Quasilinear Maxwell Variational Inequalities in Ferromagnetic Shielding

joint work with Gabriele Caselli, Irwin Yousept

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Obstacle Problem in Ferromagnetic Shielding

Electromagnetic shielding

Effect of redirecting or blocking electromagnetic fields by barriers made of conductive or magnetic materials.

Shielding of EM-waves

Electromagnetic shielding
Effect of redirecting or blocking electromagnetic fields by barriers made of conductive or magnetic materials.

Ferromagnetic shielding
Special case of Electromagnetic shielding: redirecting or blocking *magnetic fields* by ferromagnetic materials. Ferromagnetic materials are materials with high (relative) magnetic permeability, for example:

- Iron ($\mu/\mu_0 \approx 200.000$)
- Permalloy ($\mu/\mu_0 \approx 100.000$)
- Mu-metal ($\mu/\mu_0 \approx 50.000$)

The obstacle problem

To model the ferromagnetic shielding effect, we combine a Maxwell-structured elliptic VI of the first kind with a nonlinearity $\nu = \mu^{-1}\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$, resulting in the problem

$$
\text{(VI)} \qquad \begin{cases} \text{Find } (A,\phi) \in K \times H_0^1(\Omega), \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl}(v - A) + \int_\Omega \nabla\phi \cdot (v - A) \geq \int_\Omega J \cdot (v - A) \quad \forall v \in K \\ \displaystyle\int_\Omega A \cdot \nabla\psi = 0 \quad \forall \psi \in H_0^1(\Omega) \end{cases}
$$

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\& \qquad K := \{v \in H_0(\operatorname{curl})\colon |\operatorname{curl} v| \leq d(\cdot) \text{ a.e. on } \Omega\}.
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$$\& \qquad K := \{v \in H_0(\operatorname{\mathbf{curl}})\colon |\operatorname{\mathbf{curl}} v| \leq d(\cdot) \ \text{a.e. on } \Omega\}.$$

- $\Omega \subseteq \mathbb{R}^3$ open, bounded, Lipschitz, simply connected
- $(J, d) \in \boldsymbol{L}^2(\Omega) \times L^2(\Omega)$
- ν is ‘standard’, i.e. Carathéodory, strictly positive, bounded, strongly monotone and Lipschitz

We investigate:

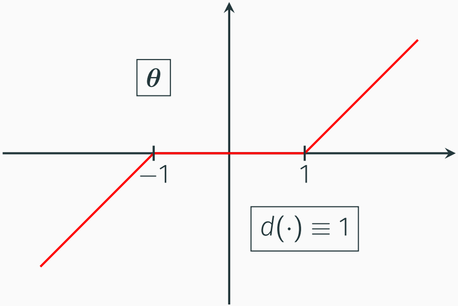
- Is (VI) well-posed?
- How regular is its dual multiplier?
- Optimal control of (VI)

Main ingredient: A Moreau-Yosida type penalization of (VI).

Regularization of the Variational Inequality

For $\gamma > 0$ we define

$$\boldsymbol{\theta} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x, s) \mapsto \begin{cases} \max(|s| - d(x), 0)\dfrac{s}{|s|}, & s \neq 0 \\ 0, & s = 0. \end{cases}$$



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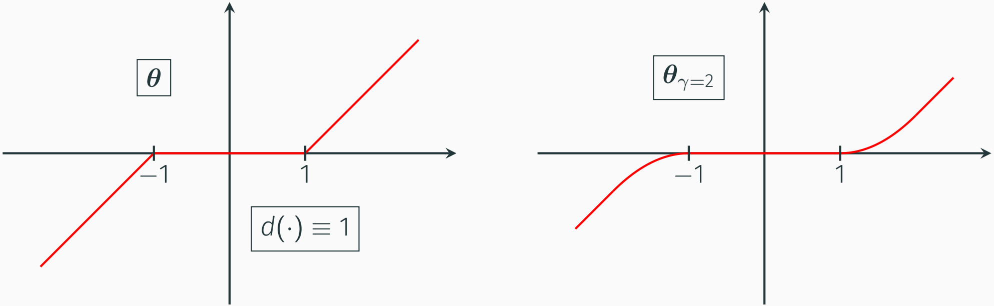

