic materials. For instance, every ferromagnetic material with high magnetic permeability (cobalt, nickel, etc.) can realize magnetic shielding by diverting the magnetic flux to another path. Nowadays, EM shielding is indispensable in many technological and daily applications, including microwave ovens, mobile phones, aircraft, MRI, circuits, semiconductor chips, and many other electronic devices. In fact, EM shielding is utterly required in every application demanding the reduction of undesired electromagnetic interference. From the mathematical perspective, EM shielding falls into the class of obstacle problems: In the free region, the electromagnetic fields satisfy the fundamental Maxwell equations, whereas in the shielded area they are constrained to stay below a certain threshold.

This paper is devoted to the mathematical analysis of ferromagnetic shielding in the static regime through the magnetic vector potential formulation. In the free region, as a particular case of Maxwell's equations, magnetostatic equations read as

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$$
\begin{cases} \mathbf{curl}(\nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}) + \nabla\phi = \boldsymbol{J} \text{ in } \Omega, \\ \operatorname{div}\,\boldsymbol{A} = 0 \text{ in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{A} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain with a connected boundary, $\boldsymbol{A}:\Omega\to\mathbb{R}^3$ denotes the magnetic vector potential, $\boldsymbol{J}:\Omega\to\mathbb{R}^3$ the current density, $\phi:\Omega\to\mathbb{R}$ the Lagrange multiplier, and $\boldsymbol{n}$ the unit outward normal to $\partial\Omega$. Furthermore, $\nu:\Omega\times\mathbb{R}^+_0\to\mathbb{R}$ describes the nonlinear magnetic reluctivity modelling the physical dependency of ferromagnetic materials on the magnetic induction $\mathbf{curl}\,\boldsymbol{A}$. From among many works on (1.1), we refer the reader to [3, 24, 43]. In the present paper, we consider a variational inequality of the first kind for (1.1), in which the magnetic induction strength $|\,\mathbf{curl}\,\boldsymbol{A}|$ is constrained to lie underneath a certain level leading to the following feasible set:

$$
\boldsymbol{K} := \{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl})\;|\;|\,\mathbf{curl}\,\boldsymbol{v}|\le d \text{ a.e. in } \Omega\} \quad \text{for a nonnegative } d\in L^2(\Omega). \tag{1.2}
$$

Following the celebrated theory of variational inequalities [25, 27, 33], we formulate the first kind variational inequality for the quasilinear magnetostatic field equations (1.1)–(1.2) as follows:

$$
\begin{cases} \text{Find } (\boldsymbol{A},\phi)\in\boldsymbol{K}\times H^1_0(\Omega) \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x + \int_\Omega \nabla\phi\cdot\boldsymbol{v}\,\mathrm{d}x \\ \displaystyle\ge \int_\Omega \boldsymbol{J}\cdot(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{K}, \\ \displaystyle\int_\Omega \boldsymbol{A}\cdot\nabla\psi\,\mathrm{d}x = 0 \quad \forall\psi\in H^1_0(\Omega). \end{cases} \tag{VI}
$$

The analysis of variational inequalities (obstacle problems) for Maxwell's equations goes back to Duvaut and Lions [11]. Years after their first investigation, the study of Maxwell variational inequalities (MVI) has gained more and more attention due to their paramount applications in superconductivity (see [5, 9, 23, 26, 34, 35, 44]). Miranda, Rodrigues, and Santos [30] established a general framework for the well-posedness of parabolic MVI and Maxwell quasi-variational inequalities (MQVI). Furthermore, we refer to the recent works of the third author [46, 47, 48] regarding hyperbolic MVI and MQVI with applications in superconductivity and EM shielding. While the aforementioned contributions are primarily devoted to the well-posedness analysis, numerical methods for MVI were proposed and analyzed in [12, 41, 42]. See also [17, 18] for the eddy current approximation and the numerical analysis of a hyperbolic Maxwell obstacle problem in electric shielding.

Motivated by the technological applications of ferromagnetic shielding, this paper makes the first attempt to analyze (VI) and the corresponding optimal control problem (1.3). Due to the involved $\boldsymbol{H}(\mathbf{curl})$-quasilinearity and the nonsmooth character in (VI), the analysis is genuinely nonstandard and challenging. In particular, it requires substantial extension of developed techniques from the existing literature. First, this paper develops a regularization approach (VE_γ) for (VI) by means of the Helmholtz decomposition and a reformulation of the first-order constraint (1.2) through the zeroth-order one (2.3) in combination with a smoothed Yosida penalization. By the limiting analysis of (VE_γ), we establish the well-posedness of (VI) and its dual formulation (Theorem 3.1). The second novelty of this paper is the dual multiplier regularity result (Theorem 4.4). To the best of the authors' knowledge, Theorem 4.4 is the first result in the multiplier regularity analysis for MVI. Our proof is realized using Maxwell techniques in combination with the L^p-Helmholtz decomposition [15] and elliptic regularity results for Dirichlet and Neumann problems [22, 49].

After analyzing both the well-posedness and regularity for (VI), the final part of this paper is devoted to the optimal control problem. Our aim is to find an optimal current source in the ferromagnetic shielding process (VI) which minimizes the L^2-distance between the induced magnetic induction and the desired one. This leads to the following minimization problem:

$$\begin{cases} \min \displaystyle\int_\Omega |\mathbf{curl}\, \boldsymbol{A} - \boldsymbol{B}_{\mathrm{d}}|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |\boldsymbol{J}|^2 \,\mathrm{d}x \\ \text{s.t. (VI)}, \end{cases} \tag{1.3}$$

where the vector field $\boldsymbol{B}_{\mathrm{d}} \in \boldsymbol{L}^2(\Omega)$ denotes the desired magnetic induction, and $\lambda > 0$ denotes the control cost parameter. Let us emphasize that the primary difficulty of (1.3) lies not only in the $\boldsymbol{H}(\mathbf{curl})$-quasilinearity and the bilateral vector **curl**-constraint (1.2) but also in the lack of differentiability. Even for the simpler H^1-case, the directional differentiability of the solution mapping of the corresponding variational inequality in the presence of bilateral or gradient constraints cannot be expected. All these aspects together make the analysis of (1.3) particularly delicate. While the mathematical analysis for the optimal control of $H^1(\Omega)$-type variational inequalities seems to have reached an advanced stage of development (cf. [4, 7, 8, 14, 19, 20, 21, 28, 29]), this paper is the first to address (1.3). In fact, we are not aware of any previous contributions toward optimal control of MVI. The final novelty of this paper is therefore the derivation of necessary optimality conditions for the nonsmooth optimal control problem (1.3) (see Theorem 5.6). In particular, our proof extends established Maxwell techniques for optimal control [31, 38, 39, 43, 45] and develops new ideas to cope with the aforementioned complexity involved in (1.3). Last but not least, we note that the results of this paper are not restricted to the objective functional (1.3) involving only the first-order term $\mathbf{curl}\,\boldsymbol{A}$. Following [43, Theorem 3.8 and Remark 3.9], we obtain analogous results for objective functionals involving the zeroth-order term $\|\boldsymbol{A} - \boldsymbol{A}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)}$ with a given $\boldsymbol{A}_{\mathrm{d}} \in \boldsymbol{L}^2(\Omega)$.

1.1. Preliminaries. Given a real Hilbert space H, we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ its scalar product and induced norm, respectively. In the case of $H = \mathbb{R}^d$, we simply write a dot and $|\cdot|$ for the Euclidean scalar product and norm. Discussing problems of Maxwell type, there naturally arise function spaces of $\mathbb{R}^3$-valued functions. We will therefore use a bold typeface to indicate them. Given some open set $\mathcal{O} \subset \mathbb{R}^3$, let $\boldsymbol{L}^2(\mathcal{O})$ denote the space of all (equivalence classes of) $\mathbb{R}^3$-valued Lebesgue square-integrable functions. Moreover, we introduce the Hilbert spaces

$$\boldsymbol{H}(\mathbf{curl}, \mathcal{O}) := \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) \mid \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O})\},$$
$$\boldsymbol{H}(\mathrm{div}, \mathcal{O}) := \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) \mid \mathrm{div}\ \boldsymbol{u} \in L^2(\mathcal{O})\}$$

endowed with their natural graph norms. Here the **curl** and the div operators are to be understood in the sense of distributions. Furthermore, let $\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O})$ denote the space of infinitely differentiable $\mathbb{R}^3$-valued functions with compact support in $\mathcal{O}$. The space $\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O})$ stands for the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\mathcal{O})$ with respect to the $\boldsymbol{H}(\mathbf{curl}, \mathcal{O})$-topology. Analogously defined is the space $\boldsymbol{H}_0(\mathrm{div}, \mathcal{O})$. With $\boldsymbol{H}(\mathrm{div}{=}0, \mathcal{O})$ and $\boldsymbol{H}_0(\mathrm{div}{=}0, \mathcal{O})$

we denote the kernels of the divergence in the respective spaces, which are henceforth endowed with the $\boldsymbol{L}^2(\mathcal{O})$-topology. Note that if $\mathcal{O}$ is additionally bounded and Lipschitz, it is known (see [40]) that both the embeddings

$$\boldsymbol{H}_0(\mathbf{curl},\mathcal{O})\cap\boldsymbol{H}(\mathrm{div},\mathcal{O})\hookrightarrow\boldsymbol{L}^2(\mathcal{O}) \quad \text{and} \quad \boldsymbol{H}(\mathbf{curl},\mathcal{O})\cap\boldsymbol{H}_0(\mathrm{div},\mathcal{O})\hookrightarrow\boldsymbol{L}^2(\mathcal{O}) \tag{1.4}$$

are compact. Let us now review some orthogonal decomposition results in $\boldsymbol{L}^2(\mathcal{O})$. First of all, by the classical Hilbert projection theorem, we have the elementary Helmholtz decompositions

$$\boldsymbol{L}^2(\mathcal{O})=\boldsymbol{H}_0(\mathrm{div}{=}0,\mathcal{O})\oplus\nabla H^1(\mathcal{O}), \tag{1.5}$$

$$\boldsymbol{L}^2(\mathcal{O})=\boldsymbol{H}(\mathrm{div}{=}0,\mathcal{O})\oplus\nabla H_0^1(\mathcal{O}). \tag{1.6}$$

Then, aiming to write the divergence-free part as the **curl** of a vector potential, the above decompositions can be refined. In particular, if the boundary $\partial\mathcal{O}$ is connected, it holds that (cf. [1])

$$\boldsymbol{L}^2(\mathcal{O})=\mathbf{curl}\,(\boldsymbol{H}(\mathbf{curl},\mathcal{O})\cap\boldsymbol{H}_0(\mathrm{div}{=}0,\mathcal{O}))\oplus\nabla H_0^1(\mathcal{O}). \tag{1.7}$$

For the case $\mathcal{O}=\Omega$, we agree not to specify the domain when stating the introduced function spaces. In the given context, let us also introduce the space

$$\boldsymbol{X}_{N,0}:=\boldsymbol{H}_0(\mathbf{curl})\cap\boldsymbol{H}(\mathrm{div}{=}0),$$

which plays a pivotal role in our analysis. Especially, owing to (1.4) and recalling that Ω features a connected boundary, there exists a constant $C_\mathrm{p}>0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\le C_\mathrm{p}\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall\boldsymbol{u}\in\boldsymbol{X}_{N,0}. \tag{1.8}$$

Finally, $C>0$ represents a generic constant whose value can vary from line to line. We close this section by presenting the basic (physical) assumptions for our analysis. We assume the magnetic reluctivity $\nu:\Omega\times\mathbb{R}_0^+\to\mathbb{R}$ to be a Carathéodory function: For every $s\in\mathbb{R}_0^+$, the function $\nu(\cdot,s)$ is measurable, and, for almost every $x\in\Omega$, the function $\nu(x,\cdot)$ is continuous. By $\nu_0>0$, we denote the magnetic reluctivity in a vacuum. Further conditions on the nonlinearity (cf. [3, 24]) are collected in the following assumption which we assume to be valid throughout the whole document.

Assumption 1.1. There exist constants $\underline{\nu},\overline{\nu}\in(0,\nu_0)$ such that

$$\begin{aligned} \underline{\nu}\le\nu(x,s)\le\nu_0 \qquad & \text{for a.e. } x\in\Omega \text{ and every } s\in\mathbb{R}_0^+,\\ (\nu(x,s)s-\nu(x,\hat{s})\hat{s})(s-\hat{s})\ge\underline{\nu}(s-\hat{s})^2 \quad & \forall s,\hat{s}\in\mathbb{R}_0^+,\\ |\nu(x,s)s-\nu(x,\hat{s})\hat{s}|\le\overline{\nu}|s-\hat{s}| \qquad & \forall s,\hat{s}\in\mathbb{R}_0^+ \end{aligned}$$

holds true.

