P22: Multi-Physics Phenomena in High-Temperature Superconductivity: Analysis, Numerics and Optimization

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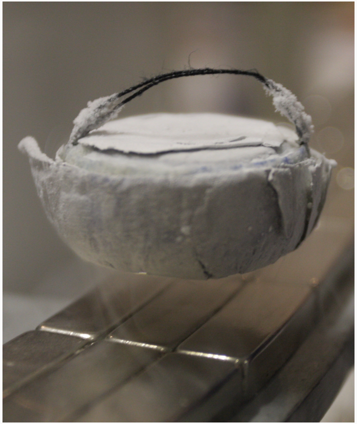
Superconductivity

What is superconductivity?

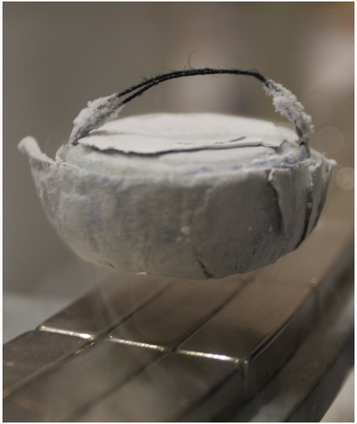
Superconductivity comprises physical properties of certain materials which cause the material to lose its electrical resistance. This causes any magnetic field to be expelled from the material (Meissner effect).
A material with such property is called *superconductor*.



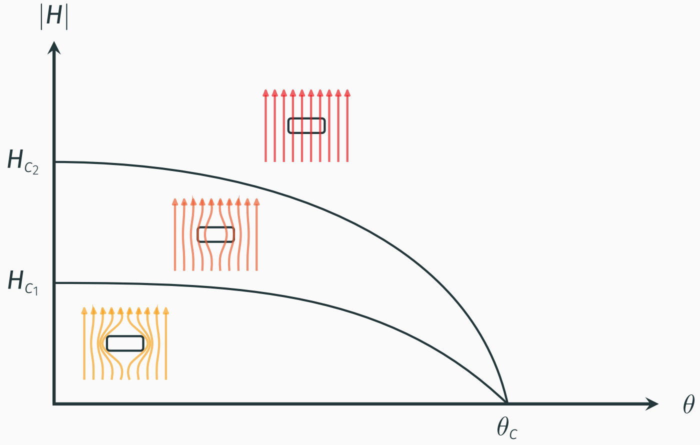
What is superconductivity?

Superconductivity comprises physical properties of certain materials which cause the material to lose its electrical resistance. This causes any magnetic field to be expelled from the material (Meissner effect). A material with such property is called *superconductor*. We are particularly interested in *high-temperature type-II* superconductors, i.e. those superconductors in which

- the superconducting state occurs below $-196.2°C$ (high-temperature)
- the transition between the superconducting and non-superconducting state is not abrupt (type-II)



Type-II Superconductors



- $|H|$ the magnetic field strength
- H_{c_1}, H_{c_2} the critical magnetic fields strengths
- θ the temperature
- θ_c the critical temperature

To model the behaviour of type-II superconductors we first consider the standard Maxwell's equations

$$\begin{cases} \epsilon\partial_t E - \operatorname{curl} H + J = f & \text{in } (0,T)\times\Omega \\ \mu\partial_t H + \operatorname{curl} E = 0 & \text{in } (0,T)\times\Omega. \end{cases}$$

If the medium Ω is a good conductor, we have Ohm's law

$$J = \sigma E.$$

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If the medium Ω is a good conductor, we have Ohm's law

$$J = \sigma E.$$

In the presence of a superconductor $\Omega_{sc} \subset \Omega$, Ohm's law needs to be replaced by a nonlinear and non-smooth constitutive relation between the electric field E and the current density J.

Bean's critical state model postulates that

- the current density strength $|\boldsymbol{J}|$ cannot exceed some critical value $j_c \in \mathbb{R}^+$
- the electric field $\boldsymbol{E}$ vanishes if $|\boldsymbol{J}| < j_c$
- the electric field $\boldsymbol{E}$ is parallel to $\boldsymbol{J}$.

