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Part I: existence and uniqueness

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ANALYSIS OF A VISCOUS TWO-FIELD GRADIENT DAMAGE MODEL
PART I: EXISTENCE AND UNIQUENESS

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Abstract. The paper deals with a viscous damage model including two damage variables, a local and a non-local one, which are coupled through a penalty term in the free energy functional. Under certain regularity conditions for linear elasticity equations, existence and uniqueness of the solution is proven, provided that the penalization parameter is chosen sufficiently large. Moreover, the regularity of the unique solution is investigated, in particular the differentiability w.r.t. time.

Key words. Viscous damage evolution, $W^{1,p}$-theory, penalization

1. Introduction. This paper is concerned with the mathematical analysis of a particular gradient enhanced damage model. The special feature of the model under consideration is that it contains two damage variables, which are connected through a penalty term in the free energy functional. For this reason we call our model 'two-field damage model'. It is inspired by the one presented in [6], which is a popular model that is widely used in computational mechanics. While one damage variable provides a local character and carries the non-smooth time evolution, the other one accounts for nonlocal effects. The goal of this work and the companion paper [21] is to show that this model is well posed from a mathematical point of view. To be more precise, we first prove existence and uniqueness for fixed penalty parameter. Afterwards we turn our attention to the limit analysis for penalty parameter tending to infinity.

From a mathematical perspective, the damage model in [6] has two main drawbacks. Firstly, it is rate-independent and the corresponding dissipation functional is unbounded. Secondly, the coupling between damage evolution and balance of momentum is realized via the less regular one of the two damage variables. To make the problem amenable to a rigorous mathematical analysis, we therefore slightly modify the model. In order to guarantee existence and uniqueness of a solution, we add a viscosity term to the damage evolution, which turns the rate-independent model in [6] into a rate-dependent one. Moreover, we couple the damage evolution and the balance of momentum through the more regular damage variable in order to enable the use of compact embeddings which are essential for the proof of existence. The overall model arising in this way consists of an elliptic system for nonlocal damage and displacement field and a non-smooth evolution equation for the local damage variable.

In the present paper, we focus on proving existence and uniqueness for our modified model for a fixed penalization parameter. An essential tool in this context is the $W^{1,p}$–theory with $p > 2$ for nonlinear elasticity equations from [15]. Convex analysis results play a crucial role as well, since they enable us to give an equivalent formulation of the damage evolution in terms of an ordinary differential equation in Banach space. In combination with a classical contraction argument, this allows us to prove the unique solvability of the damage model under consideration. The ODE character of the damage evolution along with elliptic regularity results from [13] are then employed for the investigation of the regularity of the solution, in particular regarding its

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differentiability w.r.t. time. Here, the characterization via ODE turns out to be very useful, as continuously differentiability w.r.t. time of the damage can be immediately followed. This carries on to the other variables, on account of the smoothness of the elliptic system. The findings of this paper constitute the basis for the limit analysis for penalization parameter tending to infinity, which is addressed in the companion paper [21]. The passage to the limit is performed by means of an equivalent reformulation of the model in terms of an energy identity in the spirit of [17]. In the limit both damage variables coincide, and the limit model is in accordance with the class of classical partial damage models introduced in [10].

Let us put our work into perspective. Numerous damage models have been addressed by many authors under different aspects. In [1–3, 9] various viscous damage models have been analyzed with regard to existence and regularity of solutions. The concept of viscosity also plays an important role in the mathematical treatment of rate-independent damage models, as the vanishing viscosity approach is a prominent method to establish solutions for rate-independent problems. We only refer to [7, 16–19, 23–25], and the references therein. Various notions of solutions are known for rate-independent models, such as e.g. global energetic solutions and balanced viscosity (BV) solutions. An overview thereof is given in [22], in the framework of generalized gradient systems. Under suitable assumptions BV solutions are obtained via a vanishing viscosity analysis, which has been demonstrated in [17] for a gradient damage model in the spirit of [10]. However, to the best of our knowledge, a damage model containing two damage variables has never been investigated so far with regard to a rigorous mathematical analysis, although these models are frequently used for numerical simulations, cf. e.g. [20, 26, 27, 30, 31]. This concerns the existence and regularity of solutions, let alone the behavior of the damage variables and the displacement field, as the penalty vanishes.

The paper is organized as follows. In Section 2.1 we introduce the two-field damage model from [6], which serves as a basis for our damage model. Section 2.2 is devoted to the modifications of the model from [6], which were already indicated above. We describe their mathematical motivation in detail and compare our model to the one from [6]. It turns out that the modified coupling between damage evolution and balance of momentum is expected to have only little influence in practice, cf. Remark 2.5, whereas the viscous regularization is a standard procedure in computational mechanics. Section 2.3 then gives an overview of the variables, operators and function spaces and collects the notations and standing assumptions. In Section 3.1 we address the existence and uniqueness for the elliptic system as part of the complete damage model. Based on these results, Section 3.2 deals with the complete model including the evolutionary equation for the local damage variable. This turns out to be equivalent to an operator differential equation, which allows us to apply standard contraction arguments for the proof of existence and uniqueness. For convenience of the reader, some results on Nemyczkii operators, which are used in Section 3 are stated in Appendix A. Sections 4 and 5 are devoted to improving the regularity of the solution. We first address the higher spatial regularity of the nonlocal damage variable, and prove its Lipschitz continuity as a function of the local one. In Section 5 we show that the operators mapping the local damage variable to the nonlocal damage parameter and displacement are continuously Fréchet-differentiable. This finally allows us to prove that the overall solution is continuously differentiable w.r.t. time in appropriate spaces.