Under Assumption 1.1, it holds for a.e. $x\in\Omega$ that

$$\begin{aligned} (\nu(x,|s|)s-\nu(x,|\hat{s}|)\hat{s})\cdot(s-\hat{s})\ge\underline{\nu}|s-\hat{s}|^2 \quad & \forall s,\hat{s}\in\mathbb{R}^3,\\ |\nu(x,|s|)s-\nu(x,|\hat{s}|)\hat{s}|\le L|s-\hat{s}| \quad & \forall s,\hat{s}\in\mathbb{R}^3, \end{aligned} \tag{1.9}$$

where $L=2\nu_0+\overline{\nu}$. A proof for (1.9) can be found in [43, Lemma 2.2].

2. Regularization of (VI). We propose a regularization approach for (VI) based on three main steps:

1. reduction of (VI) to the lower level problem ($\mathrm{VI}_{\mathrm{sol}}$) with a divergence-free source term;
2. reformulation of the first-order constraint (1.2) by the zeroth-order one (2.3) and the application of the Yosida regularization to the subdifferential of the indicator function for the zeroth-order obstacle set;
3. smoothing of the maximum function (2.4).

For the first step, let us consider a solenoidal source term $\boldsymbol{J}_{\mathrm{sol}}$, i.e., a function $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$ and test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$ in (VI). In this particular case, since

$$\int_\Omega \boldsymbol{v} \cdot \nabla\psi \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}{=}0), \quad \forall \psi \in H_0^1(\Omega), \tag{2.1}$$

(VI) leads to the following problem:

$$(\mathrm{VI}_{\mathrm{sol}}) \qquad \begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0) \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}) \,\mathrm{d}x \\ \displaystyle\geq \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{v} - \boldsymbol{A}) \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0). \end{cases}$$

Let us remark that the auxiliary problem ($\mathrm{VI}_{\mathrm{sol}}$) is indeed helpful for our investigation and serves as a basis for our well-posedness result for (VI). More precisely, applying the Helmholtz decomposition (1.6) to the source term

$$\boldsymbol{L}^2(\Omega) \ni \boldsymbol{J} = \boldsymbol{J}_{\mathrm{sol}} + \nabla\phi_{\boldsymbol{J}} \in \boldsymbol{H}(\mathrm{div}{=}0) \oplus \nabla H_0^1(\Omega), \tag{2.2}$$

we show in Theorem 3.1 that the solution $\boldsymbol{A}$ to ($\mathrm{VI}_{\mathrm{sol}}$) for $\boldsymbol{J}_{\mathrm{sol}}$ as in (2.2) turns out to be the unique solution to (VI) with the corresponding (unique) multiplier given by $\phi_{\boldsymbol{J}}$ from (2.2). For the second step, we introduce the zeroth-order obstacle set

$$\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)} := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid |\boldsymbol{v}| \leq d \text{ a.e. on } \Omega\},$$

with which we can reformulate our first-order constraint as

$$\boldsymbol{A} \in \boldsymbol{K} \quad \Leftrightarrow \quad \mathbf{curl}\,\boldsymbol{A} \in \boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}. \tag{2.3}$$

Based on the proposed reformulation, we invoke the Yosida regularization of the subdifferential of the indicator function $I_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}}$ which is given by $\gamma(\mathrm{Id} - \mathbb{P}_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}})$ (cf. [36, p. 137] and [6, Corollary 12.30]) with $\gamma > 0$ being the regularization parameter. Here, $\mathbb{P}_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}}$ denotes the Hilbert projection onto the nonempty, closed, and convex set $\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)} \subset \boldsymbol{L}^2(\Omega)$. The simplified $\boldsymbol{L}^2(\Omega)$ structure of $\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}$ now allows us to find an explicit expression (cf. [19, Example 4.2]) for the associated Yosida approximation as follows:

$$\gamma(\mathrm{Id} - \mathbb{P}_{\boldsymbol{K}_{\boldsymbol{L}^2(\Omega)}})(\boldsymbol{v}) = \gamma\boldsymbol{\theta}(\cdot, \boldsymbol{v}(\cdot)),$$

with

$$\boldsymbol{\theta} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\theta}(x, s) := \begin{cases} \max(|s| - d(x), 0)\dfrac{s}{|s|} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases} \tag{2.4}$$

Note that for a.e. $x \in \Omega$, the function $\boldsymbol{\theta}(x,\cdot)$ is continuous but not differentiable. Thus, for our final step, we regularize the nonsmooth function $\boldsymbol{\theta}$ by

$$\boldsymbol{\theta}_\gamma : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,s) \mapsto \begin{cases} \max_\gamma(|s| - d(x), 0)\dfrac{s}{|s|} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0, \end{cases}$$

where

$$\max{}_\gamma(\cdot,0) : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} x - \gamma^{-1} & \text{if } x \geq 2\gamma^{-1}, \\ \dfrac{\gamma}{4}x^2 & \text{if } x \in (0, 2\gamma^{-1}), \\ 0 & \text{if } x \leq 0. \end{cases}$$

Geometrically speaking, the function $\max_\gamma(\cdot,0)$ is a continuously differentiable regularization of $\max(\cdot,0)$, which approximates the kink at 0 by a quadratic function in the interval $(0,2\gamma^{-1})$. Altogether, for every $\gamma > 0$, we consider the following regularized problem:

$$(\mathrm{VE}_\gamma) \qquad \begin{cases} \text{Find } \boldsymbol{A}_\gamma \in \boldsymbol{X}_{N,0}, \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ \displaystyle = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \end{cases}$$

We shall see later in section 5 that (VE_γ) serves as the state equation for the regularized optimal control problem (P_γ).

LEMMA 2.1. *Let $\gamma > 0$. Then, the mapping $\boldsymbol{\theta}_\gamma$ is continuously differentiable with respect to the second variable, with derivative $\mathrm{D}_s\boldsymbol{\theta}_\gamma : \Omega \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ given by*

$$(2.5) \quad \mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s) := \begin{cases} \dfrac{s\otimes s}{|s|^2} + \dfrac{|s| - d(x) - \gamma^{-1}}{|s|}\left(\mathrm{Id} - \dfrac{s \otimes s}{|s|^2}\right) & \text{if } |s| \geq d(x) + 2\gamma^{-1}, \\ \dfrac{\gamma}{2}(|s| - d(x))\dfrac{s\otimes s}{|s|^2} + \dfrac{\gamma(|s|-d(x))^2}{4|s|}\left(\mathrm{Id} - \dfrac{s\otimes s}{|s|^2}\right) & \text{if } |s| \in (d(x), d(x) + 2\gamma^{-1}), \\ 0 & \text{if } |s| \leq d(x). \end{cases}$$

For all $s \in \mathbb{R}^3$ and almost every $x \in \Omega$, the matrix $\mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s) \in \mathbb{R}^{3\times 3}$ is symmetric and positive semidefinite. Moreover, $\mathrm{D}_s\boldsymbol{\theta}_\gamma : \Omega \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ is uniformly bounded. Finally, for almost every $x \in \Omega$, $\boldsymbol{\theta}_\gamma(x,\cdot) : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ is monotone and Lipschitz-continuous, and it holds that

$$(2.6) \qquad |\boldsymbol{\theta}_\gamma(x,s) - \boldsymbol{\theta}(x,s)| \leq \frac{3}{\gamma} \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}^3.$$

Proof. The proof is rather straightforward and only given for the convenience of the reader. It is apparent that $\boldsymbol{\theta}_\gamma$ is continuously differentiable at $s \neq 0$ since it is the product and composition of $\mathcal{C}^1$-mappings, and the same is true if $s = 0$ and $d(x) > 0$. If s and $d(x)$ are both zero, it suffices to check that $s \mapsto s|s|$ is continuously differentiable at the origin, which is easily verified. A direct computation shows that the Jacobian is given by (2.5). Note that, for each $s \in \mathbb{R}^3$, the matrices

$$\frac{s \otimes s}{|s|^2} \quad \text{and} \quad \mathrm{Id} - \frac{s\otimes s}{|s|^2}$$

are (symmetric) projection matrices. Thus, they have spectrum $\{0,1\}$ so that, for every $s \in \mathbb{R}^3$ and almost every $x \in \Omega$, the matrix $\mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s) \in \mathbb{R}^{3\times 3}$ is positive semidefinite. Thus, we can apply [32, Theorem 12.3] to conclude that $\boldsymbol{\theta}_\gamma$ is monotone with respect to the second variable. For the uniform boundedness of $\mathrm{D}_s\boldsymbol{\theta}_\gamma$, we observe that

$$\frac{|s\otimes s|_2}{|s|^2} = 1 \quad \text{and} \quad \frac{|s| - d(x) - \gamma^{-1}}{|s|} \leq 1,$$

where $|\cdot|_2$ denotes the spectral norm. Therefore, there exists a constant $C > 0$ such that $|\mathrm{D}_s\boldsymbol{\theta}_\gamma(x,s)| \leq C$ for almost all $x \in \Omega$ and all $s \in \mathbb{R}^3$. Combining [32, Theorem 9.2] and [32, Theorem 9.7], this also implies the Lipschitz continuity of $\boldsymbol{\theta}_\gamma$. To finish the proof we calculate

$$|\boldsymbol{\theta}_\gamma(x,s) - \boldsymbol{\theta}(x,s)| \leq \begin{cases} \Big| |s| - d(x) - \gamma^{-1} - (|s| - d(x)) \Big| & \text{if } |s| \geq d(x) + 2\gamma^{-1}, \\ \Big| \frac{\gamma}{4}(|s| - d(x))^2 - (|s| - d(x)) \Big| & \text{if } |s| \in (d(x), d(x) + 2\gamma^{-1}), \end{cases}$$

which yields the desired estimate (2.6). ∎

LEMMA 2.2. *For every* $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$*, the regularized problem* (VE$_\gamma$) *admits a unique solution* $\boldsymbol{A}_\gamma \in \boldsymbol{X}_{N,0}$*.*

Proof. In view of the Browder–Minty theorem, we define an operator $\mathcal{M}_\gamma : \boldsymbol{X}_{N,0} \to \boldsymbol{X}_{N,0}^*$ by

$$\begin{aligned} &\langle \mathcal{M}_\gamma \boldsymbol{A}, \boldsymbol{v}\rangle_{\boldsymbol{X}_{N,0}^*, \boldsymbol{X}_{N,0}} \\ &:= \int_\Omega \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}|) \operatorname{\mathbf{curl}} \boldsymbol{A} \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} \,\mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \operatorname{\mathbf{curl}} \boldsymbol{A}) \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{A}, \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \end{aligned}$$

A combination of Assumption 1.1, the Poincaré–Friedrichs inequality (1.8), and Lemma 2.1 (the monotonicity and continuity of $\boldsymbol{\theta}_\gamma$) implies that $\mathcal{M}_\gamma$ is strongly monotone and hemicontinuous. Indeed, for any $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \boldsymbol{X}_{N,0}$, it holds that

$$\begin{aligned} &\langle \mathcal{M}_\gamma \boldsymbol{A}_1 - \mathcal{M}_\gamma \boldsymbol{A}_2, \boldsymbol{A}_1 - \boldsymbol{A}_2 \rangle_{\boldsymbol{X}_{N,0}^*, \boldsymbol{X}_{N,0}} \\ &= \int_\Omega (\nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_1|) \operatorname{\mathbf{curl}} \boldsymbol{A}_1 - \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_2|) \operatorname{\mathbf{curl}} \boldsymbol{A}_2) \cdot \operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2) \,\mathrm{d}x \\ &\quad + \gamma \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \operatorname{\mathbf{curl}} \boldsymbol{A}_1) - \boldsymbol{\theta}_\gamma(\cdot, \operatorname{\mathbf{curl}} \boldsymbol{A}_2)) \cdot \operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2) \,\mathrm{d}x \\ &\geq \int_\Omega (\nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_1|) \operatorname{\mathbf{curl}} \boldsymbol{A}_1 - \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_2|) \operatorname{\mathbf{curl}} \boldsymbol{A}_2) \cdot \operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2) \,\mathrm{d}x \\ &\underbrace{\geq}_{(1.9)} \underline{\nu} \|\operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2)\|^2_{\boldsymbol{L}^2(\Omega)} \underbrace{\geq}_{(1.8)} \frac{\underline{\nu}\min\{1, C_{\mathrm{p}}^{-2}\}}{2} \|\boldsymbol{A}_1 - \boldsymbol{A}_2\|^2_{\boldsymbol{X}_{N,0}}, \end{aligned} \tag{2.7}$$

which implies the strong monotonicity of $\mathcal{M}_\gamma$. The hemicontinuity of $\mathcal{M}_\gamma$ follows immediately from the continuity properties of the nonlinearities ν and $\boldsymbol{\theta}_\gamma$ in combination with Lebesgue's dominated convergence theorem. Since the right-hand side in (VE$_\gamma$) induces a functional in $\boldsymbol{X}_{N,0}^*$, the usage of the Browder–Minty theorem completes the proof. ∎