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- the electric field $\boldsymbol{E}$ is parallel to $\boldsymbol{J}$.

Bean's model assumes that j_c is a constant. To account for the dependence on θ and $\boldsymbol{H}$ (as found through experiments by Kim et al.), we consider a non-negative

$$j_c \colon \Omega \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}.$$

The generalized Bean-Kim model then formalizes to

$$\begin{cases} \boldsymbol{J}(x,t) \cdot \boldsymbol{E}(x,t) = j_c(x, \theta(x,t), \boldsymbol{H}(x,t))|\boldsymbol{E}(x,t)| & \text{in } (0,T) \times \Omega_{\text{sc}} \\ |\boldsymbol{J}(x,t)| \leq j_c(x, \theta(x,t), \boldsymbol{H}(x,t)) & \text{in } (0,T) \times \Omega_{\text{sc}} \\ \boldsymbol{J}(x,t) = 0 & \text{in } (0,T) \times (\Omega \setminus \Omega_{\text{sc}}). \end{cases}$$

Including the Bean-Kim model into Maxwell's equations we end up with the following nonlinear and non-smooth system

$$\begin{cases} \epsilon\partial_t \boldsymbol{E} - \operatorname{\mathbf{curl}} \boldsymbol{H} + \boldsymbol{J} = \boldsymbol{f} & \text{in } (0,T)\times\Omega \\ \mu\partial_t \boldsymbol{H} + \operatorname{\mathbf{curl}} \boldsymbol{E} = 0 & \text{in } (0,T)\times\Omega. \\ \boldsymbol{J}(x,t)\cdot\boldsymbol{E}(x,t) = j_c(x,\theta(x,t),\boldsymbol{H}(x,t))|\boldsymbol{E}(x,t)| & \text{in } (0,T)\times\Omega_{\text{sc}} \\ |\boldsymbol{J}(x,t)| \le j_c(x,\theta(x,t),\boldsymbol{H}(x,t)) & \text{in } (0,T)\times\Omega_{\text{sc}} \\ \boldsymbol{J}(x,t) = 0 & \text{in } (0,T)\times(\Omega\setminus\Omega_{\text{sc}}) \\ \boldsymbol{E}\times\boldsymbol{n} = 0 & \text{in } (0,T)\times\Omega \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0) & \text{in } \Omega. \end{cases}$$

Finally, we formulate our system in a weak sense and end up with the following hyperbolic QVI of the second kind:

Find $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{H}(\mathbf{curl})\cap\mu^{-1}\boldsymbol{H}_0(\mathrm{div}{=}0))$:

$$
\text{(QVI)}\qquad
\begin{cases}
\displaystyle\int_\Omega \epsilon\partial_t\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \mathbf{curl}\,\boldsymbol{H}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\
\qquad \displaystyle + \int_\Omega j_c(\cdot,\theta(t),\boldsymbol{H}(t))(|\boldsymbol{v}|-|\boldsymbol{E}(t)|)\,\mathrm{d}x \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\;\mathrm{d}x \\
\text{for a.e. } t\in(0,T) \text{ and all } \boldsymbol{v}\in\boldsymbol{L}^2(\Omega), \\
\mu\partial_t\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \text{for a.e. } t\in(0,T), \\
(\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0).
\end{cases}
$$

Previous Related Work

Maxwell VI of the second kind

- Well-posedness in the case of a general proper, convex and lower semicontinuous nonlinearity $\psi \colon \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ satisfying a local boundedness assumption.

  (I. Yousept. Hyperbolic Maxwell Variational Inequalities of the Second Kind. *ESAIM: COCV*, 2020)

- AFEM for the underlying L^1-structured elliptic VI.

  (M. Winckler, I. Yousept, and J. Zou. Adaptive Edge Element Approximation for H(curl)-elliptic Variational Inequalities of Second Kind. *SIAM J. Numer. Anal.*, 2020)

- Shape optimization for the underlying L^1-structured elliptic VI.

  (A. Laurain, M. Winckler, I. Yousept. Shape Optimization for Superconductors Governed by H(curl)-elliptic Variational Inequalities. *SIAM J. Control Optim.*, 2021)

UDE

Maxwell VI of the first kind

- Well-posedness for the Maxwell obstacle problem, i.e. the case $\psi = I_K$.