2. Formulation of the Model and Standing Assumptions. In this section we first motivate our damage model, by formally presenting the inspiration thereof. In the second part, we introduce the precise model, while in the third one, the function spaces and the variables are defined. At the end of the section we state the general assumptions on the data.

2.1. A Two-Field Gradient Enhanced Damage Model. The model analysed throughout this paper was inspired by a damage model presented in [6]. Therein, two damage variables are introduced, which the authors call 'local' and 'nonlocal' damage. In the free energy, a gradient term and a term, which penalizes the difference between local and nonlocal damage, are included. To be precise, the energy functional $\mathcal{E} : [0,T] \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ according to [6] is given by

$$\mathcal{E}(t,\boldsymbol{u},\varphi,d) := \frac{1}{2}\int_\Omega g(d)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\, dx - \langle \ell(t), \boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 + \frac{\beta}{2}\|\varphi - d\|_2^2,$$

where V is an appropriate Sobolev space and $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)$ is the linearized strain tensor. We refer to Section 2.3 for more details. The parameters $\alpha, \beta > 0$ are weighting parameters for the gradient regularization and for the penalization, respectively, see [6] for more details.

The model introduced in [6] describes the evolution of damage in an elastic body. During the process, a time dependent volume and boundary load, denoted by ℓ, is applied upon the body, which has a part of its boundary clamped. The body is described by the domain $\Omega \subset \mathbb{R}^N$, on which we impose mild smoothness assumptions, see Section 2.3. The load induces a certain displacement $\boldsymbol{u} : [0,T] \times \Omega \to \mathbb{R}^N$, as well as local and nonlocal damage. The latter one is denoted by $\varphi : [0,T] \times \Omega \to \mathbb{R}$, while the local damage is called $d : [0,T] \times \Omega \to \mathbb{R}$. Its values measure the degree of the material rigidity loss. Therefore, $d(t,x) = 0$ means that the body is completely sound, while $d(t,x) \to \infty$ means that the body is so damaged that there is no more opponence from its side. The function g is supposed to be smooth and it measures the influence of the damage on the elastic behaviour of the body. For the precise assumptions on the function g, see Assumption 2.9. Finally, $\mathbb{C}$ is the elasticity tensor, which is assumed to be coercive and bounded, see Assumption 2.10.

At each time point, the displacement and the nonlocal damage are supposed to minimize the stored energy, i.e.,

$$(\boldsymbol{u}(t), \varphi(t)) \in \underset{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)}{\arg\min}\ \mathcal{E}(t,\boldsymbol{u},\varphi,d(t)). \tag{2.1}$$

The evolution of local damage in the rate independent case is modeled by the differential inclusion

$$-\partial_d\mathcal{E}(t,\boldsymbol{u}(t),\varphi(t),d(t)) \in \partial\mathcal{R}_1(\dot{d}(t)) \qquad \text{f.a.a. } t \in (0,T), \tag{2.2}$$

where the function $\mathcal{R}_1$ denotes the dissipated energy.

Definition 2.1 (Dissipation Functional). *The dissipation $\mathcal{R}_1 : L^2(\Omega) \to [0,\infty]$ is defined as*

$$\mathcal{R}_1(\eta) := \begin{cases} r\int_\Omega \eta\, dx, & \textit{if } \eta \geq 0 \textit{ a.e. in } \Omega, \\ \infty, & \textit{otherwise,} \end{cases}$$

where $r > 0$ stands for the fracture toughness of the material.

Thanks to the positive homogeneity of $\mathcal{R}_1$, the considered process is rate independent, which means that the values of the damage do not depend on the rate with which ℓ changes in time. As a consequence, one ignores inertial and viscosity effects.

The system (2.1)–(2.2) is equivalent to the damage model [6, (6), (7) and (18)]. Note that [6, (18)] corresponds to the dual formulation of the evolutionary equation (2.2). In order to see this, we refer to Section 3.2, where a similar result is proven.

Remark 2.2. *Let us point out that, in the classical literature (see for instance [9,10]), the damage variable is frequently set to 1, if the material is fully sound, and 0, if it is completely damaged. However, since the starting point for our analysis is the model from [6], we decided to work in the framework described above, where $d \in [0, \infty)$. We underline that Section 7 in the companion paper [21] demonstrates that one can transfer our setting into the classical one [10] by a suitably chosen transformation of the damage variable.*

Remark 2.3. *The notions 'local' and 'nonlocal' damage, respectively, refer to the fact that the variable d provides a local character, as it solves an ODE in Banach space, while the equation for φ features a nonlocal gradient regularization, see (3.41) below.*

2.2. Modification of the Model. Because of theoretical reasons, we modify the energy functional $\mathcal{E}$ such that the function g depends on the nonlocal damage instead of the local damage. This modification is motivated by the fact that the local damage possesses less regularity. Therefore, we insert φ instead of d into the coefficient function g such that the coupling between the balance of momentum and the damage evolution is realized with the more regular function φ.

Definition 2.4 (Energy Functional). *The stored energy $\mathcal{E} : [0, T] \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is given by*

$$\mathcal{E}(t, \boldsymbol{u}, \varphi, d) := \frac{1}{2} \int_\Omega g(\varphi) \mathbb{C} \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx - \langle \ell(t), \boldsymbol{u} \rangle_V + \frac{\alpha}{2} \|\nabla \varphi\|_2^2 + \frac{\beta}{2} \|\varphi - d\|_2^2.$$

Remark 2.5. *As the penalty approach aims to minimize the deviation between φ and d, we expect the two models to yield similar results, at least for large values of β. This is also confirmed by the limit analysis for $\beta \to \infty$ in the companion paper [21], which shows that both damage variables equal in the limit.*

We will also work with a different dissipation functional, namely a viscous regularization of the dissipation functional from Definition 2.1. Although (weak) solvability results for rate-independent damage processes with non-convex energy functional as in our case may be proven, one can neither expect the solutions to be unique nor smooth in time, see [17,22]. To overcome this issue, we apply a viscous regularization, which is frequently used in the context of damage modelling. This consists of adding an L^2-viscosity term in the dissipation functional, which leads to a rate-dependent process, since the dissipation loses its positive homogeneity.

