Numerical Analysis for Maxwell Obstacle Problems in Faraday Shielding

jointly with Irwin Yousept

Maurice Hensel

February 7, 2022

University of Duisburg-Essen

Obstacle problem in Faraday shielding

Maxwell obstacle problem

Faraday shielding (recap): Effect of redirecting or blocking certain electric fields.

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Measurement of an electric field by an EMF-meter with and without Faraday shielding

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- $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in (\boldsymbol{K}\cap \boldsymbol{H}_0(\mathbf{curl})) \times \boldsymbol{H}(\mathbf{curl})$ initial data

Theorem

The obstacle problem (P) *admits a unique solution*

$$(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega))$$

satisfying the local magnetic regularity

$$\boldsymbol{H}_{|\Omega\setminus\overline{\omega}} \in L^\infty((0,T),\boldsymbol{H}(\mathbf{curl},\Omega\setminus\overline{\omega})).$$

⇝ Result by Irwin Yousept [1]

[1] I. Yousept. Well-posedness theory for electromagnetic obstacle problems. *J. Differential Equations*, 269(10):8855–8881, 2020

First Attempt: Mixed FEM and implicit Euler

- Denote by $\{\mathcal{T}_h\}_{h>0}$ a quasi-uniform family of triangulations, s.t.

$$\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T, \quad \overline{\omega} = \bigcup_{T\in\mathcal{T}_h^\omega} T,$$

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UDE

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- Time partition:

$$\tau = \frac{T}{N}, \quad t_n = n\tau \quad \forall n \in \{0, \dots, N\}, \quad N \in \mathbb{N}$$

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$\rightsquigarrow$ decoupling techniques:

$$\begin{cases} \text{Find } \{(\boldsymbol{E}_h^n, \boldsymbol{H}_h^n)\}_{n=1}^N \subset (K \cap \mathrm{ND}_h) \times \mathrm{DG}_h, \text{ s.t.} \\ \displaystyle\int_\Omega (\epsilon + \tau\sigma)\boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^n) + \tau^2\mu^{-1} \operatorname{\mathbf{curl}} \boldsymbol{E}_h^n \cdot \operatorname{\mathbf{curl}}(\boldsymbol{v}_h - \boldsymbol{E}_h^n)\,\mathrm{d}x \\ \displaystyle\geq \int_\Omega (\tau\boldsymbol{f}^n + \boldsymbol{E}_h^{n-1}) \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^n) + \tau\boldsymbol{H}_h^{n-1} \cdot \operatorname{\mathbf{curl}}(\boldsymbol{v}_h - \boldsymbol{E}_h^n) \quad \forall \boldsymbol{v}_h \in K \cap \mathrm{ND}_h \\ \boldsymbol{H}_h^n = \boldsymbol{H}_h^{n-1} - \tau\mu^{-1} \operatorname{\mathbf{curl}} \boldsymbol{E}_h^n. \end{cases}$$

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Drawback: High computational cost due to the requirement of a nonsmooth solver!

Second Attempt: Mixed FEM and Yee stepping

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  - the Amperé-Maxwell VI in (P) at $t_{n-\frac{1}{2}} := t_n - \frac{\tau}{2}$
  - the Faraday equation in (P) at t_n

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  - the Faraday equation in (P) at t_n
- Central difference and mean value approximation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t_{n-\frac{1}{2}}) \approx \frac{\boldsymbol{E}(t_n) - \boldsymbol{E}(t_{n-1})}{\tau}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t_n) \approx \frac{\boldsymbol{H}(t_{n+\frac{1}{2}}) - \boldsymbol{H}(t_{n-\frac{1}{2}})}{\tau}, \quad \boldsymbol{E}(t_{n-\frac{1}{2}}) \approx \frac{\boldsymbol{E}(t_n) + \boldsymbol{E}(t_{n-1})}{2}$$

[2] K. Yee. Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media. *IEEE Transactions on Antennas and Propagation*, 14(3):302–307, 1966

$$
(\mathrm{P}_{N,h}) \qquad \begin{cases}
\text{Find } \{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^{N} \subset (\boldsymbol{K} \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ s.t.} \\
\displaystyle\int_\Omega \epsilon \delta \boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) + \sigma \boldsymbol{E}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\
\displaystyle\geq \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{K} \cap \mathbf{DG}_h \quad \forall n \in \{1, \ldots, N\} \\
\displaystyle\int_\Omega \mu \delta \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h + \boldsymbol{E}_h^n \cdot \mathbf{curl}\, \boldsymbol{w}_h \,\mathrm{d}x = \mathbf{0} \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h \quad \forall n \in \{1, \ldots, N\},
\end{cases}
$$

with

$$
\delta \boldsymbol{E}_h^n := \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau}, \quad \delta \boldsymbol{H}_h^{n+\frac{1}{2}} := \frac{\boldsymbol{H}_h^{n+\frac{1}{2}} - \boldsymbol{H}_h^{n-\frac{1}{2}}}{\tau}, \quad \boldsymbol{E}_h^n := 2\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^{n-1}.
$$