  (I. Yousept. Well-posedness Theory for Electromagnetic Obstacle Problems. *J. Differential Equations*, 2020)
- Eddy Current approximation of the Maxwell obstacle problem.

  (M. Hensel and I. Yousept. Eddy Current Approximation in Maxwell Obstacle Problems. *Interfaces Free Bound.*, 2022)
- Numerical analysis of the Maxwell obstacle problem.

  (M. Hensel and I. Yousept. Numerical Analysis for Maxwell Obstacle Problems in Electric Shielding. *SIAM J. Numer. Anal.*, 2022)
- Analysis & control of an $H(\mathbf{curl})$-quasilinear first kind elliptic VI with bilateral $\mathbf{curl}$-constraints.

  (M. Hensel and I. Yousept. Quasilinear Variational Inequalities in Ferromagnetic Shielding: Well-posedness, Regularity, and Optimal Control. submitted, 2022)

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Maxwell QVI of the second kind

- Well-posedness for the introduced QVI.

  (I. Yousept. Maxwell Quasi-Variational Inequalities in Superconductivity. *ESAIM:M2AN*, 2021)

- Efficient solvers and numerical analysis for the introduced QVI.

  (M. Hensel, M. Winckler and I. Yousept. Leapfrog Scheme for Hyperbolic Maxwell Quasi-variational Inequalities. In preparation)

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Recent other contributions to QVIs

- A. Alphonse, M. Hintermüller and C.N. Rautenberg. Existence, iteration procedures and directional differentiability for parabolic QVIs. *Calc. Var. PDE*, 2020
- A. Alphonse, M. Hintermüller and C.N. Rautenberg. On the differentiability of the minimal and maximal solution maps of elliptic quasi-variational inequalities. *J. Math. Anal. Appl.*, 2021
- A. Alphonse, M. Hintermüller and C.N. Rautenberg. Optimal control and directional differentiability for elliptic quasi-variational inequalities. *Set-Valued Var. Anal.*, 2022
- C. Christof and G. Wachsmuth. Lipschitz Stability and Hadamard Directional Differentiability for Elliptic and Parabolic Obstacle-type Quasi-variational Inequalities. 2021
- G. Wachsmuth. From Resolvents to Generalized Equations and Quasi-variational Inequalities: Existence and Differentiability. *Journal of Nonsmooth Analysis and Optimization*, 2022

Back to (QVI)

Well-posedness of (QVI)

$$
\text{(QVI)} \qquad \begin{cases} \displaystyle\int_\Omega \epsilon \partial_t \boldsymbol{E}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t)) - \mathbf{curl}\, \boldsymbol{H}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t))\, \mathrm{d}x \\ \qquad + \displaystyle\int_\Omega j_c(\cdot, \theta(t), \boldsymbol{H}(t))(|\boldsymbol{v}| - |\boldsymbol{E}(t)|)\, \mathrm{d}x \geq \int_\Omega \boldsymbol{f}(t) \cdot (\boldsymbol{v} - \boldsymbol{E}(t))\, \mathrm{d}x \\ \text{for a.e. } t \in (0,T) \text{ and all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \\ \mu \partial_t \boldsymbol{H}(t) + \mathbf{curl}\, \boldsymbol{E}(t) = 0 \quad \text{for a.e. } t \in (0,T), \\ (\boldsymbol{E}, \boldsymbol{H})(0) = (\boldsymbol{E}_0, \boldsymbol{H}_0). \end{cases}
$$

- $\boldsymbol{f} \in H^1((0,T), \boldsymbol{L}^2(\Omega))$
- $\theta \in H^1((0,T), L^2(\Omega)) \cap \mathcal{C}([0,T], L^\infty(\Omega))$
- $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in \boldsymbol{H}_0(\mathbf{curl}) \times \mu^{-1}\boldsymbol{H}_0(\mathrm{div}{=}0)$

- j_c locally bounded in the 2. variable and locally Lipschitz in the 2. variable
- j_c globally bounded in the 3. variable and globally Lipschitz in the 3. variable

Theorem (Yousept '21)

There exists a unique solution

$$(\boldsymbol{E}, \boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T), \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}(\mathbf{curl}) \cap \mu^{-1}\boldsymbol{H}_0(\mathrm{div}{=}0))$$

to (QVI).