Definition 2.6 (Viscous Dissipation Functional). *We define $\mathcal{R}_\delta : L^2(\Omega) \to [0, \infty]$ as*

$$\mathcal{R}_\delta(\eta) := \begin{cases} r \int_\Omega \eta \, dx + \frac{\delta}{2} \|\eta\|_2^2, & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty, & \text{otherwise,} \end{cases}$$

where $\delta > 0$ is the viscosity parameter.

To summarize, the viscous 'two-field damage model' arising from the above considerations reads:

$$\left.\begin{aligned} (\boldsymbol{u}(t),\varphi(t)) \in \operatorname*{arg\,min}_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t,\boldsymbol{u},\varphi,d(t)), \\ 0 \in \partial\mathcal{R}_\delta(\dot{d}(t)) + \partial_d\mathcal{E}(t,\boldsymbol{u}(t),\varphi(t),d(t)) \end{aligned}\right\} \tag{P}$$

for almost all $t \in (0,T)$ with the initial condition $d(0) = d_0$ a.e. in Ω.

2.3. Notation and Standing Assumptions. Throughout the paper, C denotes a generic positive constant. If X and Y are two linear normed spaces, the space of linear and bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$. The dual of a linear normed space X will be denoted by X^*. For the dual pairing between X and X^* we write $\langle .,.\rangle_X$ and, if it is clear from the context, which dual pairing is meant, we just write $\langle .,.\rangle$. By $\|\cdot\|_p$ we denote the $L^p(\Omega)-$norm for $p\in[1,\infty]$ and by $(\cdot,\cdot)_2$ the $L^2(\Omega)-$scalar product. If X is compactly embedded in Y, we write $X \hookrightarrow\hookrightarrow Y$, and $X \overset{d}{\hookrightarrow} Y$ means that X is dense in Y. In the rest of the paper $N \in \{2,3\}$ denotes the spatial dimension. By bold-face case letters we denote vector valued variables and vector valued spaces. (Partial) derivatives w.r.t. time are frequently denoted by a dot.

Definition 2.7. *For $p \in [1,\infty]$ we define the following subspace of $\boldsymbol{W}^{1,p}(\Omega)$:*

$$\boldsymbol{W}^{1,p}_D(\Omega) := \{v \in \boldsymbol{W}^{1,p}(\Omega) : v|_{\Gamma_D} = 0\},$$

where Γ_D is a part of the boundary of the domain Ω, see Assumption 2.8 below. The dual space of $\boldsymbol{W}^{1,p'}_D(\Omega)$ is denoted by $\boldsymbol{W}^{-1,p}_D(\Omega)$, where p' is the conjugate exponent of p. If $p = 2$, we abbreviate $V := \boldsymbol{W}^{1,2}_D(\Omega)$.

Assumption 2.8. *The domain $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, is a bounded Lipschitz domain in the sense of [12, Chap. 1.2]. Its boundary is denoted by Γ and consists of two disjoint measurable parts Γ_N and Γ_D such that $\Gamma = \Gamma_N \cup \Gamma_D$. While Γ_N is a relatively open subset, Γ_D is a relatively closed subset of Γ with positive measure.*

In addition, the set $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [13]. That is, for every point $x \in \Gamma$, there exists an open neighborhood $\mathcal{U}_x \subset \mathbb{R}^N$ of x and a bi-Lipschitz map (a Lipschitz continuous and bijective map with Lipschitz continuous inverse) $\Psi_x : \mathcal{U}_x \to \mathbb{R}^N$ such that $\Psi_x(x) = 0 \in \mathbb{R}^N$ and $\Psi_x\big(\mathcal{U}_x \cap (\Omega\cup\Gamma_N)\big)$ equals one of the following sets:

$$\begin{aligned} E_1 &:= \left\{y \in \mathbb{R}^N : |y| < 1,\ y_N < 0\right\}, \\ E_2 &:= \left\{y \in \mathbb{R}^N : |y| < 1,\ y_N \le 0\right\}, \\ E_3 &:= \{y \in E_2 : y_N < 0 \text{ or } y_1 > 0\}. \end{aligned}$$

A detailed characterization of Gröger-regular sets in two and three spatial dimensions is given in [14].

Assumption 2.9. *The function $g : \mathbb{R} \to [\epsilon, 1]$ satisfies*

$$g \in C^{1,1}(\mathbb{R}) \tag{2.3}$$

with $\epsilon > 0$. With a little abuse of notation the Nemystkii-operators associated with g and g', considered with different domains and ranges, will be denoted by the same symbol.

The coefficient function g measures how the elastic properties of the body are preserved depending on the value of the damage. Therefore, from a mechanical point of view, it would make sense to impose g to be monotonically decreasing. This property of g is needed, if one aims to show that the nonlocal damage variable admits just positive values, as the local damage variable does. (In fact, it suffices that g is monotonically decreasing on $\mathbb{R}^-$ to prove this result.) However, since we do not need this result in our analysis, we do not require that g is monotonically decreasing.

We emphasize that, due to the condition $g \geq \epsilon$, our model constitutes a partial damage model. By contrast, $\lim_{\varphi\to\infty} g(\varphi) = 0$ is assumed in [6, (2)], which assures that complete material rigidity loss occures in the case of complete damage. However, in order to guarantee coercivity of the bilinear form associated with the balance of momentum in (3.9), we have to impose a positive lower bound on g.