LEMMA 2.3. *For every* $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$*, the problem* (VI$_{\mathrm{sol}}$) *admits a unique solution* $\boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$*. Furthermore, the unique solution* $\boldsymbol{A}_\gamma \in \boldsymbol{X}_{N,0}$ *of* (VE$_\gamma$) *converges strongly in* $\boldsymbol{X}_{N,0}$ *to the unique solution* $\boldsymbol{A}$ *of* (VI$_{\mathrm{sol}}$) *as* $\gamma \to \infty$*.*

Proof. Let $\gamma > 0$ be given. Testing (VE_γ) with $\boldsymbol{v} = \boldsymbol{A}_\gamma$ leads to

$$\begin{aligned}(2.8) \qquad &\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{A}_\gamma \,\mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{A}_\gamma \,\mathrm{d}x \\ &\quad = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot \boldsymbol{A}_\gamma \,\mathrm{d}x.\end{aligned}$$

Utilizing Assumption 1.1, the Poincaré–Friedrichs inequality (1.8), and Lemma 2.1 (the monotonicity of $\boldsymbol{\theta}_\gamma$), a straightforward computation in the fashion of (2.7) in combination with the Hölder and Young inequalities shows that the sequence $\{\boldsymbol{A}_\gamma\}_{\gamma>0} \subset \boldsymbol{X}_{N,0}$ is bounded in $\boldsymbol{X}_{N,0}$, and consequently there exists a subsequence, still denoted in the same way, and $\boldsymbol{A} \in \boldsymbol{X}_{N,0}$ such that

$$(2.9) \qquad \boldsymbol{A}_\gamma \rightharpoonup \boldsymbol{A} \quad \text{weakly in } \boldsymbol{X}_{N,0} \quad \text{as } \gamma \to \infty.$$

By the compactness of the embedding (1.4), we also obtain that $\boldsymbol{A}_\gamma \to \boldsymbol{A}$ strongly in $\boldsymbol{L}^2(\Omega)$ as $\gamma \to \infty$. Next we shall prove that $\boldsymbol{A} \in \boldsymbol{K}$, i.e., $|\,\mathbf{curl}\,\boldsymbol{A}| \le d$ a.e. in Ω. Dividing the equation in (2.8) by γ and due to the boundedness of $\{\boldsymbol{A}_\gamma\}_{\gamma>0}$ in $\boldsymbol{X}_{N,0}$ implying the boundedness of $\{\mathbf{curl}\,\boldsymbol{A}_\gamma\}_{\gamma>0}$ in $\boldsymbol{L}^2(\Omega)$, we get

$$(2.10) \qquad \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{A}_\gamma \,\mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty,$$

while from (VE_γ) it also follows that

$$(2.11) \qquad \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v} \,\mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}.$$

As $\boldsymbol{\theta}_\gamma$ is monotone in the second variable, it holds for all $\boldsymbol{v} \in \boldsymbol{X}_{N,0}$ that

$$\begin{aligned} 0 &\le \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) - \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{v})) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v}) \,\mathrm{d}x \\ &= \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v}) \,\mathrm{d}x \\ &\quad + \int_\Omega (\boldsymbol{\theta}(\cdot, \mathbf{curl}\,\boldsymbol{v}) - \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{v})) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v}) \,\mathrm{d}x \\ &\quad - \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\,\boldsymbol{v}) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v}) \,\mathrm{d}x. \end{aligned}$$

Thanks to (2.10), (2.11), and the strong $\boldsymbol{L}^2(\Omega)$-convergence of $\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{v})$ toward $\boldsymbol{\theta}(\cdot, \mathbf{curl}\,\boldsymbol{v})$ (see (2.6)), we obtain after passing to the limit $\gamma \to \infty$ in the previous inequality that

$$(2.12) \qquad \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\,\boldsymbol{v}) \cdot \mathbf{curl}(\boldsymbol{A} - \boldsymbol{v}) \,\mathrm{d}x \le 0 \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}.$$

Now we take $s \in (0,1)$, $\widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0}$ arbitrarily fixed and set $\boldsymbol{v} = \boldsymbol{A} + s\widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0}$ in (2.12) to deduce that

$$(2.13) \qquad \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})) \cdot \mathbf{curl}\,\widetilde{\boldsymbol{v}} \,\mathrm{d}x \ge 0 \quad \forall \widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0}.$$

By the continuity of $\boldsymbol{\theta}$ with respect to the second variable, it follows that $\boldsymbol{\theta}(\cdot, \mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})) \to \boldsymbol{\theta}(\cdot, \mathbf{curl}\,\boldsymbol{A})$ a.e. in Ω as $s \to 0$. Moreover, we have

$$\begin{aligned} &|\boldsymbol{\theta}(\cdot, \mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}}))| \underbrace{=}_{(2.4)} |\max(|\mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})| - d, 0)| \\ &\quad \le ||\mathbf{curl}(\boldsymbol{A} + s\widetilde{\boldsymbol{v}})| - d| \le |\,\mathbf{curl}\,\boldsymbol{A}| + |\mathbf{curl}\,\widetilde{\boldsymbol{v}}| + d \quad \forall s \in (0,1) \text{ and a.e. in } \Omega, \end{aligned}$$

and therefore we apply Lebesgue's dominated convergence theorem to pass to the limit $s \to 0$ in (2.13). This implies

$$\int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{A}) \cdot \mathbf{curl}\, \widetilde{\boldsymbol{v}}\, \mathrm{d}x \geq 0 \quad \forall \widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0}$$
$$\Rightarrow \quad \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{A}) \cdot \mathbf{curl}\, \widetilde{\boldsymbol{v}}\, \mathrm{d}x = 0 \quad \forall \widetilde{\boldsymbol{v}} \in \boldsymbol{X}_{N,0},$$

and setting $\widetilde{\boldsymbol{v}} = \boldsymbol{A}$ in the last equation finally yields

$$0 = \int_\Omega \boldsymbol{\theta}(\cdot, \mathbf{curl}\, \boldsymbol{A}) \cdot \mathbf{curl}\, \boldsymbol{A}\, \mathrm{d}x = \int_\Omega \underbrace{\max(|\,\mathbf{curl}\, \boldsymbol{A}| - d, 0)|\,\mathbf{curl}\, \boldsymbol{A}|}_{\geq 0}\, \mathrm{d}x,$$

which implies

$$\max(|\,\mathbf{curl}\, \boldsymbol{A}(x)| - d(x), 0)|\,\mathbf{curl}\, \boldsymbol{A}(x)| = 0 \quad \text{for a.e. } x \in \Omega \quad \Rightarrow \quad \boldsymbol{A} \in \boldsymbol{K}.$$

Let us now show that the weak convergence (2.9) is strong. To this end, first we test (VE_γ) with $\boldsymbol{v} = \boldsymbol{A}_\gamma - \boldsymbol{A} \in \boldsymbol{X}_{N,0}$ to obtain

$$\begin{aligned} &\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}_\gamma|)\, \mathbf{curl}\, \boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \\ &\quad + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x. \end{aligned} \tag{2.14}$$

In view of Assumption 1.1 and (1.8), there exists a constant $C_\nu > 0$ such that

$$\begin{aligned} &\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}_\gamma|)\, \mathbf{curl}\, \boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \\ &\quad \geq C_\nu \|\boldsymbol{A}_\gamma - \boldsymbol{A}\|^2_{\boldsymbol{X}_{N,0}} + \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x. \end{aligned} \tag{2.15}$$

Moreover, it holds that

$$\begin{aligned} &\int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \\ &\quad = \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma) - \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A})) \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \geq 0, \end{aligned} \tag{2.16}$$

where we used the fact that $\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}) = 0$ since $|\,\mathbf{curl}\, \boldsymbol{A}(x)| \leq d(x)$ for a.e. $x \in \Omega$. Applying (2.15) and (2.16) to (2.14) leads to

$$\begin{aligned} C_\nu \|\boldsymbol{A}_\gamma - \boldsymbol{A}\|^2_{\boldsymbol{X}_{N,0}} \leq &\int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x \\ &- \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A})\, \mathrm{d}x. \end{aligned}$$

Since, by the convergence (2.9), the right-hand side of the above inequality tends to 0 as $\gamma \to \infty$, it follows that

$$\boldsymbol{A}_\gamma \to \boldsymbol{A} \quad \text{strongly in } \boldsymbol{X}_{N,0} \quad \text{as } \gamma \to \infty. \tag{2.17}$$

We are left to show that $\boldsymbol{A}$ is a solution to ($\mathrm{VI}_{\mathrm{sol}}$). To this end, let $\boldsymbol{v} \in \boldsymbol{K}$. Testing ($\mathrm{VE}_\gamma$) with $\boldsymbol{v} - \boldsymbol{A}_\gamma$ yields that

$$
\begin{aligned}
&\int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x \\
&\quad = \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x \\
&\qquad + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x \\
&\quad = \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x \\
&\qquad + \gamma \int_\Omega (\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) - \underbrace{\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{v})}_{=0}) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x \\
&\quad \le \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x,
\end{aligned} \tag{2.18}
$$

where we exploited the fact that $\boldsymbol{v} \in \boldsymbol{K}$ and that $\boldsymbol{\theta}_\gamma$ is monotone. In view of (2.17), after passing to the limit $\gamma \to \infty$ in (2.18), we obtain that $\boldsymbol{A}$ is a solution to ($\mathrm{VI}_{\mathrm{sol}}$). Uniqueness is obtained by a standard energy argument exploiting once again the monotonicity of $\boldsymbol{\theta}_\gamma$. This concludes the proof. ∎

3. Well-posedness.

THEOREM 3.1. *Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ be given with the associated Helmholtz decomposition*

$$
\boldsymbol{J} = \boldsymbol{J}_{\mathrm{sol}} + \nabla \phi_{\boldsymbol{J}} \in \boldsymbol{H}(\mathrm{div}{=}0) \oplus \nabla H_0^1(\Omega). \tag{3.1}
$$

Furthermore, let $\boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$ denote the unique solution to ($\mathrm{VI}_{\mathrm{sol}}$) *for $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$ given by* (3.1)*. Then, $(\boldsymbol{A}, \phi_{\boldsymbol{J}})$ is the unique solution to* (VI)*, and there exists a unique $\boldsymbol{m} \in \boldsymbol{X}_{N,0}$, the so-called dual multiplier, such that*

$$
\begin{cases}
\displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \mathbf{curl}\,\boldsymbol{m} \cdot \mathbf{curl}\,\boldsymbol{v} + \nabla \phi_{\boldsymbol{J}} \cdot \boldsymbol{v}\,\mathrm{d}x \\
\displaystyle = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\
\displaystyle\int_\Omega \boldsymbol{A} \cdot \nabla \psi\,\mathrm{d}x = 0 \quad \forall \psi \in H_0^1(\Omega), \\
\displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{m} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}) \le 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}.
\end{cases} \tag{3.2}
$$

Proof. First, the unique solution $\boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$ of ($\mathrm{VI}_{\mathrm{sol}}$) satisfies

$$
\begin{aligned}
&\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v}_{\mathrm{sol}} - \boldsymbol{A})\,\mathrm{d}x \\
&\qquad \ge \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot (\boldsymbol{v}_{\mathrm{sol}} - \boldsymbol{A})\,\mathrm{d}x \quad \forall \boldsymbol{v}_{\mathrm{sol}} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0).
\end{aligned} \tag{3.3}
$$

Recalling the Helmholtz decomposition (1.6), it holds that

$$
\forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\ \exists (\boldsymbol{v}_{\mathrm{sol}}, \phi_{\boldsymbol{v}}) \in \boldsymbol{H}(\mathrm{div}{=}0) \times H_0^1(\Omega) : \boldsymbol{v} = \boldsymbol{v}_{\mathrm{sol}} + \nabla \phi_{\boldsymbol{v}}.
$$

If $\boldsymbol{v} \in \boldsymbol{K}$ (see (1.2) for its definition), then we obtain from the above decomposition that $\boldsymbol{v}_{\text{sol}} \in \boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)$ since $|\,\mathbf{curl}\,\boldsymbol{v}_{\text{sol}}| = |\mathbf{curl}(\boldsymbol{v}_{\text{sol}} + \nabla\phi_{\boldsymbol{v}})| = |\,\mathbf{curl}\,\boldsymbol{v}| \le d$ a.e. in Ω. For this reason,

$$\forall \boldsymbol{v} \in \boldsymbol{K}\ \ \exists(\boldsymbol{v}_{\text{sol}}, \phi_{\boldsymbol{v}}) \in (\boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)) \times H_0^1(\Omega) : \boldsymbol{v} = \boldsymbol{v}_{\text{sol}} + \nabla\phi_{\boldsymbol{v}}. \tag{3.4}$$