Vector-valued penalization and smoothing

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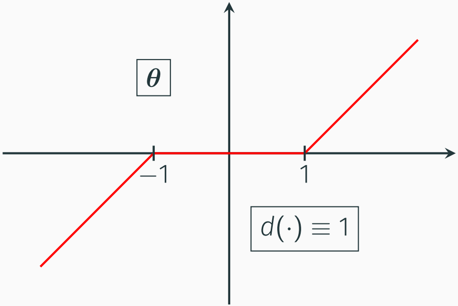

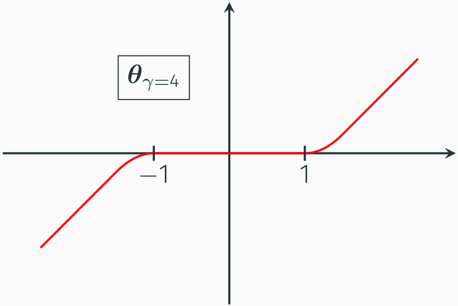

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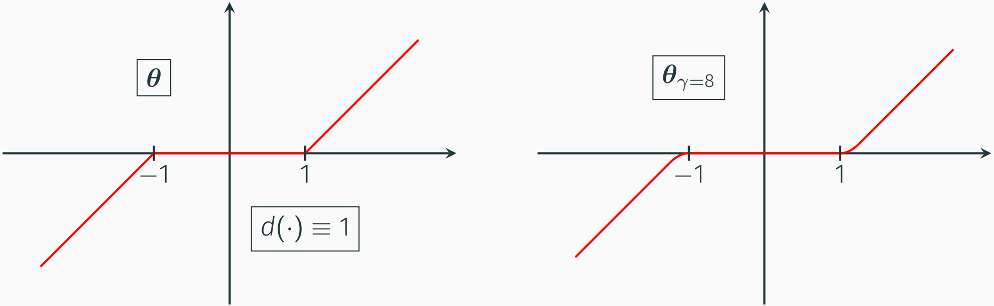

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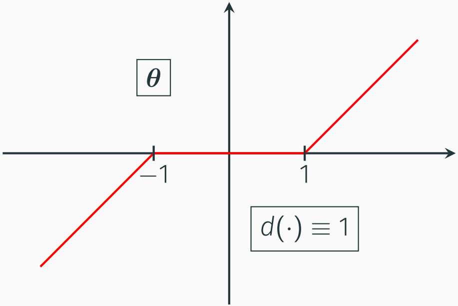

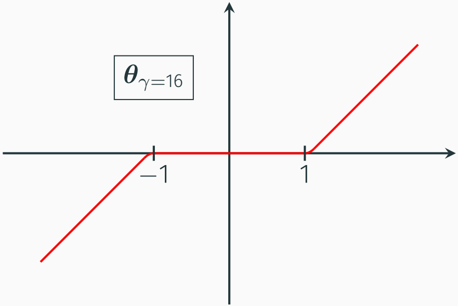

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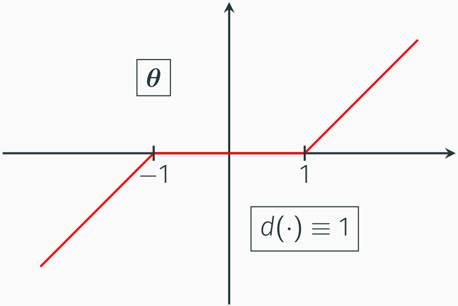

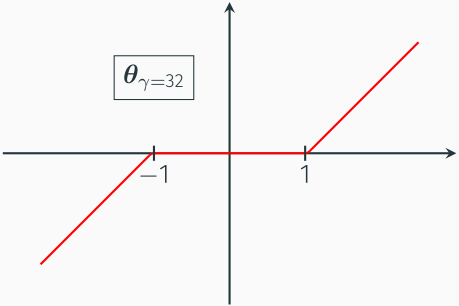

For $\gamma > 0$, we consider the regularized (unconstrained) problem

$$(\mathrm{VE}_\gamma^{\mathrm{sol}}) \qquad \begin{cases} \text{Find } \boldsymbol{A}_\gamma \in X_{N,0} := \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0), \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}\,\boldsymbol{v} + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{A}_\gamma) \cdot \mathbf{curl}\,\boldsymbol{v} \quad = \int_\Omega \boldsymbol{J}_{\mathrm{sol}} \cdot \boldsymbol{v} \\ \forall \boldsymbol{v} \in X_{N,0}. \end{cases}$$

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Lemma

For every $\boldsymbol{J}_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$, the regularized problem $(\mathrm{VE}_\gamma^{\mathrm{sol}})$ admits a unique solution $\boldsymbol{A}_\gamma$.