ASSUMPTION 2.10. *The fourth-order tensor $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}}))$ is symmetric and uniformly coercive, i.e., there is a constant $\gamma_{\mathbb{C}} > 0$ such that*

$$\mathbb{C}(x)\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \gamma_{\mathbb{C}} |\boldsymbol{\sigma}|^2 \quad \forall\, \boldsymbol{\sigma} \in \mathbb{R}^{N\times N}_{\text{sym}} \text{ and f.a.a. } x \in \Omega, \tag{2.4}$$

where $|\cdot|$ denotes the Frobenius norm on $\mathbb{R}^{N\times N}$ and $(\cdot : \cdot)$ the scalar product inducing this norm.

ASSUMPTION 2.11. *For the applied volume and boundary load we require*

$$\ell \in C^{0,1}([0,T]; \boldsymbol{W}^{-1,p}_D(\Omega)),$$

where $p > 2$ is specified below, see Lemma 3.3, Assumption 3.10, and Assumption 3.16.

Moreover, the initial damage is supposed to satisfy $d_0 \in L^2(\Omega)$.

3. Existence and Uniqueness.

In this section, we mainly focus on finding unique solutions $\boldsymbol{u}, \varphi, d$ to the problem (P) for a given load ℓ. For this purpose, we first show that the optimization problem in (P) admits solutions for fixed t and d. Such solutions will turn out to satisfy the elliptic system in (3.19) below as necessary optimality system, but only after we establish that the displacement $\boldsymbol{u}$ possesses improved space regularity. As (3.19) is uniquely solvable, if the penalization parameter β is sufficiently large, we therefore obtain unique solvability for the optimization problem in (P) with solutions characterized by (3.19). After concluding uniqueness, Lipschitz-continuity of the resulting solution maps is proven. Finally, based on these results, existence and uniqueness for the evolution equation in (P) is shown in Section 3.2. Before we begin our investigations, let us emphasize that the assumptions introduced in the previous Section 2.3 are tacitly assumed in all what follows without mentioning them every time. By contrast, additional assumptions which appear later on throughout the paper are always invoked when needed in the upcoming statements.

3.1. The Elliptic System.

Throughout this section we work with a fixed $(t, d) \in [0,T] \times L^2(\Omega)$ and deal with the optimization problem

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{J}(\boldsymbol{u}, \varphi), \tag{3.1}$$

where $\mathcal{J} : V \times H^1(\Omega) \to \mathbb{R}$ is defined as

$$\mathcal{J}(\boldsymbol{u}, \varphi) := \mathcal{E}(t, \boldsymbol{u}, \varphi, d). \tag{3.2}$$

Existence of solutions.

PROPOSITION 3.1. *The optimization problem* (3.1) *admits at least one solution.*

Proof. Thanks to $g \geq \epsilon$ and the coercivity of $\mathbb{C}$ by Assumptions 2.9 and 2.10, respectively, Korn's inequality implies that $\mathcal{J}$ is radially unbounded on $V \times H^1(\Omega)$. Moreover, $\mathcal{J}$ is weakly lower semicontinuous. To see this, consider a sequence $\{(\boldsymbol{u}_n, \varphi_n)\} \subset V \times H^1(\Omega)$ with $(\boldsymbol{u}_n, \varphi_n) \rightharpoonup (\boldsymbol{u}, \varphi)$ in $V \times H^1(\Omega)$. Note that this convergence implies

$$\varepsilon(\boldsymbol{u}_n) \rightharpoonup \varepsilon(\boldsymbol{u}) \quad \text{in } L^2(\Omega; \mathbb{R}^{N\times N}_{\text{sym}}) \qquad \text{and} \qquad \varphi_n \to \varphi \quad \text{in } L^2(\Omega). \tag{3.3}$$

We define $f : \Omega \times \mathbb{R} \times \mathbb{R}^{N\times N}_{\text{sym}} \to \mathbb{R}$ as $f(x, y, \zeta) := g(y)\mathbb{C}(x)\zeta : \zeta$. In view of the convexity of norms squared, the positivity of g and Assumption 2.10, $\zeta \mapsto f(x, y, \zeta)$ is convex f.a.a. $x \in \Omega$ and all $y \in \mathbb{R}$ and $f(x, y, \zeta) \geq 0$. This allows us to apply [5, Theorem 3.23], which gives in turn

$$\liminf_{n\to\infty} \int_\Omega g(\varphi_n)\mathbb{C}\varepsilon(\boldsymbol{u}_n) : \varepsilon(\boldsymbol{u}_n)\, dx \geq \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\, dx,$$

in view of (3.3). This together with the weak lower semicontinuity of norm squares gives that $\mathcal{J}$ is indeed weakly lower semicontinuous. The existence of solutions for (3.1) follows by classical arguments of the direct method of variational calculus. □

Our next goal is to improve the regularity of the optimal displacement. For this purpose we need the following

DEFINITION 3.2. *For given $\varphi \in L^1(\Omega)$ we define the linear form $A_\varphi : V \to V^*$ as*

$$\langle A_\varphi \boldsymbol{u}, \boldsymbol{v}\rangle_V := \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, dx.$$

The operator A_φ considered with different domains and ranges will be denoted by the same symbol for the sake of convenience.

Note that the operator A_φ is well defined in view of Hölder's inequality and Lemma A.1.