As $\boldsymbol{J}_{\text{sol}} \in \boldsymbol{H}(\text{div}{=}0)$ and $\mathbf{curl}\,\nabla \equiv 0$, it follows by applying (3.4) to (3.3) that

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A})\,\mathrm{d}x \ge \int_\Omega \boldsymbol{J}_{\text{sol}} \cdot (\boldsymbol{v} - \boldsymbol{A})\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{3.5}$$

Applying the decomposition (3.1) to (3.5) and taking $\boldsymbol{A} \in \boldsymbol{H}(\text{div}{=}0)$ into account, we obtain that

$$\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A})\,\mathrm{d}x + \int_\Omega \nabla\phi_{\boldsymbol{J}} \cdot \boldsymbol{v}\,\mathrm{d}x \ge \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{v} - \boldsymbol{A})\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Hence, $(\boldsymbol{A}, \phi_{\boldsymbol{J}}) \in (\boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)) \times H_0^1(\Omega)$ is a solution to (VI). Toward uniqueness, let $(\widetilde{\boldsymbol{A}}, \widetilde{\phi}) \in (\boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)) \times H_0^1(\Omega)$ be another solution to (VI). Considering only test functions $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\text{div}{=}0)$ in (VI), we obtain due to (3.1) and (2.1) that $\widetilde{\boldsymbol{A}}$ is a solution to (VI$_{\text{sol}}$), which by the uniqueness of the solution to (VI$_{\text{sol}}$) implies that $\widetilde{\boldsymbol{A}} = \boldsymbol{A}$. Next, for any $\varphi \in H_0^1(\Omega)$, we have that $\boldsymbol{A} + \nabla\varphi \in \boldsymbol{K}$ since $|\mathbf{curl}(\boldsymbol{A} + \nabla\varphi)| = |\,\mathbf{curl}\,\boldsymbol{A}| \le d$ a.e. in Ω. Thus, testing the variational inequality for the solution $(\boldsymbol{A}, \widetilde{\phi})$ to (VI) with $\boldsymbol{v} = \boldsymbol{A} + \nabla\varphi \in \boldsymbol{K}$, we obtain due to $\mathbf{curl}\,\nabla \equiv 0$ that

$$\int_\Omega \nabla\widetilde{\phi} \cdot \nabla\varphi\,\mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \nabla\varphi\,\mathrm{d}x \quad \forall \varphi \in H_0^1(\Omega). \tag{3.6}$$

Applying (3.1) to (3.6), we end up with

$$\int_\Omega \nabla(\widetilde{\phi} - \phi_{\boldsymbol{J}}) \cdot \nabla\varphi\,\mathrm{d}x = 0 \quad \forall \varphi \in H_0^1(\Omega) \quad \Rightarrow \quad \widetilde{\phi} = \phi_{\boldsymbol{J}}.$$

In conclusion, $(\boldsymbol{A}, \phi_{\boldsymbol{J}})$ is the unique solution to (VI).

Let us now prove that $(\boldsymbol{A}, \phi_{\boldsymbol{J}})$ satisfies the dual characterization (3.2). In view of Lemma 2.3, $\{\boldsymbol{A}_\gamma\}_{\gamma>0} \subset \boldsymbol{H}_0(\mathbf{curl})$ is bounded in $\boldsymbol{H}(\mathbf{curl})$, and hence it follows from (VE$_\gamma$) that $\{\gamma\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma)\}_{\gamma>0}$ is bounded in $[\mathbf{curl}\,\boldsymbol{X}_{N,0}]^*$. Therefore, we find $\boldsymbol{\Psi} \in [\mathbf{curl}\,\boldsymbol{X}_{N,0}]^*$ such that after selecting a subsequence

$$\gamma\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \rightharpoonup \boldsymbol{\Psi} \quad \text{weakly in } [\mathbf{curl}\,\boldsymbol{X}_{N,0}]^* \quad \text{as } \gamma \to \infty. \tag{3.7}$$

At the same time, since $\mathbf{curl}\,\boldsymbol{X}_{N,0} \subset \boldsymbol{L}^2(\Omega)$ is closed, Riesz's representation theorem implies the existence of $\boldsymbol{m} \in \boldsymbol{X}_{N,0}$ such that

$$\boldsymbol{\Psi}(\mathbf{curl}\,\boldsymbol{v}) = \int_\Omega \mathbf{curl}\,\boldsymbol{m} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \tag{3.8}$$

Combining (3.7) with (3.8), it follows that

$$\gamma\int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \to \int_\Omega \mathbf{curl}\,\boldsymbol{m} \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0} \quad \text{as } \gamma \to \infty. \tag{3.9}$$

Due to (3.9) and the strong convergence $\boldsymbol{A}_\gamma \to \boldsymbol{A}$ in $\boldsymbol{X}_{N,0}$ as $\gamma \to \infty$ (see Lemma 2.3), we obtain after passing to the limit $\gamma \to \infty$ in (VE$_\gamma$) that

$$\begin{aligned}\int_\Omega \nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{m}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ = \int_\Omega \boldsymbol{J}_{\mathrm{sol}}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{X}_{N,0}.\end{aligned} \tag{3.10}$$

As a result of (1.8), (3.10) implies that $\boldsymbol{m}\in\boldsymbol{X}_{N,0}$ is unique. Indeed, assuming that there exist $\boldsymbol{m}_1,\boldsymbol{m}_2\in\boldsymbol{X}_{N,0}$ satisfying (3.10), it follows that

$$\int_\Omega \mathbf{curl}(\boldsymbol{m}_1-\boldsymbol{m}_2)\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x = 0 \quad \forall \boldsymbol{v}\in\boldsymbol{X}_{N,0}.$$

Then, inserting $\boldsymbol{v}=\boldsymbol{m}_1-\boldsymbol{m}_2\in\boldsymbol{X}_{N,0}$ and taking (1.8) into account, we obtain that $\boldsymbol{m}_1=\boldsymbol{m}_2$. Now, since $\mathbf{curl}\,\nabla\equiv 0$ and $\boldsymbol{J}_{\mathrm{sol}}\in\boldsymbol{H}(\mathrm{div}{=}0)$, it follows from the Helmholtz decomposition $\boldsymbol{H}_0(\mathbf{curl}) = \boldsymbol{X}_{N,0}\oplus\nabla H_0^1(\Omega)$ that the variational equality (3.10) is valid for all test functions in $\boldsymbol{H}_0(\mathbf{curl})$, i.e., it holds for all $\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl})$ that

$$\begin{aligned}\int_\Omega \nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{m}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ = \int_\Omega \boldsymbol{J}_{\mathrm{sol}}\cdot\boldsymbol{v}\,\mathrm{d}x \underbrace{=}_{(3.1)} \int_\Omega (\boldsymbol{J}-\nabla\phi_{\boldsymbol{J}})\cdot\boldsymbol{v}\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

For the last part in (3.2) we take $\boldsymbol{v}\in\boldsymbol{K}\subset\boldsymbol{H}_0(\mathbf{curl})$ and test equation (3.11) with $\boldsymbol{v}-\boldsymbol{A}$ to deduce that

$$\begin{aligned}&\int_\Omega \mathbf{curl}\,\boldsymbol{m}\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x \\ &\quad= \int_\Omega (\boldsymbol{J}-\nabla\phi_{\boldsymbol{J}})\cdot(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x - \int_\Omega \nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{A})\,\mathrm{d}x \underbrace{\le}_{(\mathrm{VI})} 0.\end{aligned}$$

To summarize, we have proved that there is a unique $\boldsymbol{m}\in\boldsymbol{X}_{N,0}$ such that the unique solution $(\boldsymbol{A},\phi_{\boldsymbol{J}})$ of (VI) satisfies the dual characterization (3.2). This completes the proof. ∎

4. Dual multiplier regularity. This section is devoted to the dual multiplier regularity analysis. Important tools for our study include elliptic regularity results for Dirichlet and Neumann problems [22, 49] (cf. [13]) and the L^p-Helmholtz decomposition [15], which we recall in the following lemmata.

LEMMA 4.1 (Jerison and Kenig [22]). *There exists a $p_0>3$ such that for any $p\in(p_0',p_0)$ the Dirichlet problem*

$$\int_\Omega \nabla u\cdot\nabla\psi\,\mathrm{d}x = F(\psi) \quad \forall\psi\in W_0^{1,p'}(\Omega)$$

admits for every $F\in[W_0^{1,p'}(\Omega)]^*$ a unique solution $u\in W_0^{1,p}(\Omega)$. If Ω is of class $\mathcal{C}^1$, then the claim holds true for all $p\in(1,\infty)$.

LEMMA 4.2 (Zanger [49]). *There exists a $p_0>3$ such that for any $p\in(p_0',p_0)$ the Neumann problem*

$$\int_\Omega \nabla u\cdot\nabla\psi\,\mathrm{d}x = F(\psi) \quad \forall\psi\in W^{1,p'}(\Omega)$$

admits, for every $F\in[W^{1,p'}(\Omega)]^*$ with $F(1)=0$, a solution $u\in W^{1,p}(\Omega)$.

LEMMA 4.3 (Fujiwara–Morimoto [15]; cf. [37]). *Let Ω be of class $\mathcal{C}^1$ and $p \in (1,\infty)$. Then, every $\boldsymbol{q} \in \boldsymbol{L}^p(\Omega)$ admits the following decomposition:*

$$\boldsymbol{q} = \nabla\eta + \boldsymbol{u}$$

for some $\eta \in L^p_{\mathrm{loc}}(\Omega)$ satisfying $\nabla\eta \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \cap \boldsymbol{H}_0(\mathrm{div}{=}0)$.

THEOREM 4.4. *Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, and let $(\boldsymbol{A}, \boldsymbol{m}, \phi_{\boldsymbol{J}}) \in \boldsymbol{X}_{N,0} \times \boldsymbol{X}_{N,0} \times H^1_0(\Omega)$ denote the unique solution to the dual formulation* (3.2)*. Then, the following multiplier regularity results hold true:*

(i) *If $\boldsymbol{J} \in \boldsymbol{L}^p(\Omega)$ and $d \in L^p(\Omega)$ for $p \in [2,3]$, then $\phi_{\boldsymbol{J}} \in W^{1,p}_0(\Omega)$ and $\mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$.*

(ii) *If $\boldsymbol{J} \in \boldsymbol{L}^p(\Omega)$ and $d \in L^p(\Omega)$ for $p \in [2,6]$, and Ω is of class $\mathcal{C}^{1,1}$, then $\phi_{\boldsymbol{J}} \in W^{1,p}_0(\Omega)$ and $\mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$.*

(iii) *If $\boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl})$, $d \in L^p(\Omega)$ for $p \in [2,\infty)$, and Ω is of class $\mathcal{C}^{2,1}$, then $\mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$.*

(iv) *If $\boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl})$, $d \in L^\infty(\Omega)$, $\nu(\cdot, |\mathbf{curl}\,\boldsymbol{A}|) \in \mathcal{C}^{0,1}(\overline{\Omega})$, and Ω is of class $\mathcal{C}^{2,1}$, then $\mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{L}^\infty(\Omega)$.*

Remark 4.5. The third assumption in (iv) holds, for example, in the linear case $\nu(\cdot, s) = \nu_0(\cdot)$ for some $\nu_0 \in \mathcal{C}^{0,1}(\overline{\Omega})$.

Proof. The proof is split into four parts.