UDE

$$
(\mathrm{P}_{N,h}) \qquad
\begin{cases}
\text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^{N} \subset (K \cap \mathbf{DG}_h) \times \mathbf{ND}_h \ \text{ s.t.} \\
\int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \mathbf{curl}\, H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\,\mathrm{d}x \\
\geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\,\mathrm{d}x \quad \forall v_h \in K \cap \mathbf{DG}_h \quad \forall n \in \{1,\ldots,N\} \\
\int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \mathbf{curl}\, w_h\,\mathrm{d}x = 0 \quad \forall w_h \in \mathbf{ND}_h \quad \forall n \in \{1,\ldots,N\},
\end{cases}
$$

with

$$
\delta E_h^n := \frac{E_h^n - E_h^{n-1}}{\tau}, \quad \delta H_h^{n+\frac{1}{2}} := \frac{H_h^{n+\frac{1}{2}} - H_h^{n-\frac{1}{2}}}{\tau}, \quad E_h^n := 2E_h^{n-\frac{1}{2}} - E_h^{n-1}.
$$

Note: Obstacle discretization at $t_{n-\frac{1}{2}}$ rather than t_n & L^2-structure

$$
(\mathrm{P}_{N,h}) \qquad \begin{cases} \text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (K \cap \mathrm{DG}_h) \times \mathrm{ND}_h \text{ s.t.} \\ \displaystyle\int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \operatorname{curl} H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \\ \displaystyle\geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall v_h \in K \cap \mathrm{DG}_h \quad \forall n \in \{1, \dots, N\} \\ \displaystyle\int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \operatorname{curl} w_h \, \mathrm{d}x = 0 \quad \forall w_h \in \mathrm{ND}_h \quad \forall n \in \{1, \dots, N\}, \end{cases}
$$

with

$$
\delta E_h^n := \frac{E_h^n - E_h^{n-1}}{\tau}, \quad \delta H_h^{n+\frac{1}{2}} := \frac{H_h^{n+\frac{1}{2}} - H_h^{n-\frac{1}{2}}}{\tau}, \quad E_h^n := 2E_h^{n-\frac{1}{2}} - E_h^{n-1}.
$$

Note: Obstacle discretization at $t_{n-\frac{1}{2}}$ rather than t_n & L^2-structure

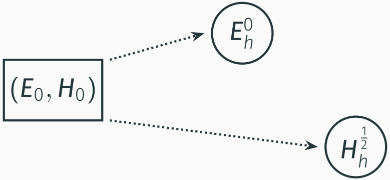
Mixed FEM and Yee stepping

$$(\mathrm{P}_{N,h}) \quad \begin{cases} \text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (K \cap \mathrm{DG}_h) \times \mathrm{ND}_h \text{ s.t.} \\ \int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \operatorname{curl} H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\,\mathrm{d}x \\ \geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\,\mathrm{d}x \quad \forall v_h \in K \cap \mathrm{DG}_h \quad \forall n \in \{1,\ldots,N\} \\ \int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \operatorname{curl} w_h \,\mathrm{d}x = 0 \quad \forall w_h \in \mathrm{ND}_h \quad \forall n \in \{1,\ldots,N\}, \end{cases}$$

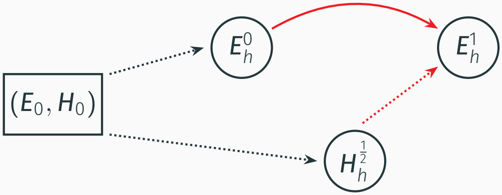
$$\boxed{(E_0, H_0)}$$

Mixed FEM and Yee stepping

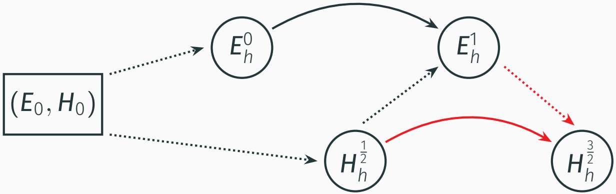
$$
(\mathrm{P}_{N,h}) \quad \begin{cases} \text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (K \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ s.t.} \\ \displaystyle\int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \mathbf{curl}\, H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \\ \displaystyle\geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall v_h \in K \cap \mathbf{DG}_h \quad \forall n \in \{1, \ldots, N\} \\ \displaystyle\int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \mathbf{curl}\, w_h\, \mathrm{d}x = \mathbf{0} \quad \forall w_h \in \mathbf{ND}_h \quad \forall n \in \{1, \ldots, N\}, \end{cases}
$$



$$
(\mathrm{P}_{N,h}) \quad \begin{cases} \text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (K \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ s.t.} \\ \int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \mathbf{curl}\, H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \\ \geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall v_h \in K \cap \mathbf{DG}_h \quad \forall n \in \{1, \ldots, N\} \\ \int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \mathbf{curl}\, w_h\, \mathrm{d}x = 0 \quad \forall w_h \in \mathbf{ND}_h \quad \forall n \in \{1, \ldots, N\}, \end{cases}
$$



$$
(\mathrm{P}_{N,h}) \qquad \begin{cases} \text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^{N} \subset (K \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ s.t.} \\ \displaystyle\int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \mathbf{curl}\, H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \\ \displaystyle\geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall v_h \in K \cap \mathbf{DG}_h \quad \forall n \in \{1, \ldots, N\} \\ \displaystyle\int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \mathbf{curl}\, w_h\, \mathrm{d}x = \mathbf{0} \quad \forall w_h \in \mathbf{ND}_h \quad \forall n \in \{1, \ldots, N\}, \end{cases}
$$