LEMMA 3.3. *There exists $p > 2$ such that, for all $\overline{p} \in [2, p]$ and all $\varphi \in L^1(\Omega)$, the operator $A_\varphi : \boldsymbol{W}^{1,\overline{p}}_D(\Omega) \to \boldsymbol{W}^{-1,\overline{p}}_D(\Omega)$ is continuously invertible. Moreover, there exists a constant $c > 0$, independent of φ and $\overline{p}$, such that*

$$\|A_\varphi^{-1} h\|_{\boldsymbol{W}^{1,\overline{p}}_D(\Omega)} \leq c\, \|h\|_{\boldsymbol{W}^{-1,\overline{p}}_D(\Omega)} \quad \forall\, h \in \boldsymbol{W}^{-1,\overline{p}}_D(\Omega), \quad \forall\, \varphi \in L^1(\Omega) \tag{3.4}$$

holds for all $\overline{p} \in [2, p]$.

Proof. The result follows by applying [15, Proposition 1.2]. To this end, we have to verify [15, Assumption 1.5]. First Assumption 2.8 guarantees the conditions on the domain from [15, Assumption 1.5(1)]. Moreover, the family of functions $\{b_\varphi\}_{\varphi \in L^1(\Omega)}$, $b_\varphi : \Omega \times \mathbb{R}^{N\times N}_{\text{sym}} \to \mathbb{R}^{N\times N}_{\text{sym}}$, defined by

$$b_\varphi(x, \varepsilon) := g(\varphi(x))\mathbb{C}(x)\varepsilon. \tag{3.5}$$

is uniformly bounded and coercive by Assumptions 2.10 and 2.9, which in turn implies [15, Assumption 1.5(2)]. Thus, [15, Proposition 1.2] gives that A_φ is continuously invertible for every $\varphi \in L^1(\Omega)$ and moreover tells us that the norm of the inverse can be estimated independently of φ and $\overline{p}$. □

Lemma 3.4 (Partial Fréchet-differentiability of $\mathcal{E}$). *The functional $\mathcal{E}$ is partially Fréchet differentiable w.r.t. $\boldsymbol{u}$ and d on $[0,T] \times V \times H^1(\Omega) \times L^2(\Omega)$, and its partial derivatives are given by*

$$\partial_u \mathcal{E}(t, \boldsymbol{u}, \varphi, d)(\boldsymbol{\delta u}) = \langle A_\varphi \boldsymbol{u}, \boldsymbol{\delta u} \rangle_V - \langle \ell(t), \boldsymbol{\delta u} \rangle_V, \tag{3.6}$$

$$\partial_d \mathcal{E}(t, \boldsymbol{u}, \varphi, d) = \beta(d - \varphi). \tag{3.7}$$

Furthermore, if considered as a mapping in $[0,T] \times \boldsymbol{W}_D^{1,r}(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ with $r > 2$ for $N = 2$ and $r > 12/5$ in case of $N = 3$, then $\mathcal{E}$ is also partially Fréchet-differentiable w.r.t. φ. Its partial derivative reads

$$\begin{aligned} \partial_\varphi \mathcal{E}(t, \boldsymbol{u}, \varphi, d)(\delta\varphi) = \frac{1}{2} \int_\Omega g'(\varphi) \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \delta\varphi \, dx \\ + \int_\Omega \alpha \nabla\varphi \cdot \nabla\delta\varphi + \beta(\varphi - d)\delta\varphi \, dx. \end{aligned} \tag{3.8}$$

Proof. The results regarding the partial Fréchet differentiability w.r.t. d and $\boldsymbol{u}$ are obvious to see. For the latter one, keep in mind that g maps $H^1(\Omega)$ into $L^\infty(\Omega)$, see Lemma A.1. Concerning the partial Fréchet differentiability w.r.t. φ we first observe that, for every $\boldsymbol{u} \in \boldsymbol{W}_D^{1,r}(\Omega)$, the linear functional

$$L^{\frac{r}{r-2}}(\Omega) \ni w \mapsto \frac{1}{2} \int_\Omega w \, \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx \in \mathbb{R}$$

is bounded on account of Hölder's inequality with $(r-2)/r + 2/r = 1$ and thus an element of $L^{r/(r-2)}(\Omega)^*$. Moreover, the conditions on r and Sobolev embeddings imply $H^1(\Omega) \hookrightarrow L^s(\Omega)$ with some $s > r/(r-2)$ so that, in view of Lemma A.1, g is Fréchet-differentiable from $H^1(\Omega)$ to $L^{r/(r-2)}(\Omega)$. The result then follows from the chain rule. □

Proposition 3.5 (Improved regularity of the optimal displacement). *For every $\varphi \in H^1(\Omega)$, the optimization problem $\min_{\boldsymbol{u} \in V} \mathcal{J}(\boldsymbol{u}, \varphi)$ admits a unique solution $\bar{\boldsymbol{u}} \in \boldsymbol{W}_D^{1,p}(\Omega)$, which is characterized by*

$$\langle A_\varphi \bar{\boldsymbol{u}}, \boldsymbol{v} \rangle_{\boldsymbol{W}_D^{1,p'}(\Omega)} = \langle \ell(t), \boldsymbol{v} \rangle_{\boldsymbol{W}_D^{1,p'}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{W}_D^{1,p'}(\Omega). \tag{3.9}$$

Proof. The assertion follows from the standard direct method of calculus of variations combined with Lemma 3.3 and (3.6). □

Definition 3.6 (Solution operator of (3.9)). *We define the operator $\mathcal{U} : [0,T] \times H^1(\Omega) \to \boldsymbol{W}_D^{1,p}(\Omega)$ by*

$$\mathcal{U}(t, \varphi) := A_\varphi^{-1} \ell(t).$$

As an immediate consequence of Lemma 3.3 and the regularity of ℓ in Assumption 2.11 one obtains the following

Corollary 3.7. *There exists a constant $c > 0$, independent on t and φ such that*

$$\|\mathcal{U}(t, \varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \le c \quad \forall (t, \varphi) \in [0,T] \times H^1(\Omega).$$

Proposition 3.8 (Lipschitz continuity of $\mathcal{U}$). *Let $r \in [2p/(p-2), \infty]$ be given, where $p > 2$ is the integrability exponent from Lemma 3.3. Then there exists $L > 0$ such that for all $\varphi_1, \varphi_2 \in H^1(\Omega) \cap L^r(\Omega)$ and all $t_1, t_2 \in [0, T]$ it holds*

$$\|\mathcal{U}(t_1, \varphi_1) - \mathcal{U}(t_2, \varphi_2)\|_{\boldsymbol{W}_D^{1,\pi}(\Omega)} \le L(|t_1 - t_2| + \|\varphi_1 - \varphi_2\|_r), \tag{3.10}$$

where $1/\pi = 1/p + 1/r$.