Step 1. Suppose that $\boldsymbol{J} \in \boldsymbol{L}^p(\Omega)$ and $d \in L^p(\Omega)$ for $p \in [2,3]$. Since $\partial\Omega$ is connected, (1.7) yields the decomposition

$$\boldsymbol{J} = \nabla y + \mathbf{curl}\,\boldsymbol{w} \tag{4.1}$$

for some $y \in H^1_0(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}{=}0)$. On the other hand, since $\boldsymbol{J} \in \boldsymbol{L}^p(\Omega)$ with $p \in [2,3]$, Lemma 4.1 implies that the Dirichlet problem

$$\int_\Omega \nabla u \cdot \nabla\psi \,\mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \nabla\psi \,\mathrm{d}x \quad \forall \psi \in W^{1,p'}_0(\Omega)$$

admits a unique solution $u \in W^{1,p}_0(\Omega)$. Setting $\psi = u - y \in H^1_0(\Omega) \hookrightarrow W^{1,p'}_0(\Omega)$ and exploiting that $\mathbf{curl}\,\nabla \equiv 0$, we obtain

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla(u-y)\,\mathrm{d}x &= \int_\Omega \boldsymbol{J} \cdot \nabla(u-y)\,\mathrm{d}x \\ &= \int_\Omega (\boldsymbol{J} - \mathbf{curl}\,\boldsymbol{w}) \cdot \nabla(u-y)\,\mathrm{d}x = \int_\Omega \nabla y \cdot \nabla(u-y)\,\mathrm{d}x,\end{aligned}$$

and consequently $y = u \in W^{1,p}_0(\Omega)$ follows from the standard Poincaré–Friedrichs inequality in $H^1_0(\Omega)$. Now, applying (4.1) to (3.2) results in

$$\begin{aligned}&\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\boldsymbol{v} + \mathbf{curl}\,\boldsymbol{m} \cdot \mathbf{curl}\,\boldsymbol{v} + \nabla\phi_{\boldsymbol{J}} \cdot \boldsymbol{v}\,\mathrm{d}x \\ &\quad = \int_\Omega (\nabla y + \mathbf{curl}\,\boldsymbol{w}) \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_\Omega (\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} - \boldsymbol{w}) \cdot \mathbf{curl}\,\boldsymbol{v} \\ &\quad = \int_\Omega \nabla(y - \phi_{\boldsymbol{J}}) \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}).\end{aligned} \tag{4.2}$$

Therefore, inserting $\boldsymbol{v} = \nabla(y - \phi_{\boldsymbol{J}}) \in \boldsymbol{H}_0(\mathbf{curl})$ in (4.2), it follows that

$$\phi_{\boldsymbol{J}} = y = u \in W_0^{1,p}(\Omega). \tag{4.3}$$

Invoking (4.3) in (4.2) and then making use of the distributional definition of the curl operator, we infer that

$$\begin{aligned}&\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} - \boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl})\\ &\quad \text{with} \quad \mathbf{curl}(\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} - \boldsymbol{w}) = 0.\end{aligned}$$

The above regularity property allows us to employ (1.5) to attain the following decomposition:

$$\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} - \boldsymbol{w} = \boldsymbol{z} + \nabla\varphi \tag{4.4}$$

for some $\varphi \in H^1(\Omega)$ and $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\text{div=0})$. Next, in view of the regularity property $d \in L^p(\Omega)$, $\boldsymbol{A} \in \boldsymbol{K}$, (1.2), and Assumption 1.1, we have

$$\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \in \boldsymbol{L}^p(\Omega),$$

and hence the functional

$$F(\psi) := \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \nabla\psi \,\mathrm{d}x \quad \forall \psi \in W^{1,p'}(\Omega)$$

is well defined in $[W^{1,p'}(\Omega)]^*$ and satisfies $F(1) = 0$. For this reason, as $p \in [2,3]$, Lemma 4.2 yields the existence of $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla\psi \,\mathrm{d}x = F(\psi) \quad \forall \psi \in W^{1,p'}(\Omega).$$

Inserting $\psi = u - \varphi \in H^1(\Omega) \hookrightarrow W^{1,p'}(\Omega)$ and taking $\mathbf{curl}\,\boldsymbol{m} - \boldsymbol{w} - \boldsymbol{z} \in \boldsymbol{H}_0(\text{div=0})$ into account, it follows that

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla(u - \varphi)\,\mathrm{d}x &= \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \nabla(u - \varphi)\,\mathrm{d}x\\ &= \int_\Omega (\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} - \boldsymbol{w} - \boldsymbol{z}) \cdot \nabla(u - \varphi)\,\mathrm{d}x\\ &\underbrace{=}_{(4.4)} \int_\Omega \nabla\varphi \cdot \nabla(u - \varphi)\,\mathrm{d}x,\end{aligned}$$

and so $\nabla\varphi = \nabla u \in \boldsymbol{L}^p(\Omega)$. At last, since $p \in [2,3]$, we know from the embedding result [10, Theorem 2] that

$$\boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\text{div=0}) \hookrightarrow \boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega).$$

Altogether, $\mathbf{curl}\,\boldsymbol{m} = \nabla u + \boldsymbol{w} + \boldsymbol{z} - \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \in \boldsymbol{L}^p(\Omega)$. In conclusion, (i) is valid.

Step 2. Suppose that $\boldsymbol{J} \in \boldsymbol{L}^p(\Omega)$ and $d \in L^p(\Omega)$ for $p \in [2,6]$, and Ω is of class $\mathcal{C}^{1,1}$. Then, by the second claim of Lemma 4.1, we may deduce as in Step 1 that $\phi_{\boldsymbol{J}} \in W_0^{1,p}(\Omega)$ and

$$\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} - \boldsymbol{w} = \nabla\varphi + \boldsymbol{z} \tag{4.5}$$

for some $\boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\text{div=0})$ and $\varphi \in H^1(\Omega)$. Furthermore, the $\mathcal{C}^1$-property of Ω allows us to apply Lemma 4.3 to the vector field $\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \in \boldsymbol{L}^p(\Omega)$. In this way, we find $\eta \in L^p_{\text{loc}}(\Omega)$ satisfying $\nabla\eta \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \cap \boldsymbol{H}_0(\text{div=0})$ such that

$$\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} = \nabla\eta + \boldsymbol{u}. \tag{4.6}$$

Then, applying (4.6) to (4.5) results in $\mathbf{curl}\,\boldsymbol{m} + \boldsymbol{u} - \boldsymbol{w} - \boldsymbol{z} = \nabla(\varphi - \eta)$, and so

$$0 = \int_\Omega \underbrace{(\mathbf{curl}\,\boldsymbol{m} + \boldsymbol{u} - \boldsymbol{w} - \boldsymbol{z})}_{\in \boldsymbol{H}_0(\text{div=0})} \cdot \nabla(\varphi - \eta)\,\mathrm{d}x = \int_\Omega |\nabla(\varphi - \eta)|^2\,\mathrm{d}x.$$

In conclusion,

$$\mathbf{curl}\,\boldsymbol{m} = \boldsymbol{w} + \boldsymbol{z} - \boldsymbol{u} \in \boldsymbol{L}^p(\Omega),$$

since $\boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\text{div=0}) \hookrightarrow \boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$ holds true as Ω is of class $\mathcal{C}^{1,1}$ (see [2, Theorem 2.9]). This finishes the proof for (ii).

Step 3. Suppose that $\boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl})$, $d \in L^p(\Omega)$ for $p \in [2, \infty)$, and Ω is of class $\mathcal{C}^{2,1}$. According to (3.2),

$$\begin{aligned} &\int_\Omega (\boldsymbol{J} - \nabla\phi_{\boldsymbol{J}}) \cdot \nabla\psi\,\mathrm{d}x = 0 \quad \forall\psi \in H^1_0(\Omega) \\ &\quad\Rightarrow \quad \boldsymbol{J} - \nabla\phi_{\boldsymbol{J}} \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\text{div=0}) \hookrightarrow \boldsymbol{H}^1(\Omega), \end{aligned}$$

where the last embedding holds as before since Ω is of class $\mathcal{C}^{1,1}$ (see [2, Theorem 2.12]). Applying the above regularity to (3.2), it follows again by the distributional definition of the curl operator that

$$\mathbf{curl}(\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m}) \in \boldsymbol{H}^1(\Omega). \tag{4.7}$$

This particularly implies that $\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} \in \boldsymbol{H}(\mathbf{curl})$ such that (1.5) yields

$$\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m} = \nabla\varphi + \boldsymbol{z} \tag{4.8}$$

for some $\varphi \in H^1(\Omega)$ and $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\text{div=0})$. Combining (4.7) and (4.8) together leads to

$$\mathbf{curl}\,\boldsymbol{z} = \mathbf{curl}(\nabla\varphi + \boldsymbol{z}) = \mathbf{curl}(\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} + \mathbf{curl}\,\boldsymbol{m}) \in \boldsymbol{H}^1(\Omega).$$

Therefore, since Ω is of class $\mathcal{C}^{2,1}$ and by a well-known embedding result [2, Corollary 2.15], the above regularity property along with $\boldsymbol{z} \in \boldsymbol{H}_0(\text{div=0})$ implies that $\boldsymbol{z} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{\mathcal{C}}(\overline{\Omega})$. Now, as in the previous step, we apply Lemma 4.3 to the vector field $\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \in \boldsymbol{L}^p(\Omega)$ to obtain $\eta \in L^p_{\text{loc}}(\Omega)$ satisfying $\nabla\eta \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{U} \in \boldsymbol{L}^p(\Omega) \cap \boldsymbol{H}_0(\text{div=0})$ such that

$$\nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} = \nabla\eta + \boldsymbol{u} \quad \underbrace{\Rightarrow}_{(4.8)} \quad \mathbf{curl}\,\boldsymbol{m} + \boldsymbol{U} - \boldsymbol{z} = \nabla(\varphi - \eta).$$

By the same argumentation as before, the regularity property $\mathbf{curl}\,\boldsymbol{m} + \boldsymbol{U} - \boldsymbol{z} \in \boldsymbol{H}_0(\text{div=0})$ implies that $\nabla(\varphi - \eta) = 0$. In conclusion, $\mathbf{curl}\,\boldsymbol{m} = \boldsymbol{z} - \boldsymbol{U} \in \boldsymbol{L}^p(\Omega)$.

Step 4. Suppose that all the assumptions from Step 3 are satisfied with $p=\infty$ and $\nu(\cdot,|\mathbf{curl}\,\boldsymbol{A}|)\in\mathcal{C}^{0,1}(\overline{\Omega})$. By standard arguments, the latter regularity assumption implies

$$\begin{aligned}(4.9)\qquad &\underbrace{\nu(\cdot,|\mathbf{curl}\,\boldsymbol{A}|)}_{\in\mathcal{C}^{0,1}(\overline{\Omega})}\,\underbrace{\mathbf{curl}\,\boldsymbol{A}}_{\in\boldsymbol{H}_0(\mathrm{div})}\in\boldsymbol{H}_0(\mathrm{div})\\ &\text{with}\quad \mathrm{div}\,(\nu(\cdot,|\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A})=\nabla\nu(\cdot,|\mathbf{curl}\,\boldsymbol{A}|)\cdot\mathbf{curl}\,\boldsymbol{A}.\end{aligned}$$

Also, we already know that (4.8) holds true with $\varphi\in H^1(\Omega)$ and $\boldsymbol{z}\in\boldsymbol{\mathcal{C}}(\overline{\Omega})$. We apply now (1.5) and obtain that $\nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}=\nabla\theta+\boldsymbol{E}$ for $\theta\in H^1(\Omega)$ and $\boldsymbol{E}\in\boldsymbol{H}_0(\mathrm{div}{=}0)$. Then, thanks to (4.9), we see that $u:=\theta\in H^1(\Omega)$ solves the elliptic problem

$$(4.10)\qquad \begin{cases}(-\Delta+1)u=-\nabla\nu(\cdot,|\mathbf{curl}\,\boldsymbol{A}|)\cdot\mathbf{curl}\,\boldsymbol{A}+\theta \text{ in }\Omega,\\ \qquad\nabla u\cdot\boldsymbol{n}=0\text{ on }\partial\Omega.\end{cases}$$

By the regularity of Ω, [16, Theorem 2.2.2.5] implies that $\theta=u\in H^2(\Omega)\hookrightarrow\mathcal{C}(\overline{\Omega})$. Thus, since $\nabla\nu(\cdot,|\mathbf{curl}\,\boldsymbol{A}|)\in\boldsymbol{L}^\infty(\Omega)$ and $\mathbf{curl}\,\boldsymbol{A}\in\boldsymbol{L}^\infty(\Omega)$, the right-hand side of (4.10) lies in $L^\infty(\Omega)$. For this reason, [16, Theorem 2.4.2.7] yields the regularity property $\theta=u\in W^{2,p}(\Omega)$ for all $p\in[1,\infty)$ such that

$$\begin{aligned}\nabla\theta\in\boldsymbol{W}^{1,4}(\Omega)\hookrightarrow\boldsymbol{\mathcal{C}}(\overline{\Omega})\;\Rightarrow\;\boldsymbol{E}&=\nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A}-\nabla\theta\in\boldsymbol{L}^\infty(\Omega)\\ &\underbrace{\Rightarrow}_{(4.8)\&\nabla\varphi=\nabla\theta}\quad\mathbf{curl}\,\boldsymbol{m}=\boldsymbol{z}-\boldsymbol{E}\in\boldsymbol{L}^\infty(\Omega),\end{aligned}$$

where $\nabla\varphi=\nabla\theta$ is obtained as in the previous step. This completes the proof. ∎

5. Optimal control.

We denote the control-to-state mapping for (VI) by

$$\boldsymbol{G}:\boldsymbol{L}^2(\Omega)\to\boldsymbol{X}_{N,0},\quad \boldsymbol{J}\mapsto\boldsymbol{A}.$$

In view of Theorem 3.1, the restriction of $\boldsymbol{G}$ onto the subspace $\boldsymbol{H}(\mathrm{div}{=}0)$, i.e., $\boldsymbol{G}:\boldsymbol{H}(\mathrm{div}{=}0)\to\boldsymbol{X}_{N,0}$, serves as the control-to-state mapping for ($\mathrm{VI_{sol}}$). Invoking $\boldsymbol{G}$, we reformulate the optimal control problem (1.3) as

$$(\mathrm{P})\qquad \min_{\boldsymbol{J}\in\boldsymbol{L}^2(\Omega)}F(\boldsymbol{J}):=\frac{1}{2}\|\mathbf{curl}\,\boldsymbol{G}(\boldsymbol{J})-\boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{\lambda}{2}\|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

LEMMA 5.1. *The optimal control problem* (P) *admits an optimal solution. Every optimal solution to* (P) *enjoys a higher regularity property in* $\boldsymbol{H}(\mathrm{div}{=}0)$.