$$
(\mathrm{P}_{N,h}) \quad
\begin{cases}
\text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^{N} \subset (K \cap \mathbf{DG}_h) \times \mathbf{ND}_h \ \text{ s.t.} \\
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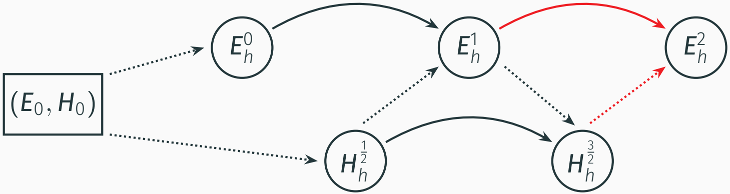
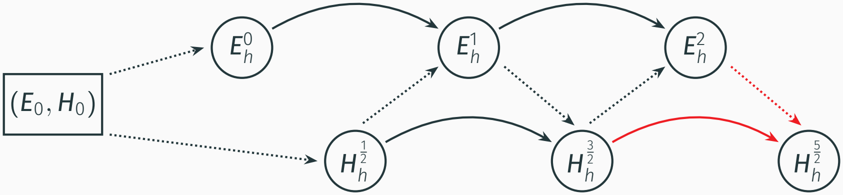
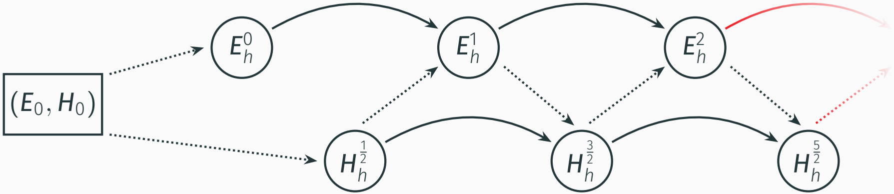
\geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) \,\mathrm{d}x \quad \forall v_h \in K \cap \mathbf{DG}_h \quad \forall n \in \{1, \ldots, N\} \\
\int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \operatorname{\mathbf{curl}} w_h \,\mathrm{d}x = 0 \quad \forall w_h \in \mathbf{ND}_h \quad \forall n \in \{1, \ldots, N\},
\end{cases}
$$



$$
(\mathrm{P}_{N,h}) \quad \begin{cases} \text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (K \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ s.t.} \\ \displaystyle\int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \mathbf{curl}\, H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \\ \displaystyle\geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall v_h \in K \cap \mathbf{DG}_h \quad \forall n \in \{1, \dots, N\} \\ \displaystyle\int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \mathbf{curl}\, w_h\, \mathrm{d}x = 0 \quad \forall w_h \in \mathbf{ND}_h \quad \forall n \in \{1, \dots, N\}, \end{cases}
$$



$$(\mathrm{P}_{N,h}) \quad \begin{cases} \text{Find } \{(E_h^{n-\frac{1}{2}}, H_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (K \cap \mathbf{DG}_h) \times \mathbf{ND}_h \text{ s.t.} \\ \int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) + \sigma E_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}}) - \mathbf{curl}\, H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \\ \geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall v_h \in K \cap \mathbf{DG}_h \quad \forall n \in \{1, \dots, N\} \\ \int_\Omega \mu \delta H_h^{n+\frac{1}{2}} \cdot w_h + E_h^n \cdot \mathbf{curl}\, w_h\, \mathrm{d}x = 0 \quad \forall w_h \in \mathbf{ND}_h \quad \forall n \in \{1, \dots, N\}, \end{cases}$$



Theorem

The problem $(\mathrm{P}_{N,h})$ *admits a unique solution* $\{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^N \subset (\boldsymbol{K} \cap \mathbf{DG}_h) \times \mathbf{ND}_h$ *with*

$$\boldsymbol{E}_h^{n-\frac{1}{2}} = \begin{cases} \dfrac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} & \text{on } \mathcal{M}_h^{n-\frac{1}{2}} \\ \left(\dfrac{2\epsilon}{\tau} + \sigma\right)^{-1} \boldsymbol{g}_h^{n-\frac{1}{2}} & \text{on } \Omega \setminus \mathcal{M}_h^{n-\frac{1}{2}}, \end{cases}$$

with right-hand sides and strict superlevel sets

$$\boldsymbol{g}_h^{n-\frac{1}{2}} := \boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau} \boldsymbol{E}_h^{n-1} \quad \text{and} \quad \mathcal{M}_h^{n-\frac{1}{2}} := \left\{ x \in \omega \,\middle|\, \left(\frac{2\epsilon}{\tau} + \sigma\right)^{-1} |\boldsymbol{g}_h^{n-\frac{1}{2}}(x)| > d \right\}.$$