Left-hand side induces a monotone and coercive operator $\boldsymbol{X}_{N,0} \to \boldsymbol{X}_{N,0}^*$.

Theorem

For $J_{\mathrm{sol}} \in \boldsymbol{H}(\mathrm{div}{=}0)$*, the unique solution* $\boldsymbol{A}_\gamma$ *of* $(\mathrm{VE}_\gamma^{\mathrm{sol}})$ *converges strongly in* $\boldsymbol{X}_{N,0}$ *to the unique solution of the problem*

$$
(\mathrm{VI}_{\mathrm{sol}}) \qquad \begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0), \ \text{s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl}(v - A) \geq \int_\Omega J_{\mathrm{sol}} \cdot (v - A) \quad \forall \boldsymbol{v} \in \boldsymbol{K} \cap \boldsymbol{H}(\mathrm{div}{=}0). \end{cases}
$$

Well-Posedness and Regularity

UDE

Corollary

For every $\boldsymbol{J} \in \mathbf{L}^2(\Omega)$*, there exists a unique solution* $(\boldsymbol{A}, \phi) \in \boldsymbol{K} \times H_0^1(\Omega)$ *to* (VI)*. Moreover, there exists a unique multiplier* $\boldsymbol{m} \in \boldsymbol{X}_{N,0}$ *such that the solution* $(\boldsymbol{A}, \phi)$ *is characterized by the dual formulation*

$$\begin{cases} \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} \boldsymbol{A}|) \operatorname{curl} \boldsymbol{A} \cdot \operatorname{curl} \boldsymbol{v} + \nabla\phi \cdot \boldsymbol{v} + \operatorname{curl} \boldsymbol{m} \cdot \operatorname{curl} \boldsymbol{v} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl}) \\ \displaystyle\int_\Omega \boldsymbol{A} \cdot \nabla\psi = 0 \quad \forall \psi \in H_0^1(\Omega) \\ \displaystyle\int_\Omega \operatorname{curl} \boldsymbol{m} \cdot \operatorname{curl}(\boldsymbol{v} - \boldsymbol{A}) \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \end{cases}$$

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Corollary

For every $J \in L^2(\Omega)$, *there exists a unique solution* $(A, \phi) \in K \times H_0^1(\Omega)$ *to* (VI). *Moreover, there exists a unique multiplier* $\boldsymbol{m} \in X_{N,0}$ *such that the solution* (A, ϕ) *is characterized by the dual formulation*

$$\begin{cases} \int_\Omega \nu(\cdot, |\operatorname{curl} A|) \operatorname{curl} A \cdot \operatorname{curl} v + \nabla\phi \cdot v + \operatorname{curl} m \cdot \operatorname{curl} v = \int_\Omega J \cdot v & \forall v \in H_0(\operatorname{curl}) \\ \int_\Omega A \cdot \nabla\psi = 0 & \forall \psi \in H_0^1(\Omega) \\ \int_\Omega \operatorname{curl} m \cdot \operatorname{curl}(v - A) \leq 0 & \forall v \in K. \end{cases}$$

How regular are the appearing multipliers?

Theorem

Let $\partial\Omega$ be connected. For $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, let $(\boldsymbol{A}, \phi, \boldsymbol{m}) \in \boldsymbol{X}_{N,0} \times H_0^1(\Omega) \times \boldsymbol{X}_{N,0}$ denote the unique solution to the previous dual formulation. Then, the following multiplier regularity results hold true:

$$p \in [2,3],\ \boldsymbol{J} \in \boldsymbol{L}^p(\Omega),\ d \in L^p(\Omega) \quad \Rightarrow \quad \phi \in W_0^{1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