Proof. By standard techniques (cf. [43, Proposition 3.2]), the control-to-state mapping $\boldsymbol{G}:\boldsymbol{L}^2(\Omega)\to\boldsymbol{X}_{N,0}$ is weak-strong continuous. Thus, the existence of an optimal solution to (P) follows from classical arguments. Let $\boldsymbol{J}^\star\in\boldsymbol{L}^2(\Omega)$ be an optimal solution to (P). Our goal now is to prove the higher regularity property $\boldsymbol{J}^\star\in\boldsymbol{H}(\mathrm{div}{=}0)$. In view of (1.6), $\boldsymbol{J}^\star$ admits the following orthogonal decomposition:

$$\begin{aligned}(5.1)\qquad &\boldsymbol{J}^\star=\boldsymbol{J}^\star_{\mathrm{sol}}+\nabla\phi_{\boldsymbol{J}^\star}\in\boldsymbol{H}(\mathrm{div}{=}0)\oplus\nabla H^1_0(\Omega)\\ &\Rightarrow\quad \|\boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)}=\|\boldsymbol{J}^\star_{\mathrm{sol}}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Let us now consider the optimal control problem

$$(5.2)\qquad \min_{\boldsymbol{J}\in\boldsymbol{H}(\mathrm{div}=0)}F(\boldsymbol{J}).$$

Since $\boldsymbol{G}_{|\boldsymbol{H}(\mathrm{div}=0)}$ is as well weak-strong continuous, there exists a minimizer $\widetilde{\boldsymbol{J}}^\star_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}=0)$ for (5.2), i.e.,

$$F(\widetilde{\boldsymbol{J}}^\star_{\mathrm{sol}}) \le F(\boldsymbol{J}) \qquad \forall \boldsymbol{J} \in \boldsymbol{H}(\mathrm{div}=0).$$

It then follows that

$$\begin{aligned} F(\widetilde{\boldsymbol{J}}^\star_{\mathrm{sol}}) \le F(\boldsymbol{J}^\star_{\mathrm{sol}}) = &\ \frac{1}{2}\|\mathbf{curl}\,\boldsymbol{G}(\boldsymbol{J}^\star_{\mathrm{sol}}) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}^\star_{\mathrm{sol}}\|^2_{\boldsymbol{L}^2(\Omega)} \\ \underbrace{=}_{(5.1)} &\ \frac{1}{2}\|\mathbf{curl}\,\boldsymbol{G}(\boldsymbol{J}^\star_{\mathrm{sol}}) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{\lambda}{2}\|\nabla\phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)} \\ = &\ F(\boldsymbol{J}^\star) - \frac{\lambda}{2}\|\nabla\phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)} \le F(\widetilde{\boldsymbol{J}}^\star_{\mathrm{sol}}) - \frac{\lambda}{2}\|\nabla\phi_{\boldsymbol{J}^\star}\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned}$$

where for the last inequality we used the fact that $\boldsymbol{J}^\star$ is optimal for (P). The above inequalities yield that $\nabla\phi_{\boldsymbol{J}^\star} = 0$, which in turn implies $\phi_{\boldsymbol{J}^\star} = 0$ due to $\phi_{\boldsymbol{J}^\star} \in H^1_0(\Omega)$. Thus, by (5.1), we come to the conclusion that $\boldsymbol{J}^\star = \boldsymbol{J}^\star_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}=0)$. This completes the proof. ∎

Remark 5.2. Lemma 5.1 implies that any optimal solution $\boldsymbol{J}^\star \in \boldsymbol{L}^2(\Omega)$ of (P) is also an optimal solution of the $\boldsymbol{H}(\mathrm{div}=0)$-reduced problem

$$\min_{\boldsymbol{J}\in\boldsymbol{H}(\mathrm{div}=0)} \frac{1}{2}\|\mathbf{curl}\,\boldsymbol{G}(\boldsymbol{J}) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{5.3}$$

On the other hand, given an optimal solution $\boldsymbol{J}^\star_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}=0)$ for (5.3), it is straightforward to verify that $\boldsymbol{J}^\star_{\mathrm{sol}}$ is as well an optimal solution of (P). In that sense, both problems (P) and (5.3) are equivalent, and it is therefore sufficient to focus on the derivation of optimality conditions for (5.3).

5.1. Necessary optimality conditions for (P). This section is devoted to establishing an optimality system for (P). To overcome the underlying nonsmoothness, we consider a smoothed version of (5.3) built upon the approximation (VE_γ) in the spirit of Barbu [4]: Given an arbitrarily fixed optimal solution $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}=0)$ to (P), we consider

$$\min_{\boldsymbol{J}_\gamma\in\boldsymbol{H}(\mathrm{div}=0)} F_\gamma(\boldsymbol{J}_\gamma), \tag{P_γ}$$

with

$$F_\gamma(\boldsymbol{J}_\gamma) := \frac{1}{2}\|\mathbf{curl}\,\boldsymbol{G}_\gamma(\boldsymbol{J}_\gamma) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}_\gamma\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\lambda}{2}\|\boldsymbol{J}_\gamma - \boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)}$$

for every $\boldsymbol{J}_\gamma \in \boldsymbol{H}(\mathrm{div}=0)$, where $\boldsymbol{G}_\gamma : \boldsymbol{H}(\mathrm{div}=0) \to \boldsymbol{X}_{N,0}$ denotes the control-to-state mapping for (VE_γ) based on Lemma 2.2. Note that, as a consequence of Lemma 2.3, standard arguments yield that

$$\begin{aligned} &\boldsymbol{J}_\gamma \rightharpoonup \boldsymbol{J} \quad \text{weakly in } \boldsymbol{H}(\mathrm{div}=0) \quad \text{as } \gamma\to\infty \\ &\quad \Rightarrow \quad \boldsymbol{G}_\gamma(\boldsymbol{J}_\gamma) \to \boldsymbol{G}(\boldsymbol{J}) \quad \text{strongly in } \boldsymbol{X}_{N,0} \quad \text{as } \gamma\to\infty. \end{aligned} \tag{5.4}$$

Lemma 5.3. *For all $\gamma > 0$, there exists an optimal solution $\boldsymbol{J}^\star_\gamma \in \boldsymbol{H}(\mathrm{div}=0)$ to the problem* (P_γ).

Proof. As before, by well–known techniques, the mapping $\boldsymbol{G}_\gamma : \boldsymbol{H}(\mathrm{div}{=}0) \to \boldsymbol{X}_{N,0}$ is weak-strong continuous. Therefore, the existence of an optimal solution $\boldsymbol{J}_\gamma^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ of (P_γ) is guaranteed. ∎

We note that in general the uniqueness of optimal solutions to (P_γ) cannot be guaranteed since $\boldsymbol{G}_\gamma$ is nonlinear. Next, for the ease of notation, we introduce a vector version of the nonlinearity ν by means of the mapping

$$\boldsymbol{\mathcal{F}} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, s) \mapsto \nu(x, |s|)s,$$

for which we require the following regularity assumption to hold.

Assumption 5.4. For almost every $x \in \Omega$, the mappings $\nu(x, \cdot) : (0, \infty) \to \mathbb{R}$ and $\boldsymbol{\mathcal{F}}(x, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ are continuously differentiable. Moreover, there is a constant $C > 0$ such that

$$\left| \frac{\partial \boldsymbol{\mathcal{F}}_{\boldsymbol{i}}}{\partial s_j}(x, s) \right| \le C \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}^3,$$

for all $i, j \in \{1, 2, 3\}$.

Assumption 5.4 is obviously satisfied for $\nu \equiv 1$, i.e., for the case where the quasilinearity is not present. A nontrivial example for ν satisfying both Assumption 1.1 and Assumption 5.4 can be found in [43, Example 3.5].

Now, let $\gamma > 0$ be arbitrarily fixed. Further let $\overline{\boldsymbol{J}}, \boldsymbol{J} \in \boldsymbol{H}(\mathrm{div}{=}0)$ and let $\overline{\boldsymbol{A}}_\gamma = \boldsymbol{G}_\gamma(\overline{\boldsymbol{J}})$ be the corresponding state. To obtain differentiability properties of $\boldsymbol{G}_\gamma$, we introduce an auxiliary linear problem:

$$\text{(5.5)} \quad \begin{cases} \text{Find } \mathfrak{A}_\gamma \in \boldsymbol{X}_{N,0} \text{ s.t.} \\ \displaystyle\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \overline{\boldsymbol{A}}_\gamma) \mathbf{curl}\, \mathfrak{A}_\gamma \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x + \gamma \int_\Omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \overline{\boldsymbol{A}}_\gamma) \mathbf{curl}\, \mathfrak{A}_\gamma \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x \\ \displaystyle = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \end{cases}$$

Now, [43, Proposition 3.7] provides us with

$$\text{(5.6)} \qquad \mathrm{D}_s \boldsymbol{\mathcal{F}}(x, s) y \cdot y \ge \underline{\nu} |y|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } s, y \in \mathbb{R}^3.$$

Furthermore, as a consequence of Lemma 2.1, it holds that

$$\text{(5.7)} \qquad \mathrm{D}_s \boldsymbol{\theta}_\gamma(x, s) y \cdot y \ge 0 \quad \text{for a.e. } x \in \Omega \text{ and all } s, y \in \mathbb{R}^3.$$

Since $\overline{\boldsymbol{A}}_\gamma$ is fixed, the left-hand side of (5.5) induces a bilinear form. According to the properties (5.6) and (5.7), this bilinear form is coercive. Thanks to the uniform boundedness of $\mathrm{D}_s \boldsymbol{\mathcal{F}}$ from Assumption 5.4 and the uniform boundedness of $\mathrm{D}_s \boldsymbol{\theta}_\gamma$ from Lemma 2.1, the resulting bilinear form is also bounded. Hence, by the Lax–Milgram theorem, (5.5) admits a unique solution $\mathfrak{A}_\gamma \in \boldsymbol{X}_{N,0}$. Taking into account Assumption 5.4, it is readily well known to verify the weak Gâteaux differentiability of $\boldsymbol{G}_\gamma : \boldsymbol{L}^2(\Omega) \to \boldsymbol{X}_{N,0}$ (see the proof of [43, Proposition 3.7]). The weak Gâteaux derivative $\boldsymbol{G}_\gamma'(\overline{\boldsymbol{J}})\boldsymbol{J}$ is given by the unique solution $\mathfrak{A}_\gamma$ to (5.5).

As a consequence of the weak Gâteaux differentiability, standard adjoint techniques imply necessary optimality conditions for (P_γ) which are collected in the following lemma.