Proof is based on time discretization together with fixed point techniques.

Numerical Analysis of (QVI)

Actually, why not implicit Euler?

If we discretize in a standard way by implicit Euler, i.e.

- $\mathbf{ND}_h$ for the electric field
- $\mathbf{DG}_h$ for the magnetic field,

we end up with the following structure:

$$\begin{cases} \text{Find } \{(\boldsymbol{E}_h^n, \boldsymbol{H}_h^n)\}_{n=1}^N \subset \mathbf{ND}_h \times \mathbf{DG}_h, \text{ s.t.} \\ a(\boldsymbol{E}_h^n, \boldsymbol{v}_h - \boldsymbol{E}_h^n) + \int_\Omega j_c(\cdot, \theta_h^n, \boldsymbol{H}_h^n)(|\boldsymbol{v}_h| - |\boldsymbol{E}_h^n|) \geq F_h^n(\boldsymbol{v}_h - \boldsymbol{E}_h^n) \quad \forall \boldsymbol{v}_h \in \mathbf{ND}_h \\ \boldsymbol{H}_h^n = \boldsymbol{H}_h^{n-1} - \tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_h^n. \end{cases}$$

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Problem: Elliptic QVI pops up - no uniqueness, no efficient solve

Mixed FEM and Leapfrog Stepping

In contrast to before:

- DG_h for the electric field
- ND_h for the magnetic field
- discretize the Amperé-Maxwell VI at t_n
- discretize the Faraday equation at $t_{n-\frac{1}{2}} := t_n - \frac{\tau}{2}$
- Central difference and mean value approximation

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t_{n-\frac{1}{2}}) \approx \frac{E(t_n) - E(t_{n-1})}{\tau}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} H(t_n) \approx \frac{H(t_{n+\frac{1}{2}}) - H(t_{n-\frac{1}{2}})}{\tau},$$

$$E(t_{n-\frac{1}{2}}) \approx \frac{E(t_n) + E(t_{n-1})}{2}$$

$$
(\mathrm{QVI}_{N,h}) \quad \begin{cases} \text{Find } \{\boldsymbol{E}_h^n\}_{n=1}^N \subset \mathbf{DG}_h \text{ and } \{\boldsymbol{H}_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \mathbf{ND}_h \text{ such that} \\ \displaystyle\int_\Omega \epsilon\delta\boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) - \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{dx} \\ \displaystyle+ \int_\Omega j_c(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}})(|\boldsymbol{v}_h| - |\boldsymbol{E}_h^{n-\frac{1}{2}}|)\,\mathrm{dx} \geq \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{dx} \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h \\ \displaystyle\int_\Omega \mu\delta\boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{dx} + \int_\Omega \boldsymbol{E}_h^n \cdot \mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{dx} = 0 \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h\,. \end{cases}
$$

with

$$
\delta\boldsymbol{E}_h^n := \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau}, \quad \delta\boldsymbol{H}_h^{n+\frac{1}{2}} := \frac{\boldsymbol{H}_h^{n+\frac{1}{2}} - \boldsymbol{H}_h^{n-\frac{1}{2}}}{\tau}, \quad \boldsymbol{E}_h^{n-\frac{1}{2}} := \frac{\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1}}{2}.
$$

Mixed FEM and Leapfrog Stepping

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(\mathrm{QVI}_{N,h}) \quad \begin{cases} \text{Find } \{\boldsymbol{E}_h^n\}_{n=1}^N \subset \mathbf{DG}_h \text{ and } \{\boldsymbol{H}_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \mathbf{ND}_h \text{ such that} \\ \displaystyle\int_\Omega \epsilon \delta \boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\, \mathrm{d}x \\ \displaystyle + \int_\Omega j_c(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}})(|\boldsymbol{v}_h| - |\boldsymbol{E}_h^{n-\frac{1}{2}}|)\, \mathrm{d}x \geq \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h \\ \displaystyle\int_\Omega \mu \delta \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h\, \mathrm{d}x + \int_\Omega \boldsymbol{E}_h^n \cdot \mathbf{curl}\, \boldsymbol{w}_h\, \mathrm{d}x = 0 \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h\,. \end{cases}
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$$\boxed{(\boldsymbol{E}_0, \boldsymbol{H}_0)}$$