$$p \in [2,6],\ \boldsymbol{J} \in \boldsymbol{L}^p(\Omega),\ d \in L^p(\Omega),\ \Omega \text{ of class } \mathcal{C}^{1,1} \quad \Rightarrow \quad \phi \in W_0^{1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

$$p \in [2,\infty),\ \boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl}),\ d \in L^p(\Omega),\ \Omega \text{ of class } \mathcal{C}^{2,1} \quad \Rightarrow \quad \mathbf{curl}\, \boldsymbol{m} \in \boldsymbol{L}^p(\Omega)$$

$$\boldsymbol{J} \in \boldsymbol{H}_0(\mathbf{curl}),\ d \in L^\infty(\Omega),\ \nu(\cdot, |\mathbf{curl}\, \boldsymbol{A}|) \in \mathcal{C}^{0,1}(\overline{\Omega}),\ \Omega \text{ of class } \mathcal{C}^{2,1} \quad \Rightarrow \quad \mathbf{curl}\, \boldsymbol{m} \in \boldsymbol{L}^\infty(\Omega)$$

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The proof is mainly based on an L^p-Helmholz-decomposition and elliptic regularity theory.

Optimal Control

$$
(\mathrm{P}) \quad \begin{cases} \displaystyle\min_{(J,A)\in L^2(\Omega)\times X_{N,0}} \frac{1}{2}\|\operatorname{curl} A - B_\mathrm{d}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|J\|^2_{L^2(\Omega)} \\ \text{subject to} \\ \displaystyle\int_\Omega \nu(\cdot,|\operatorname{curl} A|)\operatorname{curl} A \cdot \operatorname{curl}(v - A) + \int_\Omega \nabla\phi \cdot (v - A) \geq \int_\Omega J \cdot (v - A) \quad \forall v \in K \\ \displaystyle\int_\Omega A \cdot \nabla\psi = 0 \quad \forall \psi \in H^1_0(\Omega). \end{cases}
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The optimal control problem

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(\mathrm{P}) \qquad \begin{cases} \displaystyle \min_{(J,A)\in H(\mathrm{div}=0)\times X_{N,0}} \frac{1}{2}\|\operatorname{curl} A - B_{\mathrm{d}}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|J\|^2_{L^2(\Omega)} \\ \text{subject to} \\ \displaystyle \int_\Omega \nu(\cdot,|\operatorname{curl} A|)\operatorname{curl} A \cdot \operatorname{curl}(v - A) \geq \int_\Omega J \cdot (v - A) \quad \forall v \in K \cap H(\mathrm{div}=0). \end{cases}
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There exists an optimal solution $J^\star \in \boldsymbol{H}(\mathrm{div}{=}0)$ *to the problem* (P).

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Task: Find optimality conditions for optimal controls $J^\star$.
Problem: The mapping G is not directionally differentiable.

The regularized optimal control problem

$$
(\mathrm{P}_\gamma) \qquad \begin{cases} \displaystyle\min_{(J_\gamma, A_\gamma)\in H(\mathrm{div}=0)\times X_{N,0}} \frac{1}{2}\|\operatorname{curl} A_\gamma - B_\mathrm{d}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|J_\gamma\|^2_{L^2(\Omega)} + \frac{\lambda}{4}\|J_\gamma - J^\star\|^2_{L^2(\Omega)} \\ \text{subject to} \\ \displaystyle\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|)\operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \theta_\gamma(\cdot, \operatorname{curl} A_\gamma)\cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \\ \forall v \in X_{N,0}. \end{cases}
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(\mathrm{P}_\gamma) \qquad \begin{cases} \min_{(J_\gamma, A_\gamma) \in H(\mathrm{div}=0) \times X_{N,0}} \frac{1}{2} \| \operatorname{curl} A_\gamma - B_\mathrm{d} \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| J_\gamma \|^2_{L^2(\Omega)} + \frac{\lambda}{4} \| J_\gamma - J^\star \|^2_{L^2(\Omega)} \\ \text{subject to} \\ \int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \theta_\gamma(\cdot, \operatorname{curl} A_\gamma) \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \\ \forall v \in X_{N,0}. \end{cases}
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The solution mapping

$$
G_\gamma : H(\mathrm{div}=0) \to X_{N,0}, \quad J_\gamma \mapsto A_\gamma
$$

is weak-strong continuous, i.e. there exists a minimizer $(J_\gamma, A_\gamma) \in H(\mathrm{div}=0) \times X_{N,0}$ for (P_γ). Especially, as a result of our smoothing process, G_γ is weakly Gâteaux differentiable.