UDE

Theorem

The problem $(\mathrm{P}_{N,h})$ *admits a unique solution* $\{(\boldsymbol{E}_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n+\frac{1}{2}})\}_{n=1}^{N} \subset (\mathbf{K} \cap \mathbf{DG}_h) \times \mathbf{ND}_h$ *with*

$$\boldsymbol{E}_h^{n-\frac{1}{2}} = \begin{cases} \dfrac{d\boldsymbol{g}_h^{n-\frac{1}{2}}}{|\boldsymbol{g}_h^{n-\frac{1}{2}}|} & \text{on } \mathcal{M}_h^{n-\frac{1}{2}} \\ \left(\dfrac{2\epsilon}{\tau} + \sigma\right)^{-1} \boldsymbol{g}_h^{n-\frac{1}{2}} & \text{on } \Omega \setminus \mathcal{M}_h^{n-\frac{1}{2}}, \end{cases}$$

with right-hand sides and strict superlevel sets

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We recall the inverse estimate

$$\exists C_{\text{inv}} > 0 \text{ s.t. } \|\,\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)} \leq \frac{C_{\text{inv}}}{h}\|\boldsymbol{v}\|_{L^2(\Omega)} \quad \forall \boldsymbol{v} \in \mathbf{ND}_h$$

and assume

Stability

We recall the inverse estimate

$$\exists C_{\text{inv}} > 0 \text{ s.t. } \|\operatorname{\mathbf{curl}} \boldsymbol{v}\|_{L^2(\Omega)} \leq \frac{C_{\text{inv}}}{h} \|\boldsymbol{v}\|_{L^2(\Omega)} \quad \forall \boldsymbol{v} \in \mathbf{ND}_h$$

and assume

- a linear CFL-condition (to deal with the inverse estimate)

$$\tau \leq \frac{\sqrt{\underline{\epsilon}}\sqrt{\underline{\mu}}}{2C_{\text{inv}}} h$$

We recall the inverse estimate

$$\exists C_{\text{inv}} > 0 \text{ s.t. } \|\,\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)} \leq \frac{C_{\text{inv}}}{h}\|\boldsymbol{v}\|_{L^2(\Omega)} \quad \forall \boldsymbol{v} \in \mathbf{ND}_h$$

and assume

- a linear CFL-condition (to deal with the inverse estimate)

$$\tau \leq \frac{\sqrt{\epsilon}\sqrt{\mu}}{2C_{\text{inv}}}h$$

- additional regularity on the initial electric field

$$\boldsymbol{E}_0 \in \boldsymbol{K} \cap \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}^1(\Omega)$$

$\rightsquigarrow$ Main ingredients for stability.

UDE

Theorem

There exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ with $N \geq 2$ and $h > 0$ the unique solution to $(\mathrm{P}_{N,h})$ *satisfies*

$$\max_{n\in\{1,\dots,N\}} \|\delta \boldsymbol{E}_h^n\|_{L^2(\Omega)} + \max_{n\in\{2,\dots,N\}} \|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} \leq C$$

$$\max_{n\in\{1,\dots,N\}} \|\boldsymbol{E}_h^n\|_{L^2(\Omega)} + \|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} \leq C$$

$$\max_{n\in\{1,\dots,N-1\}} \|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^1(\omega)} + \|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega\setminus\omega)} \leq C$$

Theorem

There exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ with $N \geq 2$ and $h > 0$ the unique solution to $(\mathrm{P}_{N,h})$ satisfies

$$\max_{n\in\{1,\dots,N\}} \|\delta \boldsymbol{E}_h^n\|_{L^2(\Omega)} + \max_{n\in\{2,\dots,N\}} \|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} \leq C$$

$$\max_{n\in\{1,\dots,N\}} \|\boldsymbol{E}_h^n\|_{L^2(\Omega)} + \|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} \leq C$$

$$\max_{n\in\{1,\dots,N-1\}} \|\operatorname{curl} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^1(\omega)} + \|\operatorname{curl} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega\setminus\omega)} \leq C$$

UDE

Theorem

There exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ with $N \geq 2$ and $h > 0$ the unique solution to $(\mathrm{P}_{N,h})$ *satisfies*

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$$\max_{n\in\{1,\dots,N\}} \|\boldsymbol{E}_h^n\|_{L^2(\Omega)} + \|\boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega)} \leq C$$

$$\max_{n\in\{1,\dots,N-1\}} \|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^1(\omega)} + \|\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}\|_{L^2(\Omega\setminus\omega)} \leq C$$

Lack of global $\boldsymbol{L}^2$-stability for $\operatorname{\mathbf{curl}} \boldsymbol{H}_h^{n-\frac{1}{2}}$: Justified by low regularity issue in (P).

We set up the following **piecewise linear interpolations**

$$E_{N,h} : [0,T] \to \mathbf{DG}_h, \quad t \mapsto \begin{cases} E_h^0 & \text{if } t = 0 \\ E_h^{n-1} + (t - t_{n-1})\delta E_h^n & \text{if } t \in (t_{n-1}, t_n], \end{cases}$$

$$H_{N,h} : [0,T] \to \mathbf{ND}_h, \quad t \mapsto \begin{cases} H_h^{\frac{1}{2}} & \text{if } t = 0 \\ H_h^{n-\frac{1}{2}} + (t - t_{n-1})\delta H_h^{n+\frac{1}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n \in \{1, \dots, N-1\} \\ H_h^{N-\frac{3}{2}} & \text{if } t \in (t_{n-1}, t_n] \text{ for } n = N. \end{cases}$$