Proof. We abbreviate $\boldsymbol{u}_i := \mathcal{U}(t_i, \varphi_i)$, $i = 1, 2$. Subtracting the equations associated with $\boldsymbol{u}_i$, $i = 1, 2$, yields

$$A_{\varphi_1}(\boldsymbol{u}_1 - \boldsymbol{u}_2) = (A_{\varphi_2} - A_{\varphi_1})\boldsymbol{u}_2 + \ell(t_1) - \ell(t_2) \quad \text{in } \boldsymbol{W}_D^{-1,p}(\Omega). \tag{3.11}$$

For given $\mu, \rho, \tau \ge 1$ such that $1/\mu = 1/\rho + 1/\tau$, Hölder's inequality and Assumption 2.10 imply

$$\|\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{w})\|_\mu \le C\|\boldsymbol{u}\|_{\boldsymbol{W}_D^{1,\rho}(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{W}_D^{1,\tau}(\Omega)} \quad \forall\, \boldsymbol{u} \in \boldsymbol{W}_D^{1,\rho}(\Omega), \boldsymbol{w} \in \boldsymbol{W}_D^{1,\tau}(\Omega), \tag{3.12}$$

We further apply Hölder's inequality with $1/\pi' + 1/r + 1/p = 1$ to the first term on the right-hand side in (3.11). This gives together with Lemma A.1, (3.12), and Corollary 3.7 the following estimate

$$\begin{aligned} \|(A_{\varphi_2} - A_{\varphi_1})\boldsymbol{u}_2\|_{\boldsymbol{W}_D^{-1,\pi}(\Omega)} &\le C\,\|g(\varphi_1) - g(\varphi_2)\|_r\,\|\boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \\ &\le C\,\|\varphi_1 - \varphi_2\|_r. \end{aligned} \tag{3.13}$$

Now, since $1/r \le (p-2)/(2p)$, it holds $\pi \in [2, p]$. Thus, we are allowed to apply estimate (3.4) to A_{φ_1}, when considered as an operator from $\boldsymbol{W}_D^{1,\pi}(\Omega)$ to $\boldsymbol{W}_D^{-1,\pi}(\Omega)$. Therewith we deduce from (3.11) and (3.13)

$$\begin{aligned} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,\pi}(\Omega)} &\le C\,\|\varphi_1 - \varphi_2\|_r + \|\ell(t_1) - \ell(t_2)\|_{\boldsymbol{W}_D^{-1,\pi}(\Omega)} \\ &\le L\,(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|), \end{aligned}$$

where we used $\ell \in C^{0,1}([0,T]; \boldsymbol{W}_D^{-1,\pi}(\Omega))$ for the last inequality. Note that the constant $L > 0$ is independent of (t_i, φ_i). ◻

We finish the discussion concerning the optimal displacement with a result which will be very useful later on in Section 5 below.

Lemma 3.9. *Let $\{t_n, \varphi_n\} \subset [0,T] \times H^1(\Omega)$ and $(t, \varphi) \in [0,T] \times H^1(\Omega)$ be given such that $(t_n, \varphi_n) \to (t, \varphi)$ in $\mathbb{R} \times L^1(\Omega)$. Then it holds $\mathcal{U}(t_n, \varphi_n) \to \mathcal{U}(t, \varphi)$ in $\boldsymbol{W}_D^{1,s}(\Omega)$ as $n \to \infty$ for every $s \in [2, p)$.*

Proof. We again abbreviate $\boldsymbol{u}_n := \mathcal{U}(t_n, \varphi_n)$ and $\boldsymbol{u} := \mathcal{U}(t, \varphi)$. By subtracting the equations associated with $\boldsymbol{u}_n$ and $\boldsymbol{u}$ we obtain for all $n \in \mathbb{N}$

$$A_\varphi(\boldsymbol{u} - \boldsymbol{u}_n) = (A_{\varphi_n} - A_\varphi)\boldsymbol{u}_n + \ell(t) - \ell(t_n) \quad \text{in } \boldsymbol{W}_D^{-1,p}(\Omega). \tag{3.14}$$

Completely analogously to (3.13), one derives the estimate

$$\|(A_{\varphi_n} - A_\varphi)\boldsymbol{u}_n\|_{\boldsymbol{W}_D^{-1,s}(\Omega)} \le C\,\|g(\varphi_n) - g(\varphi)\|_\varrho \|\boldsymbol{u}_n\|_{\boldsymbol{W}_D^{1,p}(\Omega)}, \tag{3.15}$$

with $\varrho \in [1, \infty)$ such that $1/\varrho + 1/p + 1/s' = 1$. Notice that the existence of ϱ is due to $1/s' \in [1/2, 1/p')$. Lemma A.1, Corollary 3.7, Assumption 2.11 and (3.15) now lead to

$$\|(A_{\varphi_n} - A_\varphi)\boldsymbol{u}_n + \ell(t) - \ell(t_n)\|_{\boldsymbol{W}_D^{-1,s}(\Omega)} \to 0 \text{ as } n \to \infty.$$