Theorem

$J_\gamma \in H(\mathrm{div}{=}0)$ *optimal control for* (P_γ). *Then, there exists* $(A_\gamma, Q_\gamma) \in X_{N,0} \times X_{N,0}$, *s.t.*

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A_\gamma|) \operatorname{curl} A_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \boldsymbol{\theta}_\gamma(\cdot, \operatorname{curl} A_\gamma) \cdot \operatorname{curl} v = \int_\Omega J_\gamma \cdot v \quad \forall v \in X_{N,0}$$

$$\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\,[\operatorname{curl} A_\gamma])^\top \operatorname{curl} Q_\gamma \cdot \operatorname{curl} v + \gamma \int_\Omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \operatorname{curl} A_\gamma) \operatorname{curl} Q_\gamma \cdot \operatorname{curl} v$$

$$= \int_\Omega (\operatorname{curl} A_\gamma - B_\mathrm{d}) \cdot \operatorname{curl} v \quad \forall v \in X_{N,0}$$

$$J_\gamma = -\frac{2}{3}\lambda^{-1} Q_\gamma + \frac{1}{3} J^\star.$$

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Theorem

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$$J_\gamma = -\frac{2}{3}\lambda^{-1} Q_\gamma + \frac{1}{3} J^\star.$$

Given an optimal control $J^\star \in H(\mathrm{div}{=}0)$ of (P), we obtain

- a sequence $\{J_\gamma^\star\}_{\gamma>0} \subseteq H(\mathrm{div}{=}0)$ of minimizers to (P_γ) satisfying

$$J_\gamma^\star \to J^\star \quad \text{strongly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

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- a sequence

$$\big\{\big(A_\gamma^\star, Q_\gamma^\star, \boldsymbol{\xi}_\gamma^\star, \boldsymbol{\lambda}_\gamma^\star\big)\big\}_{\gamma>0} \subseteq X_{N,0} \times X_{N,0} \times L^2(\Omega) \times L^2(\Omega)$$

of states and multipliers as well as limiting fields, s.t.

$$\begin{array}{lllll} A_\gamma^\star \to A^\star & \text{strongly} & \text{in } X_{N,0} & \text{as } \gamma \to \infty \\ Q_\gamma^\star \rightharpoonup Q^\star & \text{weakly} & \text{in } X_{N,0} & \text{as } \gamma \to \infty \\ \big(\mathbb{P}_{\mathrm{curl}\, X_{N,0}} \boldsymbol{\xi}_\gamma^\star, \mathbb{P}_{\mathrm{curl}\, X_{N,0}} \boldsymbol{\lambda}_\gamma^\star\big) \rightharpoonup (\mathrm{curl}\, m^\star, \mathrm{curl}\, n^\star) & \text{weakly} & \text{in } L^2(\Omega) \times L^2(\Omega) & \text{as } \gamma \to \infty. \end{array}$$

Theorem

The limiting fields $(A^\star, Q^\star, \operatorname{curl} m^\star, \operatorname{curl} n^\star) \in X_{N,0} \times X_{N,0} \times \operatorname{curl} X_{N,0} \times \operatorname{curl} X_{N,0}$ *satisfy*

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A^\star|) \operatorname{curl} A^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl} v = \int_\Omega J^\star \cdot v \quad \forall v \in X_N^0$$

$$\int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl}(v - A^\star) \leq 0 \quad \forall v \in K$$

$$\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\,[\operatorname{curl} A^\star])^\top \operatorname{curl} Q^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} n^\star \cdot \operatorname{curl} v$$

$$= \int_\Omega (\operatorname{curl} A^\star - B_\mathrm{d}) \cdot \operatorname{curl} v \quad \forall v \in X_N^0$$

$$J^\star = -\lambda^{-1} Q^\star.$$

UDE

Theorem

The limiting fields $(A^\star, Q^\star, \operatorname{curl} m^\star, \operatorname{curl} n^\star) \in X_{N,0} \times X_{N,0} \times \operatorname{curl} X_{N,0} \times \operatorname{curl} X_{N,0}$ *satisfy*