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\displaystyle + \int_\Omega j_c(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}})(|\boldsymbol{v}_h| - |\boldsymbol{E}_h^{n-\frac{1}{2}}|) \,\mathrm{d}x \geq \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) \,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h \\
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\end{cases}
$$

$$
\begin{tikzcd}
 & \boldsymbol{E}_h^0 \\
\boxed{(\boldsymbol{E}_0, \boldsymbol{H}_0)} \arrow[ur, dotted] \arrow[dr, dotted] & \\
 & \boldsymbol{H}_h^{\frac{1}{2}}
\end{tikzcd}
$$

$$
(\mathrm{QVI}_{N,h}) \quad \begin{cases}
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\displaystyle\int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac{1}{2}}) - \mathbf{curl}\, H_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \\
\displaystyle + \int_\Omega j_c(\cdot, \theta^{n-\frac{1}{2}}, H_h^{n-\frac{1}{2}})(|v_h| - |E_h^{n-\frac{1}{2}}|)\, \mathrm{d}x \geq \int_\Omega f_h^{n-\frac{1}{2}} \cdot (v_h - E_h^{n-\frac{1}{2}})\, \mathrm{d}x \quad \forall v_h \in \mathbf{DG}_h \\
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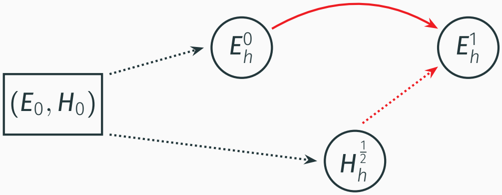

Mixed FEM and Leapfrog Stepping

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(\mathrm{QVI}_{N,h}) \quad \begin{cases}
\text{Find } \{E_h^n\}_{n=1}^N \subset \mathbf{DG}_h \text{ and } \{H_h^{n+\frac12}\}_{n=1}^{N-1} \subset \mathbf{ND}_h \text{ such that} \\
\displaystyle\int_\Omega \epsilon \delta E_h^n \cdot (v_h - E_h^{n-\frac12}) - \mathbf{curl}\, H_h^{n-\frac12} \cdot (v_h - E_h^{n-\frac12})\,\mathrm{d}x \\
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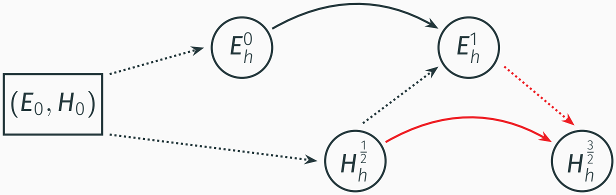

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\text{Find } \{E_h^n\}_{n=1}^N \subset \mathbf{DG}_h \text{ and } \{H_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \mathbf{ND}_h \text{ such that} \\
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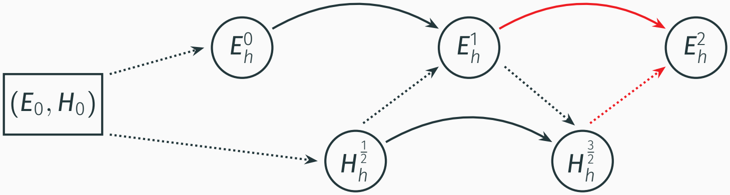

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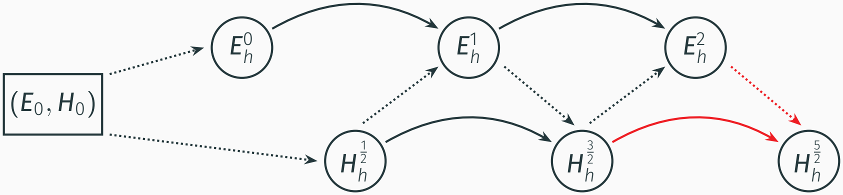