UDE

Theorem

Under the stated CFL-condition, it holds that

$$(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \overset{*}{\rightharpoonup} (\boldsymbol{E}, \boldsymbol{H}) \quad \textit{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \textit{ as } h \to 0, N \to \infty$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}, \boldsymbol{H}) \quad \textit{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \textit{ as } h \to 0, N \to \infty,$$

where $(\boldsymbol{E}, \boldsymbol{H})$ *is the unique solution to* (P)*. Assume additionally that*

$$\boldsymbol{H} \in L^1((0,T), \boldsymbol{H}(\mathbf{curl})) \quad \textit{and} \quad \max_{n \in \{1,\dots,N-1\}} \| \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \|_{\boldsymbol{L}^2(\omega)} \leq C.$$

Then it holds that

$$(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \to (\boldsymbol{E}, \boldsymbol{H}) \quad \textit{in } C([0,T], \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \textit{ as } h \to 0.$$

Convergence proof

- Stability estimates $\Rightarrow$ Existence of weakly-star converging subsequences

UDE

- Stability estimates $\Rightarrow$ Existence of weakly-star converging subsequences
- Derivation of the following system for the weak limit:

$$
(\mathrm{P}_{\text{weak}}) \quad \begin{cases}
\displaystyle\int_0^T \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) + \sigma \boldsymbol{E}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) - \boldsymbol{H}(t) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}(t)) \,\mathrm{d}x \,\mathrm{d}t \\
\displaystyle\geq \int_0^T \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) \,\mathrm{d}x \,\mathrm{d}t \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{C}_0^\infty(\Omega) \\
\displaystyle\mu \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\, \boldsymbol{E}(t) = \mathbf{0} \quad \text{for a.e. } t \in (0,T) \\
(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega)) \\
\boldsymbol{E}(t) \in \boldsymbol{K} \text{ for all } t \in [0,T] \text{ and } (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0).
\end{cases}
$$

UDE

- Stability estimates $\Rightarrow$ Existence of weakly-star converging subsequences
- Derivation of the following system for the weak limit:

$$
(\mathrm{P}_{\mathrm{weak}}) \quad \begin{cases} \displaystyle\int_0^T \int_\Omega \epsilon \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) + \sigma\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \boldsymbol{H}(t)\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle\geq \int_0^T \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x\,\mathrm{d}t \quad \forall \boldsymbol{v}\in \boldsymbol{K}\cap \boldsymbol{C}_0^\infty(\Omega) \\ \displaystyle\mu\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = \mathbf{0} \quad \text{for a.e. } t\in(0,T) \\ (\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega)) \\ \boldsymbol{E}(t)\in\boldsymbol{K} \text{ for all } t\in[0,T] \text{ and } (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases}
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$$

Main idea: Bypass missing stability by exploiting properties of **piecewise constant interpolation operator** for $\boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{C}_0^\infty(\Omega)$.

- $(\mathrm{P}_{\mathrm{weak}}) \Rightarrow (\mathrm{P})$ reduces to **enlarging** the set of test functions

$$K \cap C_0^\infty(\Omega) \quad \rightsquigarrow \quad K \cap H_0(\mathbf{curl})$$

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Question: Does there exist a mollification operator M_δ, s.t.

$$v \in K \cap H_0(\mathrm{curl}) \quad \overset{?}{\Rightarrow} \quad \begin{cases} M_\delta v \in C_0^\infty(\Omega) \\ M_\delta v \in K. \end{cases}$$

UDE

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[3] A. Ern and J.-L. Guermond. Mollification in strongly Lipschitz domains with application to continuous and discrete de Rham complexes. *Comput. Methods Appl. Math.*, 16(1):51–75, 2016

[4] S.H. Christiansen and R. Winther, Smoothed projections in finite element exterior calculus, *Math. Comp. 77* (2008), no. 262, 813–829

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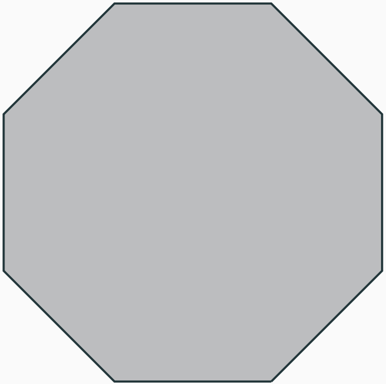
Convergence proof - constraint preserving mollification

Idea of Ern and Guermond: $v \in H_0(\mathbf{curl})$.



Convergence proof - constraint preserving mollification

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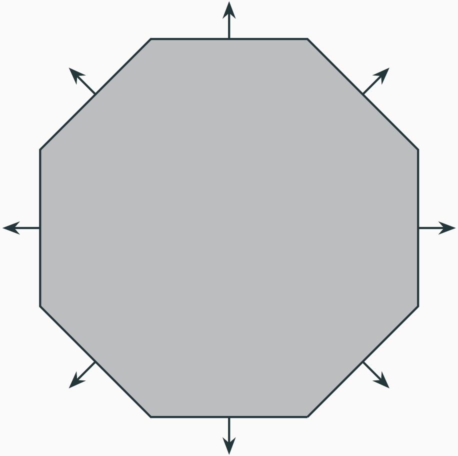
- Expand Ω by a transversal vector field