$$\int_\Omega \nu(\cdot, |\operatorname{curl} A^\star|) \operatorname{curl} A^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl} v = \int_\Omega J^\star \cdot v \quad \forall v \in X_N^0$$

$$\int_\Omega \operatorname{curl} m^\star \cdot \operatorname{curl}(v - A^\star) \leq 0 \quad \forall v \in K$$

$$\int_\Omega (\mathrm{D}_s[\nu(\cdot, |s|)s]\,[\operatorname{curl} A^\star])^\top \operatorname{curl} Q^\star \cdot \operatorname{curl} v + \int_\Omega \operatorname{curl} n^\star \cdot \operatorname{curl} v$$

$$= \int_\Omega (\operatorname{curl} A^\star - B_\mathrm{d}) \cdot \operatorname{curl} v \quad \forall v \in X_N^0$$

$$J^\star = -\lambda^{-1} Q^\star.$$

In the scalar H^1-setting (without an additional quasilinearity) with an obstacle set

$$K = \{v \in H_0^1(\Omega) \colon v \geq 0 \text{ a.e. on } \Omega\}$$

it is known that the adjoint multiplier is characterized[1] by

$$\int_\Omega (\text{adjoint multiplier}) \cdot (\text{state}) = 0$$

$$\int_\Omega (\text{adjoint multiplier}) \cdot (\text{adjoint state}) \geq 0.$$

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As a reminder, we have

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \colon |\mathbf{curl}\, \boldsymbol{v}| \leq d(\cdot) \ \text{a.e. on } \Omega\}.$$

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Further characterization of the adjoint multiplier $\operatorname{curl} n^\star$

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$$\int_\Omega \operatorname{curl} n^\star \cdot \operatorname{curl} Q^\star \geq 0 \quad (\text{under the assumption } \nu \equiv 1).$$

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$$\int_\Omega \operatorname{curl} n^\star \cdot \left(d \frac{\operatorname{curl} A^\star}{|\operatorname{curl} A^\star|} - \operatorname{curl} A^\star \right) = 0 \quad \boxed{?}$$

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We recall that

$$\mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}_\gamma^\star \rightharpoonup \operatorname{curl} n^\star \quad \text{weakly in } L^2(\Omega) \quad \text{as } \gamma \to \infty.$$

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In particular, there exist $\boldsymbol{\sigma}^\star_{d_+}, \boldsymbol{\sigma}^\star_{d_-} \in \boldsymbol{L}^2(\Omega)$, s.t.

$$\mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}^\star_\gamma \chi_{\{|\operatorname{curl} A^\star_\gamma| > d\}} \rightharpoonup \boldsymbol{\sigma}^\star_{d_+} \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty$$

$$\mathbb{P}_{\operatorname{curl} X_{N,0}} \boldsymbol{\lambda}^\star_\gamma \chi_{\{|\operatorname{curl} A^\star_\gamma| \leq d\}} \rightharpoonup \boldsymbol{\sigma}^\star_{d_-} \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty$$

and

$$\operatorname{curl} n^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}.$$

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d \frac{\operatorname{curl} A^\star}{|\operatorname{curl} A^\star|} - \operatorname{curl} A^\star \right) = 0 \quad \boxed{?}$$

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and

$$\operatorname{curl} n^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}.$$

Theorem

The adjoint multiplier $\operatorname{\mathbf{curl}} \boldsymbol{n}^\star$ *is additionally characterized by*

$$\int_\Omega \boldsymbol{\sigma}^\star_{d_+} \cdot \left(d \frac{\operatorname{\mathbf{curl}} A^\star}{|\operatorname{\mathbf{curl}} A^\star|} - \operatorname{\mathbf{curl}} A^\star \right) = 0$$

$$\operatorname{\mathbf{curl}} \boldsymbol{n}^\star = \boldsymbol{\sigma}^\star_{d_+} + \boldsymbol{\sigma}^\star_{d_-}.$$

If $\nu \equiv 1$, *i.e. the nonlinearity is not present, then*

$$\int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{n}^\star \cdot \operatorname{\mathbf{curl}} Q^\star \geq 0.$$

Thank you for your attention!