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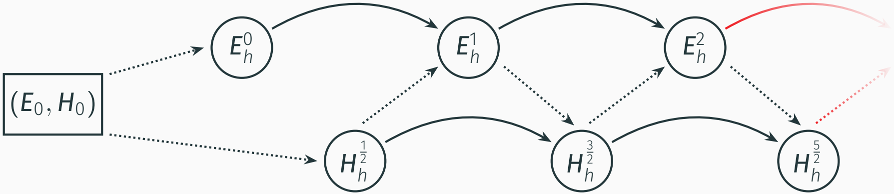
\text{Find } \{\boldsymbol{E}_h^n\}_{n=1}^N \subset \mathbf{DG}_h \text{ and } \{\boldsymbol{H}_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \mathbf{ND}_h \text{ such that} \\
\displaystyle\int_\Omega \epsilon \delta \boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}}) - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \\
\displaystyle + \int_\Omega j_c(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}})(|\boldsymbol{v}_h| - |\boldsymbol{E}_h^{n-\frac{1}{2}}|)\,\mathrm{d}x \geq \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^{n-\frac{1}{2}})\,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h \\
\displaystyle\int_\Omega \mu \delta \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{d}x + \int_\Omega \boldsymbol{E}_h^n \cdot \mathbf{curl}\, \boldsymbol{w}_h\,\mathrm{d}x = 0 \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h\,.
\end{cases}
$$



Mixed FEM and Leapfrog Stepping

UDE

Theorem

For every $h > 0$ and $N \in \mathbb{N}$, $(\mathrm{QVI}_{N,h})$ admits a unique solution $\{\boldsymbol{E}_h^n\}_{n=1}^N \subset \mathbf{DG}_h$ and $\{\boldsymbol{H}_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \mathbf{ND}_h$. In particular,

$$\boldsymbol{E}_h^n = 2\boldsymbol{E}_h^{n-\frac{1}{2}} - \boldsymbol{E}_h^n \qquad \boldsymbol{w}_h^n = \boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}$$

$$\boldsymbol{E}_h^{n-\frac{1}{2}} = \frac{\tau\epsilon^{-1}}{2}(\boldsymbol{w}_h^n - P_\partial \boldsymbol{w}_h^n) \qquad P_\partial \boldsymbol{w}_h^n = \frac{j_c(\cdot, \theta_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}})}{\max\left(|\boldsymbol{w}_h^n|, j_c(\cdot, \theta_h^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}})\right)}.$$

Theorem

For every $h > 0$ and $N \in \mathbb{N}$, $(\mathrm{QVI}_{N,h})$ admits a unique solution $\{E_h^n\}_{n=1}^N \subset \mathrm{DG}_h$ and $\{H_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \mathrm{ND}_h$. In particular,

$$E_h^n = 2E_h^{n-\frac{1}{2}} - E_h^n \qquad w_h^n = f_h^{n-\frac{1}{2}} + \operatorname{curl} H_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau} E_h^{n-1}$$

$$E_h^{n-\frac{1}{2}} = \frac{\tau\epsilon^{-1}}{2}(w_h^n - P_\partial w_h^n) \qquad P_\partial w_h^n = \frac{j_c(\cdot, \theta_h^{n-\frac{1}{2}}, H_h^{n-\frac{1}{2}})}{\max\left(|w_h^n|, j_c(\cdot, \theta_h^{n-\frac{1}{2}}, H_h^{n-\frac{1}{2}})\right)}.$$

Benefit: Easily implementable - no iterative solver needed.