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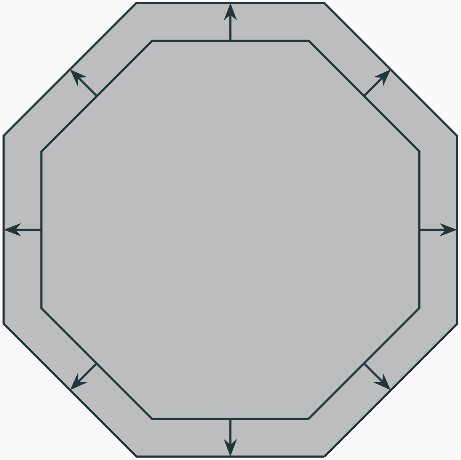

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- Expand Ω by a transversal vector field
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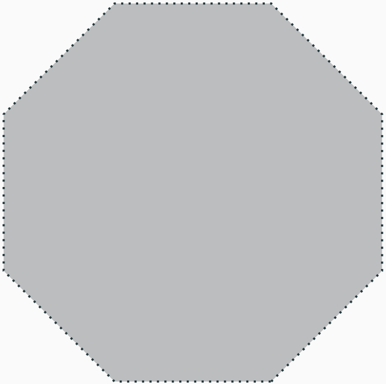

Idea of Ern and Guermond: $\boldsymbol{v} \in H_0(\mathbf{curl})$.

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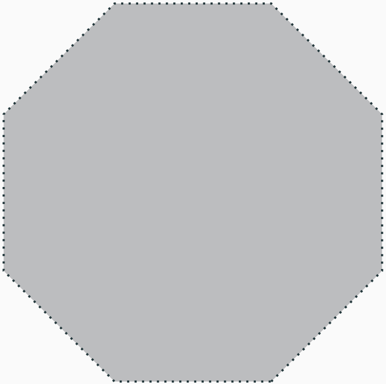
$\rightsquigarrow$ Mollify resulting field $\Rightarrow M_\delta \boldsymbol{v} \in C_0^\infty(\Omega)$



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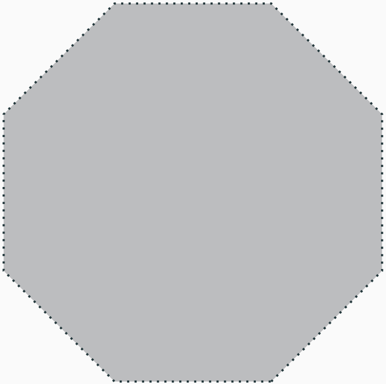
Q: And what about points in ω?



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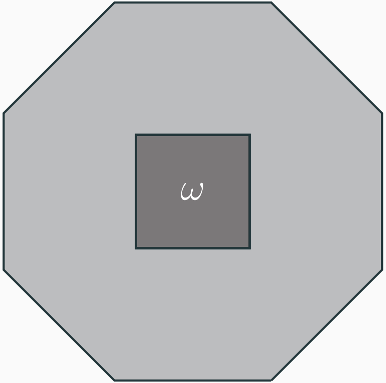

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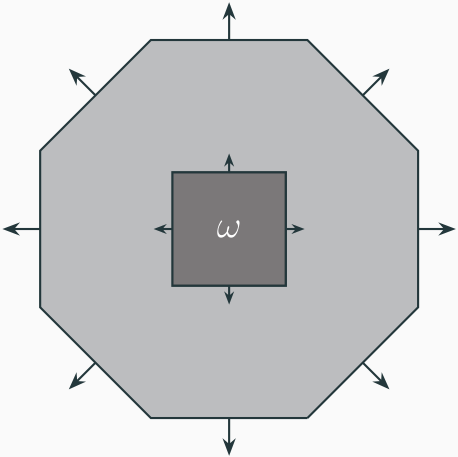

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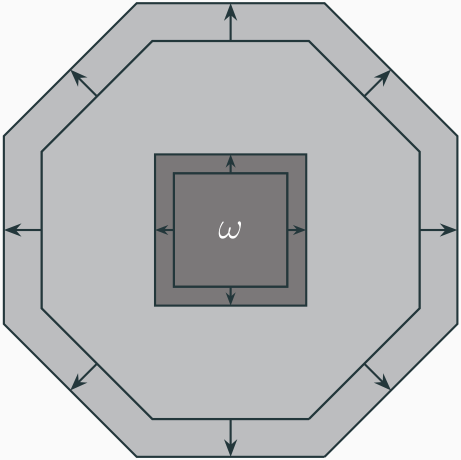

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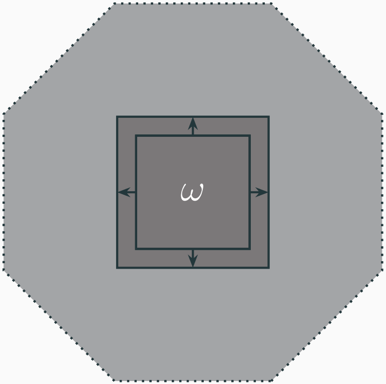

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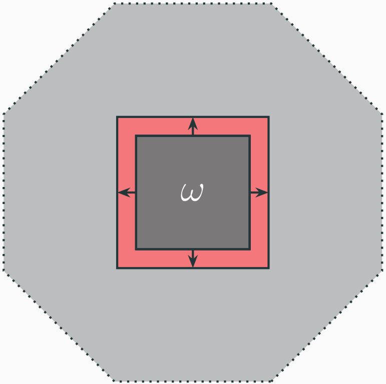
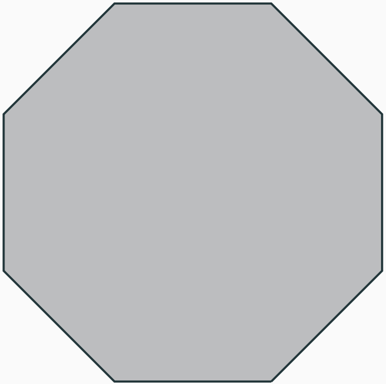


Q: And what about points in ω?