LEMMA 5.5. *Let $\gamma > 0$, and let $\boldsymbol{J}^\star_\gamma \in \boldsymbol{H}(\mathrm{div}{=}0)$ be an optimal control for* (P_γ). *Then, there exists a tuple $(\boldsymbol{A}^\star_\gamma, \boldsymbol{Q}^\star_\gamma) \in \boldsymbol{X}_{N,0} \times \boldsymbol{X}_{N,0}$ such that*

$$(5.8)\quad \begin{cases} \displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ \displaystyle = \int_\Omega \boldsymbol{J}^\star_\gamma \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}, \end{cases}$$
$$\begin{cases} \displaystyle\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ \displaystyle + \gamma \int_\Omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\ \displaystyle = \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \boldsymbol{B}_\mathrm{d}) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}, \end{cases}$$
$$\boldsymbol{J}^\star_\gamma = -\frac{1}{2}\lambda^{-1}\boldsymbol{Q}^\star_\gamma + \frac{1}{2}\boldsymbol{J}^\star.$$

In all of what follows, for every $\gamma > 0$, let $\boldsymbol{J}^\star_\gamma \in \boldsymbol{H}(\mathrm{div}{=}0)$ denote an optimal solution to (P_γ) with the associated state and adjoint state $(\boldsymbol{A}^\star_\gamma, \boldsymbol{Q}^\star_\gamma) \in \boldsymbol{X}_{N,0} \times \boldsymbol{X}_{N,0}$ satisfying (5.8). Our final goal is to establish necessary optimality conditions for (P) by means of a limit passage in the necessary optimality systems (5.8). Generally speaking, as part of necessary optimality conditions, one would expect a certain orthogonal relation between the dual multiplier and the optimal state. In the case of (P), difficulties arise due to the involved quasi-linearity and especially the first-order bilateral vector **curl**-constraint in the underlying $\boldsymbol{H}(\mathbf{curl})$-structured variational inequality. Consequently, we can only prove the boundedness of $\{\gamma\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\}_{\gamma>0}$ and $\{\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in $[\mathbf{curl}\,\boldsymbol{X}_{N,0}]^*$ and not in $\boldsymbol{L}^2(\Omega)$. We tackle this difficulty by employing the Hilbert projector into the space $\mathbf{curl}\,\boldsymbol{X}_{N,0}$,

$$(5.9)\qquad \mathbb{P}_{\mathbf{curl}} := \mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}_{N,0}} : \boldsymbol{L}^2(\Omega) \to \mathbf{curl}\,\boldsymbol{X}_{N,0},$$

taking into account that $\mathbf{curl}\,\boldsymbol{X}_{N,0} \subset \boldsymbol{L}^2(\Omega)$ is closed, and the following tailored cut-off type function:

$$(5.10)\qquad \boldsymbol{\varrho} : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{\varrho}(x,s) := s\min\left(1, \frac{d(x)}{|s|}\right) = \begin{cases} s & \text{if } |s| \le d(x), \\ d(x)\dfrac{s}{|s|} & \text{if } |s| > d(x). \end{cases}$$

THEOREM 5.6. *Let $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ be an optimal solution of* (P). *Then, there exist an optimal state $\boldsymbol{A}^\star \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)$, an adjoint state $\boldsymbol{Q}^\star \in \boldsymbol{X}_{N,0}$, a state multiplier $\boldsymbol{m}^\star \in \boldsymbol{X}_{N,0}$, and a triple of adjoint multipliers $(\boldsymbol{n}^\star, \boldsymbol{\sigma}^\star_{d_+}, \boldsymbol{\sigma}^\star_{d_-}) \in \boldsymbol{X}_{N,0} \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ such that*

$$(5.11)\qquad \int_\Omega \nu(\cdot, |\mathbf{curl}\,\boldsymbol{A}^\star|)\mathbf{curl}\,\boldsymbol{A}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{m}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{J}^\star \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0},$$

$$(5.12)\qquad \int_\Omega \mathbf{curl}\,\boldsymbol{m}^\star \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}^\star)\,\mathrm{d}x \le 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

$$(5.13)\qquad \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star)^{\mathrm{T}}\mathbf{curl}\,\boldsymbol{Q}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{n}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star - \boldsymbol{B}_\mathrm{d}) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0},$$

$$\boldsymbol{J}^\star = -\lambda^{-1}\boldsymbol{Q}^\star, \tag{5.14}$$

$$\int_\Omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \, \mathrm{d}x \geq 0, \tag{5.15}$$

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d \frac{\mathbf{curl}\, \boldsymbol{A}^\star}{|\,\mathbf{curl}\, \boldsymbol{A}^\star|} - \mathbf{curl}\, \boldsymbol{A}^\star \right) \mathrm{d}x = 0, \quad \mathbf{curl}\, \boldsymbol{n}^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}. \tag{5.16}$$

After selection of a subsequence, the triplet of adjoint multipliers $(\boldsymbol{n}^\star, \boldsymbol{\sigma}^\star_{d_+}, \boldsymbol{\sigma}^\star_{d_-})$ *is characterized by*

$$\begin{aligned} \gamma \mathbb{P}_{\mathbf{curl}} \left(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \right) &\rightharpoonup \mathbf{curl}\, \boldsymbol{n}^\star && \text{weakly in } \boldsymbol{L}^2(\Omega), \\ \gamma \mathbb{P}_{\mathbf{curl}} \left(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \right) \chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}^\star_\gamma| > d\}} &\rightharpoonup \boldsymbol{\sigma}^\star_{d_+} && \text{weakly in } \boldsymbol{L}^2(\Omega), \\ \gamma \mathbb{P}_{\mathbf{curl}} \left(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \right) \chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}^\star_\gamma| \leq d\}} &\rightharpoonup \boldsymbol{\sigma}^\star_{d_-} && \text{weakly in } \boldsymbol{L}^2(\Omega) \end{aligned}$$

as $\gamma \to \infty$*.*

Remark 5.7. We recall from Theorem 3.1 that (5.11)–(5.12) is equivalent to the primal formulation

$$\int_\Omega \nu(\cdot, |\mathbf{curl}\, \boldsymbol{A}^\star|) \mathbf{curl}\, \boldsymbol{A}^\star \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}^\star)\, \mathrm{d}x \geq \int_\Omega \boldsymbol{J}^\star \cdot (\boldsymbol{v} - \boldsymbol{A}^\star)\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Moreover, combining the variational equality (5.13) and the sign condition (5.15), we obtain a variational inequality for $(\boldsymbol{Q}^\star, \boldsymbol{n}^\star)$ as follows:

$$\begin{aligned} &\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{Q}^\star)\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \\ &\quad \geq \int_\Omega (\mathbf{curl}\, \boldsymbol{A}^\star - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{Q}^\star)\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_{N,0}. \end{aligned}$$

Proof. The proof is divided into three steps.

Step 1 *(limiting process).* Let $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ be an optimal solution of (5.3). Combining Lemma 2.3 with standard arguments from [4] taking into account the penalty term $\frac{\lambda}{2}\|\boldsymbol{J}_\gamma - \boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)}$ in (P_γ), there exists a sequence $\{\boldsymbol{J}^\star_\gamma\}_{\gamma>0} \subset \boldsymbol{H}(\mathrm{div}{=}0)$ of optimal solutions to (P_γ) such that

$$\boldsymbol{J}^\star_\gamma \to \boldsymbol{J}^\star \quad \text{strongly in } \boldsymbol{H}(\mathrm{div}{=}0) \quad \text{as } \gamma \to \infty.$$

Combining Lemma 2.3 with Theorem 3.1 and (5.4) implies that

$$\begin{aligned} \boldsymbol{A}^\star_\gamma &\to \boldsymbol{A}^\star && \text{strongly in } \boldsymbol{X}_{N,0} && \text{as } \gamma \to \infty, \\ \gamma \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma) &\rightharpoonup \mathbf{curl}\, \boldsymbol{m}^\star && \text{weakly in } [\mathbf{curl}\, \boldsymbol{X}_{N,0}]^* && \text{as } \gamma \to \infty, \end{aligned} \tag{5.17}$$

where $(\boldsymbol{A}^\star, \boldsymbol{m}^\star) \in (\boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0)) \times \boldsymbol{X}_{N,0}$ is the unique solution to the dual formulation (3.2) with right-hand side $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ and $\phi = 0$, i.e. (5.11) and (5.12). Let us now invoke the necessary optimality conditions (5.8) for the optimal control $\boldsymbol{J}^\star_\gamma$ of the regularized problem (P_γ). Inserting $\boldsymbol{v} = \boldsymbol{Q}^\star_\gamma$ in (5.8) and taking (5.6) as well as (5.7) into account, we obtain the boundedness of $\{\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in $\boldsymbol{X}_{N,0}$. Furthermore, in view of (5.8), the boundedness of $\{\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ and $\{\boldsymbol{A}^\star_\gamma\}_{\gamma>0}$ as well as Assumption 5.4 yield the boundedness of $\{\gamma \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in the dual space $[\mathbf{curl}\, \boldsymbol{X}_{N,0}]^*$. Altogether, there exist $\boldsymbol{Q}^\star \in \boldsymbol{X}_{N,0}$ and $\boldsymbol{n}^\star_0 \in \boldsymbol{X}_{N,0}$ such that, after selection of a subsequence, we obtain

$$\begin{aligned} \boldsymbol{Q}^\star_\gamma &\rightharpoonup \boldsymbol{Q}^\star && \text{weakly in } \boldsymbol{X}_{N,0} && \text{as } \gamma \to \infty, \\ \gamma \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma &\rightharpoonup \mathbf{curl}\, \boldsymbol{n}^\star_0 && \text{weakly in } [\mathbf{curl}\, \boldsymbol{X}_{N,0}]^* && \text{as } \gamma \to \infty. \end{aligned} \tag{5.18}$$

Taking into account (5.17) and Assumption 5.4, we apply Lebesgue's dominated convergence theorem to deduce

$$\begin{aligned}(5.19)\quad \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)\,\mathbf{curl}\,\boldsymbol{v} \to \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star)\,\mathbf{curl}\,\boldsymbol{v}\\ \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma\to\infty \quad \forall \boldsymbol{v}\in\boldsymbol{X}_{N,0}.\end{aligned}$$

It now follows from (5.17), (5.18), and (5.19) that $(\boldsymbol{Q}^\star, \boldsymbol{n}_0^\star)$ satisfies the adjoint equation (5.13). Moreover, passing to the limit $\gamma\to\infty$ in the representation for the optimal control in (5.8), we conclude that (5.14) is valid.

Step 2 *(orthogonality condition).* For every $\gamma>0$, employing (5.9), we decompose $\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star$ as

$$\begin{aligned}&\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star = \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star) + \boldsymbol{z}_\gamma,\\ (5.20)\qquad &\text{with}\quad \boldsymbol{z}_\gamma := (I-\mathbb{P}_{\mathbf{curl}})(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star) \in (\mathbf{curl}\,\boldsymbol{X}_{N,0})^\perp.\end{aligned}$$

By definition, for every $\gamma>0$, there exists $\boldsymbol{g}_\gamma\in\boldsymbol{X}_{N,0}$, such that

$$\mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star) = \mathbf{curl}\,\boldsymbol{g}_\gamma.$$

Inserting $\boldsymbol{v}=\boldsymbol{g}_\gamma$ in the adjoint equation in (5.8) yields

$$\begin{aligned}&\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star\cdot\mathbf{curl}\,\boldsymbol{g}_\gamma\,\mathrm{d}x + \gamma\int_\Omega(\mathbf{curl}\,\boldsymbol{g}_\gamma+\boldsymbol{z}_\gamma)\cdot\mathbf{curl}\,\boldsymbol{g}_\gamma\,\mathrm{d}x\\ &\quad = \int_\Omega(\mathbf{curl}\,\boldsymbol{A}_\gamma^\star-\boldsymbol{B}_{\mathrm{d}})\cdot\mathbf{curl}\,\boldsymbol{g}_\gamma\,\mathrm{d}x.\end{aligned}$$

Since $\boldsymbol{z}_\gamma\in(\mathbf{curl}\,\boldsymbol{X}_{N,0})^\perp$, the $\boldsymbol{L}^2(\Omega)$-inner product between $\boldsymbol{z}_\gamma$ and $\mathbf{curl}\,\boldsymbol{g}_\gamma$ vanishes such that

$$\begin{aligned}&\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star\cdot\gamma\mathbf{curl}\,\boldsymbol{g}_\gamma\,\mathrm{d}x + \gamma^2\|\mathbf{curl}\,\boldsymbol{g}_\gamma\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\quad = \int_\Omega(\mathbf{curl}\,\boldsymbol{A}_\gamma^\star-\boldsymbol{B}_{\mathrm{d}})\cdot\gamma\mathbf{curl}\,\boldsymbol{g}_\gamma\,\mathrm{d}x.\end{aligned}$$