Additional assumptions for stability and limiting analysis:

- $\boldsymbol{E}_0 \in \boldsymbol{H}^1(\Omega)$ and a linear CFL-condition of type $\tau \leq Ch$

Relaxation of previous assumptions:

- $\boldsymbol{f} \in \mathrm{BV}([0,T], \boldsymbol{L}^2(\Omega))$
- $\theta \in \mathrm{BV}([0,T], L^2(\Omega)) \cap \mathcal{C}([0,T], L^\infty(\Omega))$

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Theorem

$$(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \overset{*}{\rightharpoonup} (\boldsymbol{E}, \boldsymbol{H}) \qquad \textit{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) \overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{E}, \boldsymbol{H}) \qquad \textit{weakly-* in } L^\infty((0,T), \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))$$

$$\|(\boldsymbol{E}_{N,h}, \boldsymbol{H}_{N,h}) - (\boldsymbol{E}, \boldsymbol{H})\|_{\mathcal{C}([0,T], \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega))} \to 0$$

For our numerical test, we consider

- the VI case

$$j_c(\cdot,\theta,\boldsymbol{H}) = (1-\theta)^2 \chi_{\Omega_{sc}}(\cdot) \quad \text{with } \theta(t,x) = t$$

- the QVI case

$$j_c(\cdot,\theta,\boldsymbol{H}) = \frac{(1-\theta)^2}{1+|\boldsymbol{H}|^\beta} \chi_{\Omega_{sc}}(\cdot) \quad \text{with } \theta(t,x) = t$$

with the superconductor $\Omega_{sc} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \sqrt{x_1^2+x_2^2+x_3^2} \leq 0.2\}$.

[1] M. Ciszek, B.A. Glowacki, S.P. Ashworth, A.M. Campbell, W.Y. Liang, R. Flükiger, and R.E. Gladyshevskii. AC losses and critical currents in Ag/(Tl,Pb,Bi)-1223 tape. *Physica C: Superconductivity and its Applications*, 1996

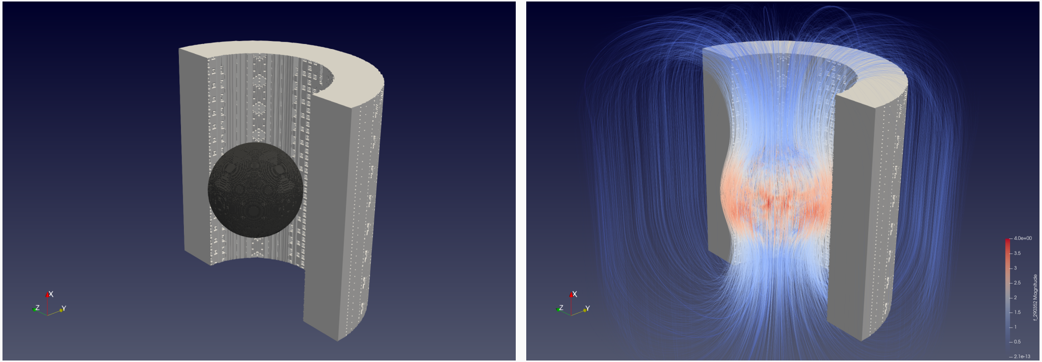

Figure 1: Source term, superconductor and the magnetic field.

Numerical Test

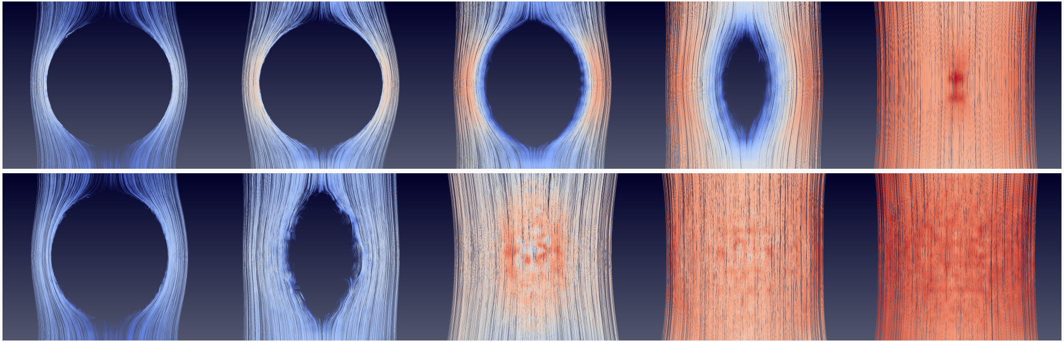

Figure 2: Top row: pure temperature dependence (VI).
Bottom row: temperature and magnetic field dependence (QVI).

Thank you for your attention!