In view of (3.14) applying (3.4) to $A_\varphi : \boldsymbol{W}_D^{1,s}(\Omega) \to \boldsymbol{W}_D^{-1,s}(\Omega)$ then gives the assertion. ◻

Uniqueness. Next we concentrate on deriving necessary optimality conditions for the optimal nonlocal damage. For this purpose one has to differentiate the function $\mathcal{J}$ w.r.t. φ. This means that one has to apply Lemma 3.4, which can be done only under the following additional

Assumption 3.10. *From now on we assume that, in case of $N = 3$, the assertion of Lemma 3.3 holds for all $\varphi \in H^1(\Omega)$ with $p > 12/5$, i.e., for every $\varphi \in H^1(\Omega)$, the operator $A_\varphi : \boldsymbol{W}_D^{1,p}(\Omega) \to \boldsymbol{W}_D^{-1,p}(\Omega)$ is continuously invertible for some $p > 12/5$ and an estimate analogous to (3.4) holds.*

Remark 3.11. *The existence of a p fulfilling Assumption 3.10 is guaranteed by the results of [15], provided that the domain is smooth enough and the difference between the boundedness and monotonicity constants of the stress strain relation is sufficiently small. In our case, the stress strain relation is given by (3.5), and thus, the assertion is ensured, if the values $\epsilon\,\gamma_{\mathbb{C}}$ and $\|\mathbb{C}\|_\infty$ are close enough to each other, see the proof of Lemma 3.3 and [15, Assumption 1.5.(2)] for more details. Recall that, in the two-dimensional case, Assumption 3.10 is automatically fulfilled.*

Remark 3.12. *Alternatively to Assumption 3.10, one can proceed as in [17] and use the Sobolev–Slobodeckij space $H^{3/2}(\Omega)$ for the nonlocal damage in three dimensions. To this end, one replaces the gradient term in the energy functional by a seminorm on $H^{3/2}(\Omega)$, cf. [17, (2.4b)]. The advantage thereof is that $H^{3/2}(\Omega) \hookrightarrow L^r(\Omega)$ for every $r \in [1, \infty)$ for both, the two- and three-dimensional case. A close inspection of the upcoming analysis shows that the embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for all $r < \infty$ in case of $N = 2$ is the key ingredient to prove the uniqueness result for (3.1) without any additional assumptions on the integrability exponent p in the two-dimensional case. Thus, working with $H^{3/2}(\Omega)$ instead of $H^1(\Omega)$ in three dimensions allows us to do the same in case of $N = 3$ so that there would be no need for making extra assumptions on p. However, we chose not to work with $H^{3/2}(\Omega)$, as the bilinear form associated with the $H^{3/2}(\Omega)$-seminorm is difficult to realize in numerical computations.*

The following definition will be useful in the sequel:

Definition 3.13 (The linear and nonlinear part of (3.19b)). *Suppose that Assumption 3.10 is fulfilled. Then we define the mappings $B : H^1(\Omega) \to H^1(\Omega)^*$ and $F : [0, T] \times H^1(\Omega) \to H^1(\Omega)^*$ by*

$$\langle B\varphi, \psi\rangle_{H^1(\Omega)} := \int_\Omega \alpha\nabla\varphi \cdot \nabla\psi + \beta\varphi\psi \, dx, \quad \phi, \psi \in H^1(\Omega), \tag{3.16}$$

$$\langle F(t, \varphi), \psi\rangle_{H^1(\Omega)} := \frac{1}{2}\int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t, \varphi)) : \varepsilon(\mathcal{U}(t, \varphi))\psi \, dx, \tag{3.17}$$
$$t \in [0, T],\ \varphi, \psi \in H^1(\Omega).$$

We emphasize that F is well defined. To see this first note that $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}}))$ and $g' \in L^\infty(\mathbb{R})$ by Assumptions 2.10 and 2.9. Moreover, Sobolev embeddings imply $H^1(\Omega) \hookrightarrow L^s(\Omega)$ with $s = 6$ in case of $N = 3$ and $s < \infty$ for $N = 2$. Therefore, the assertion directly follows from Lemma 3.3 in case of $N = 2$, whereas one needs Assumption 3.10 for $N = 3$.

With a little abuse of notation, the operators B and F considered with different domains and ranges will be denoted by the same symbol.

Proposition 3.14. *Under Assumption 3.10, every global minimizer $(\bar{\boldsymbol{u}}, \bar{\varphi})$ of (3.1) fulfills $\bar{\boldsymbol{u}} = \mathcal{U}(t, \bar{\varphi}) \in \boldsymbol{W}_D^{1,p}(\Omega)$ and*

$$B\bar{\varphi} + F(t, \bar{\varphi}) = \beta d \quad \text{in } H^1(\Omega)^*, \tag{3.18}$$

which is equivalent to the following optimality system:

$$-\operatorname{div} g(\bar\varphi)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}) = \ell(t) \quad \text{in } \boldsymbol{W}_D^{-1,p}(\Omega) \tag{3.19a}$$

$$-\alpha\Delta\bar\varphi + \beta\,\bar\varphi + \frac{1}{2}\,g'(\bar\varphi)\mathbb{C}\,\varepsilon(\bar{\boldsymbol{u}}) : \varepsilon(\bar{\boldsymbol{u}}) = \beta d \quad \text{in } H^1(\Omega)^*, \tag{3.19b}$$

where $\operatorname{div} : L^p(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}) \to \boldsymbol{W}_D^{-1,p}(\Omega)$ *denotes the distributional vector-valued divergence, i.e.,*

$$\langle \operatorname{div}\boldsymbol{\sigma}, \boldsymbol{v}\rangle := -\int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v})\,dx, \quad \boldsymbol{\sigma} \in L^p(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}),\ \boldsymbol{v} \in \boldsymbol{W}_D^{1,p'}(\Omega), \tag{3.20}$$

and $\Delta : H^1(\Omega) \to H^1(\Omega)^*$ *is the distributional Laplace operator, respectively.*