In view of Assumption 5.4, both sequences $\{\mathbf{curl}\,\boldsymbol{A}_\gamma^\star\}_{\gamma>0}$ and $\{\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star\}_{\gamma>0}$ are bounded in $\boldsymbol{L}^2(\Omega)$. Thus, an application of the Hölder and Young inequalities implies that $\{\gamma\,\mathbf{curl}\,\boldsymbol{g}_\gamma\}_{\gamma>0}$ is bounded in $\boldsymbol{L}^2(\Omega)$. As a consequence, there exists $\boldsymbol{n}^\star\in\boldsymbol{X}_{N,0}$, such that, after selecting a subsequence, it holds that

$$\begin{aligned}(5.21)\quad \gamma\mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star) = \gamma\mathbf{curl}\,\boldsymbol{g}_\gamma \rightharpoonup \mathbf{curl}\,\boldsymbol{n}^\star\\ \text{weakly in } \boldsymbol{L}^2(\Omega)\quad \text{as } \gamma\to\infty,\end{aligned}$$

where we used the fact that $\mathbf{curl}\,\boldsymbol{X}_{N,0}\subset\boldsymbol{L}^2(\Omega)$ is closed. Since, according to (5.18) and (5.20), it also holds that

$$\int_\Omega \gamma\mathbf{curl}\,\boldsymbol{g}_\gamma\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \to \int_\Omega \mathbf{curl}\,\boldsymbol{n}_0^\star\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \text{as } \gamma\to\infty \quad \forall\boldsymbol{v}\in\boldsymbol{X}_{N,0},$$

we infer that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{n}^\star\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \mathbf{curl}\,\boldsymbol{n}_0^\star\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{X}_{N,0}.$$

Next, for every $\gamma > 0$, we set

$$\begin{aligned}\boldsymbol{\sigma}_{\gamma,d_+} &:= \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star)\chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star| > d\}} = \gamma \mathbf{curl}\, \boldsymbol{g}_\gamma \chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star| > d\}},\\ \boldsymbol{\sigma}_{\gamma,d_-} &:= \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star)\chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star| \le d\}} = \gamma \mathbf{curl}\, \boldsymbol{g}_\gamma \chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star| \le d\}}.\end{aligned}$$

By definition, it holds that

$$\begin{aligned}|\boldsymbol{\sigma}_{\gamma,d_+}| &\le |\gamma \mathbf{curl}\, \boldsymbol{g}_\gamma| \quad \text{a.e. in } \Omega,\\ |\boldsymbol{\sigma}_{\gamma,d_-}| &\le |\gamma \mathbf{curl}\, \boldsymbol{g}_\gamma| \quad \text{a.e. in } \Omega.\end{aligned}$$

Consequently, as $\{\gamma \mathbf{curl}\, \boldsymbol{g}_\gamma\}_{\gamma>0} \subset \boldsymbol{L}^2(\Omega)$ is bounded, the sequences $\{\boldsymbol{\sigma}_{\gamma,d_+}\}_{\gamma>0}$ and $\{\boldsymbol{\sigma}_{\gamma,d_-}\}_{\gamma>0}$ are also bounded in $\boldsymbol{L}^2(\Omega)$. For this reason, there exist $\boldsymbol{\sigma}_{d_+}^\star, \boldsymbol{\sigma}_{d_+}^\star \in \boldsymbol{L}^2(\Omega)$ such that, after extracting a subsequence, we obtain that

$$\begin{aligned}\boldsymbol{\sigma}_{\gamma,d_+} &\rightharpoonup \boldsymbol{\sigma}_{d_+}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty,\\ \boldsymbol{\sigma}_{\gamma,d_-} &\rightharpoonup \boldsymbol{\sigma}_{d_-}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty.\end{aligned} \tag{5.22}$$

Finally, due to

$$\boldsymbol{\sigma}_{\gamma,d_+} + \boldsymbol{\sigma}_{\gamma,d_-} = \gamma \mathbf{curl}\, \boldsymbol{g}_\gamma \quad \forall \gamma > 0$$

and the weak convergence (5.21), we come to the conclusion that $\mathbf{curl}\, \boldsymbol{n}^\star = \boldsymbol{\sigma}_{d_+}^\star + \boldsymbol{\sigma}_{d_-}^\star$. Let us now make use of the cut-off type function

$$(\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\le d}(x) := \boldsymbol{\varrho}(x, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star) = \begin{cases} \mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x) & \text{if } |\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)| \le d(x),\\ d(x)\dfrac{\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)}{|\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)|} & \text{if } |\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)| > d(x)\end{cases}$$

for a.e. $x \in \Omega$. Since, for a.e. $x \in \Omega$, the mapping $\boldsymbol{\varrho}(x, \cdot)$ is Lipschitz continuous with Lipschitz constant 1, we obtain that

$$\begin{aligned}\int_\Omega |(\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\le d} - \mathbf{curl}\, \boldsymbol{A}^\star|^2\, \mathrm{d}x &\underbrace{=}_{|\,\mathbf{curl}\, \boldsymbol{A}^\star| \le d} \int_\Omega |(\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\le d} - (\mathbf{curl}\, \boldsymbol{A}^\star)^{\le d}|^2\, \mathrm{d}x\\ = \int_\Omega |\boldsymbol{\varrho}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star) - \boldsymbol{\varrho}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star)|^2\, \mathrm{d}x &\le \int_\Omega |\,\mathbf{curl}\, \boldsymbol{A}_\gamma^\star - \mathbf{curl}\, \boldsymbol{A}^\star|^2\, \mathrm{d}x \underbrace{\to}_{(5.17)} 0\end{aligned}$$

as $\gamma \to \infty$. Therefore, it holds that

$$(\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\le d} \to \mathbf{curl}\, \boldsymbol{A}^\star \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty. \tag{5.23}$$

Combining (5.23) with (5.21) and the fact that $\mathbf{curl}\, \boldsymbol{n}^\star = \boldsymbol{\sigma}_{d_+}^\star + \boldsymbol{\sigma}_{d_-}^\star$ holds, we obtain that

$$\begin{aligned}&\int_\Omega \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star) \cdot (\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\le d} \mathrm{d}x\\ &\quad \to \int_\Omega \boldsymbol{\sigma}_{d_+}^\star \cdot \mathbf{curl}\, \boldsymbol{A}^\star\, \mathrm{d}x + \int_\Omega \boldsymbol{\sigma}_{d_-}^\star \cdot \mathbf{curl}\, \boldsymbol{A}^\star\, \mathrm{d}x\end{aligned} \tag{5.24}$$

as $\gamma \to \infty$. On the other hand, in view of the construction of the cut-off mapping $\boldsymbol{\varrho}$ (see (5.10) for its definition), the left-hand side of the latter equation can be rewritten as

$$\begin{aligned}
&\int_\Omega \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma) \cdot (\mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\leq d}\, \mathrm{d}x \\
&\quad = \int_\Omega \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma) \chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}^\star_\gamma| > d\}} \cdot d \frac{\mathbf{curl}\, \boldsymbol{A}^\star_\gamma}{|\,\mathbf{curl}\, \boldsymbol{A}^\star_\gamma|}\, \mathrm{d}x \\
&\qquad + \int_\Omega \gamma \mathbb{P}_{\mathbf{curl}}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma) \chi_{\{|\,\mathbf{curl}\, \boldsymbol{A}^\star_\gamma| \leq d\}} \cdot \mathbf{curl}\, \boldsymbol{A}^\star_\gamma\, \mathrm{d}x \\
&\quad = \int_\Omega \boldsymbol{\sigma}_{\gamma, d_+} \cdot d \frac{\mathbf{curl}\, \boldsymbol{A}^\star_\gamma}{|\,\mathbf{curl}\, \boldsymbol{A}^\star_\gamma|}\, \mathrm{d}x + \int_\Omega \boldsymbol{\sigma}_{\gamma, d_-} \cdot \mathbf{curl}\, \boldsymbol{A}^\star_\gamma\, \mathrm{d}x \\
&\quad \to \int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot d \frac{\mathbf{curl}\, \boldsymbol{A}^\star}{|\,\mathbf{curl}\, \boldsymbol{A}^\star|}\, \mathrm{d}x + \int_\Omega \boldsymbol{\sigma}^\star_{d_-} \cdot \mathbf{curl}\, \boldsymbol{A}^\star\, \mathrm{d}x \quad \text{as } \gamma \to \infty,
\end{aligned} \tag{5.25}$$

where the last convergence follows again from (5.17) and (5.22). Then, comparing the convergences (5.25) and (5.24) concludes the proof for the orthogonality condition (5.16).

Step 3 *(sign condition).* Finally, let us prove the sign condition (5.15). For this last step, we test the adjoint equation in (5.8) with the adjoint state, i.e., $\boldsymbol{v} = \boldsymbol{Q}^\star_\gamma$, to obtain

$$\begin{aligned}
&\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma\, \mathrm{d}x - \int_\Omega (\mathbf{curl}\, \boldsymbol{A}^\star_\gamma - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma\, \mathrm{d}x \\
&\quad = -\gamma \int_\Omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma\, \mathrm{d}x \leq 0.
\end{aligned} \tag{5.26}$$

Next, let us estimate

$$\begin{aligned}
&\liminf_{\gamma\to\infty} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma\, \mathrm{d}x \\
&\quad = \liminf_{\gamma\to\infty} \Big[\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}} \mathbf{curl}(\boldsymbol{Q}^\star_\gamma - \boldsymbol{Q}^\star) \cdot \mathbf{curl}(\boldsymbol{Q}^\star_\gamma - \boldsymbol{Q}^\star)\, \mathrm{d}x \\
&\qquad\qquad + 2 \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star\, \mathrm{d}x \\
&\qquad\qquad - \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star\, \mathrm{d}x \Big] \\
&\quad \underbrace{\geq}_{(5.6)} 2 \liminf_{\gamma\to\infty} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star\, \mathrm{d}x \\
&\qquad - \limsup_{\gamma\to\infty} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star\, \mathrm{d}x. \\
&\quad \underbrace{\geq}_{(5.19)} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star\, \mathrm{d}x.
\end{aligned} \tag{5.27}$$

Using the limiting adjoint equation (5.13), we ultimately find that

$$
\begin{aligned}
& -\int_\Omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
\underbrace{=}_{(5.13)} & \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x - \int_\Omega (\mathbf{curl}\, \boldsymbol{A}^\star - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
\underbrace{\leq}_{(5.27)} & \liminf_{\gamma\to\infty} \Bigg(\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\mathrm{T}} \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \cdot \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \,\mathrm{d}x \\
& \qquad\qquad - \int_\Omega (\mathbf{curl}\, \boldsymbol{A}_\gamma^\star - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \,\mathrm{d}x \Bigg) \underbrace{\leq}_{(5.26)} 0.
\end{aligned}
$$

This completes the proof. ∎

COROLLARY 5.8. *Every optimal solution* $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ *of* (P) *enjoys the regularity property* $\boldsymbol{J}^\star \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0)$. *Furthermore, the following higher regularity results hold true for the associated multipliers* $\phi_{\boldsymbol{J}^\star} \in H_0^1(\Omega)$ *and* $\boldsymbol{m}^\star \in \boldsymbol{X}_{N,0}$*:*

(i) *If* $d \in L^p(\Omega)$ *for* $p \in [2,3]$, *then* $\phi_{\boldsymbol{J}^\star} \in W_0^{1,p}(\Omega)$ *and* $\mathbf{curl}\, \boldsymbol{m}^\star \in \boldsymbol{L}^p(\Omega)$.

(ii) *If* $d \in L^p(\Omega)$ *for* $p \in [2,6]$, *and* Ω *is of class* $\mathcal{C}^{1,1}$, *then* $\phi_{\boldsymbol{J}^\star} \in W_0^{1,p}(\Omega)$ *and* $\mathbf{curl}\, \boldsymbol{m}^\star \in \boldsymbol{L}^p(\Omega)$.

(iii) *If* $d \in L^p(\Omega)$ *for* $p \in [2,\infty)$, *and* Ω *is of class* $\mathcal{C}^{2,1}$, *then* $\mathbf{curl}\, \boldsymbol{m}^\star \in \boldsymbol{L}^p(\Omega)$.

(iv) *If* $d \in L^\infty(\Omega)$, $\nu(\cdot, |\mathbf{curl}\, \boldsymbol{A}^\star|) \in \mathcal{C}^{0,1}(\overline{\Omega})$, *and* Ω *is of class* $\mathcal{C}^{2,1}$, *then* $\mathbf{curl}\, \boldsymbol{m}^\star \in \boldsymbol{L}^\infty(\Omega)$.

Proof. Let $\boldsymbol{J}^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ be an optimal solution of (P). Then, Theorem 5.6 implies $\boldsymbol{J}^\star = -\lambda^{-1}\boldsymbol{Q}^\star$ so that the optimal solution satisfies $\boldsymbol{J}^\star \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0)$. Recalling the continuous embeddings (cf. [10, Theorem 2] and [2, Theorem 2.12])

$$
\begin{aligned}
&\boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0) \hookrightarrow \boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega) \quad \forall p \in [2,3], \\
&\boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0) \hookrightarrow \boldsymbol{H}^{1}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega) \quad \forall p \in [2,6] \quad \text{if } \Omega \text{ is of class } \mathcal{C}^{1,1},
\end{aligned}
$$

the claim follows therefore by applying Theorem 4.4. ∎

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