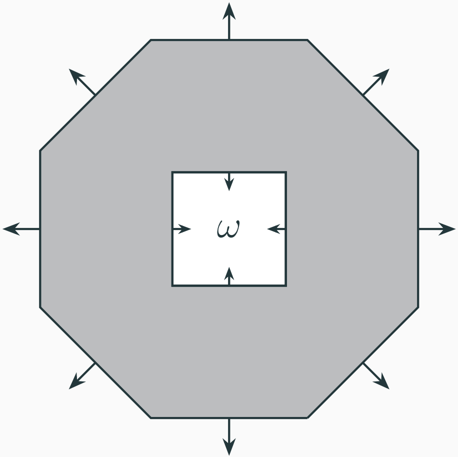
A: Could get pushed outside of ω.

Workaround:



Workaround:

- This time expand $\Omega \setminus \overline{\omega}$



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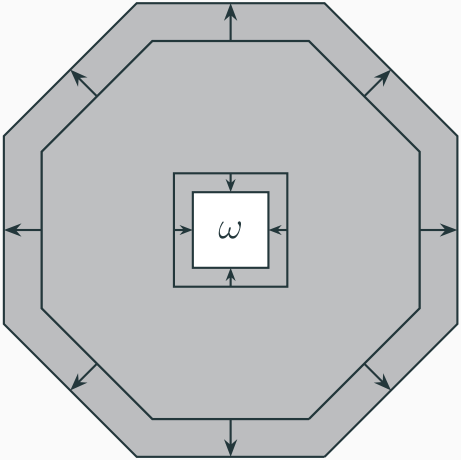

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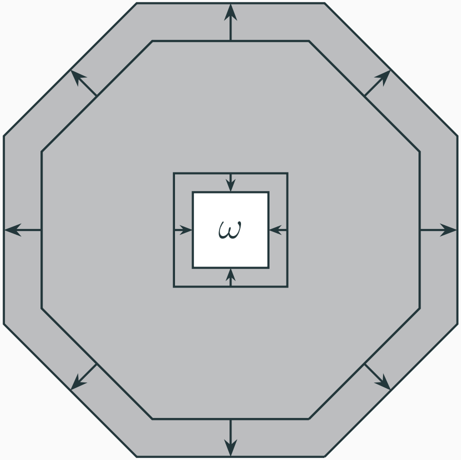

[5] S. Hofmann, M. Mitrea, and M. Taylor. Geometric and transformational properties of Lipschitz domains,... *J. Geom. Anal.*, 17(4):593–647, 2007

Workaround:

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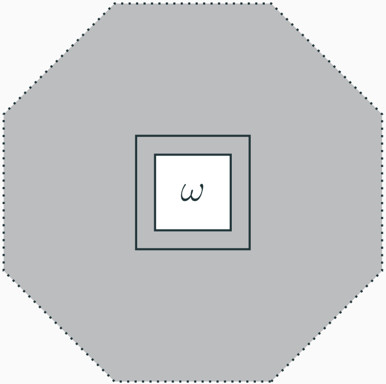
- Cut off vector field near the boundary



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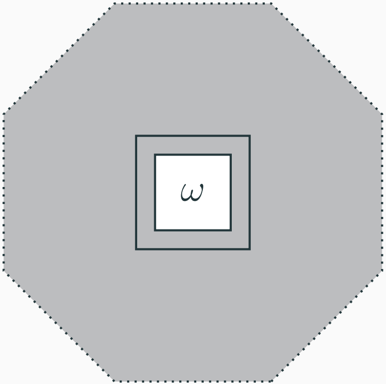

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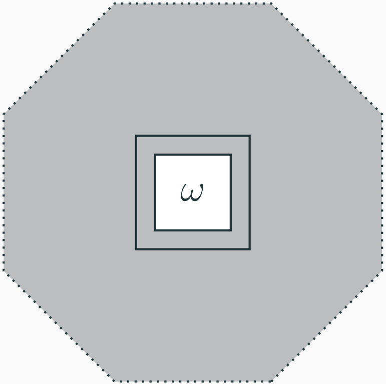
Techniques from geometrical analysis:

$$v \in K \cap H_0(\mathrm{curl}) \Rightarrow M_\delta v \in K$$



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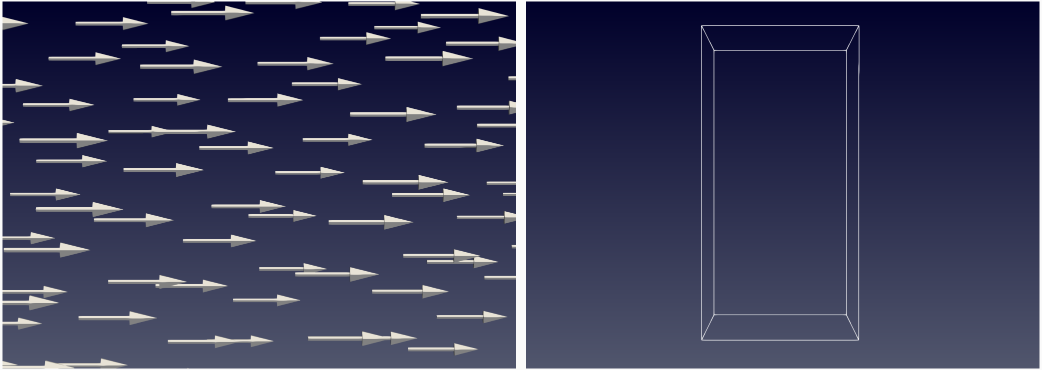
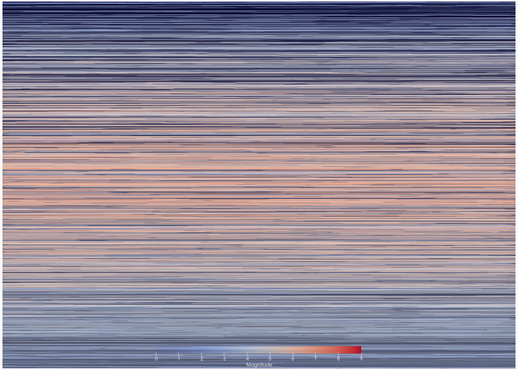
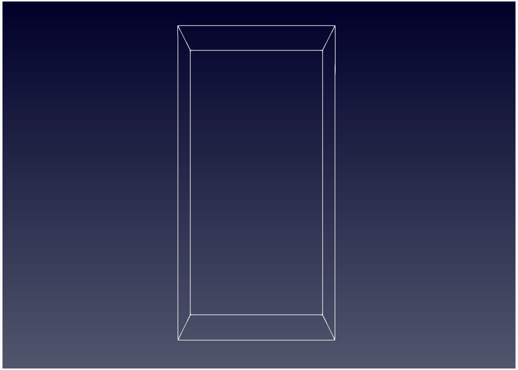
Numerical test using FEniCS



Applied current source and obstacle

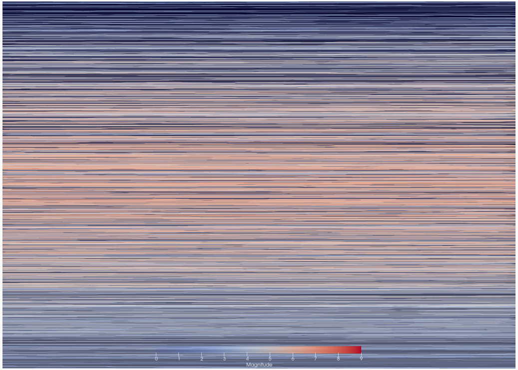
Numerical test

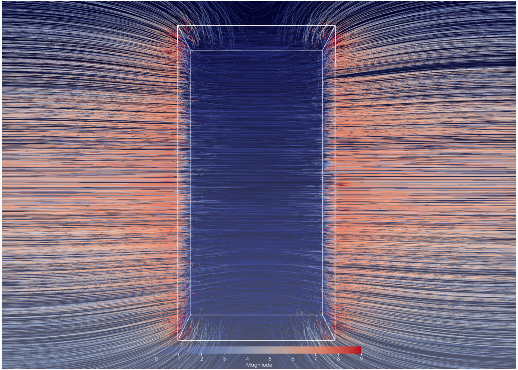




Free electric field and obstacle

Numerical test





Free and shielded electric field

Reference parameters:

$$\widetilde{h} = 1/2^6, \quad \widetilde{N} = 5 \cdot 2^6 \qquad \rightsquigarrow \qquad E := E_{\widetilde{N},\widetilde{h}}$$

Define:

$$\mathrm{RelErr}_{N,h}(E) := \frac{\|E_{N,h} - E\|_{C([0,T],L^2(\Omega))}}{\|E\|_{C([0,T],L^2(\Omega))}} \approx \frac{\max\limits_{n\in\{0,\dots,N\}} \|E_{N,h}(t_n) - E(t_n)\|_{L^2(\Omega)}}{\max\limits_{n\in\{0,\dots,N\}} \|E(t_n)\|_{L^2(\Omega)}}$$

N	$5 \cdot 2^2$	$5 \cdot 2^3$	$5 \cdot 2^4$	$5 \cdot 2^5$	$5 \cdot 2^6$
h	$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$
$\dim(\mathrm{DG}_h)$	1.152	9.216	31.024	589.824	4.718.592
$\dim(\mathrm{ND}_h)$	604	4.184	73.728	238.688	1.872.064
$\mathrm{RelErr}_{N,h}(E)$	0.3832	0.1070	0.0591	0.0248	–
$\mathrm{RelErr}_{N,h}(H)$	0.3556	0.2083	0.1480	0.1086	–

Numerical Experiment

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/F (video.mov) »,Annotations=« /Mode /Open »,T=(mmdefaultlabel1), Border=0 0 0

Thank you for your attention!