Proof. The global optimality of $(\bar{\boldsymbol{u}}, \bar\varphi)$ in particular implies that $\bar{\boldsymbol{u}}$ is a global minimizer of

$$\min_{\boldsymbol{u}\in V} \mathcal{J}(\boldsymbol{u}, \bar\varphi).$$

Therefore, $\bar{\boldsymbol{u}} = \mathcal{U}(t, \bar\varphi)$, in view of Proposition 3.5. Similarly, the global optimality of $(\bar{\boldsymbol{u}}, \bar\varphi)$ also implies that $\bar\varphi$ is a global minimizer of

$$\min_{\varphi\in H^1(\Omega)} \bar f(\varphi) := \mathcal{J}(\bar{\boldsymbol{u}}, \varphi). \tag{3.21}$$

Thanks to the improved regularity of $\bar{\boldsymbol{u}}$ by Proposition 3.5 in case of $N = 2$ and Assumption 3.10 for $N = 3$, respectively, one can differentiate $\bar f$ on $H^1(\Omega)$ by means of Lemma 3.4. This gives in turn $\bar f'(\bar\varphi) = \partial_\varphi \mathcal{J}(\bar{\boldsymbol{u}}, \bar\varphi) = 0$ as necessary optimality condition for a global minimizer of (3.21). In view of (3.8), Definition 3.13, and $\bar{\boldsymbol{u}} = \mathcal{U}(t, \bar\varphi)$, this is equivalent to (3.18). The equivalence to (3.19) directly follows from the definitions of $A_{\bar\varphi}$, B, and F. ☐

Remark 3.15. *Note that the improved regularity of the optimal displacement resulting from Proposition 3.5 combined with Assumption 3.10, i.e.,* $\bar{\boldsymbol{u}} \in \boldsymbol{W}_D^{1,p}(\Omega)$ *with* $p > 2$ *for* $N = 2$ *and* $p > 12/5$ *for* $N = 3$*, is essential for deriving necessary optimality conditions for the nonlocal damage. This is due to the fact that* $\mathcal{J}$ *is differentiable with respect to* φ *only on* $[0, T] \times \boldsymbol{W}_D^{1,p}(\Omega) \times H^1(\Omega) \times L^2(\Omega)$*, cf. Lemma 3.4.*

From Propositions 3.1 and 3.14 we know that (3.19b) has at least one solution. In the following we aim for showing that this solution is unique, which will give in turn the unique solvability of (3.1). Unfortunately, Assumption 3.10 does not suffice to prove the uniqueness of solutions to (3.19). In order to show strong monotonicity of the operator on the left-hand side of (3.19b), we additionally need that $H^1(\Omega) \hookrightarrow L^r(\Omega)$ with $r > 2p/(p-2)$, see proof of Lemma 3.18 below for more details. This motivates the first part of the following

Assumption 3.16. *From now on we require the following:*

1. *For every* $\varphi \in H^1(\Omega)$*, the assertion of Lemma 3.3, including the a priori estimate* (3.4)*, holds for some* $p > N$.
2. *The penalization parameter* β *is sufficiently large, depending only on the given data, see* (3.33) *below.*

Note that Assumption 3.16.1 is automatically fulfilled if $N = 2$, see Lemma 3.3. In case of $N = 3$, this assumption is guaranteed by imposing additional conditions on the data, see Remark 3.11 for more details. We emphasize that, as in case of Assumption 3.10 before, Assumption 3.16.1 is not needed, if one replaces the H^1-seminorm in the energy functional in Definition 2.4 by a $H^{3/2}$-seminorm, see Remark 3.12 for more details. Assumption 3.16.2 is not restrictive at all, since β is a penalization parameter, which is supposed to be large anyway and will be sent to ∞ in the companion paper [21]. We point out that the dependence of β on the given data does not affect the rest of the analysis.

We start the discussion of uniqueness with a Lipschitz-continuity result concerning the mapping F. For later purpose, we prove a slightly more general result.

Lemma 3.17. *Let $r \geq 2p/(p-2)$ and $1/s + 2/p + 1/r = 1$. Under Assumption 3.16.1 the following estimate holds for all $t_1, t_2 \in [0,T]$, $\varphi_1, \varphi_2 \in H^1(\Omega) \cap L^r(\Omega)$ and $\psi \in L^s(\Omega)$:*

$$|\langle F(t_1,\varphi_1) - F(t_2,\varphi_2), \psi\rangle| \leq C\big(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|\big)\|\psi\|_s,$$

with a constant $C > 0$ independent of $(t_i, \varphi_i)_{i=1,2}$ and ψ.

Proof. We again denote $\boldsymbol{u}_i := \mathcal{U}(t_i, \varphi_i)$ for $i = 1, 2$. The definition of F in (3.17) implies

$$\begin{aligned} &|\langle F(t_1,\varphi_1) - F(t_2,\varphi_2), \psi\rangle| \\ &\quad\leq \int_\Omega \big|(g'(\varphi_1) - g'(\varphi_2))\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1)\psi\big|\, dx \\ &\qquad + \int_\Omega \big|g'(\varphi_2)[\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1) - \mathbb{C}\varepsilon(\boldsymbol{u}_2) : \varepsilon(\boldsymbol{u}_2)]\psi\big|\, dx. \end{aligned} \tag{3.22}$$

We discuss the two terms on the right-hand side of (3.22) separately:

(i) In view of (3.12) and Corollary 3.7 we have

$$\|\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1)\|_{\frac{p}{2}} \leq c, \tag{3.23}$$

where $c > 0$ is a constant independent on (t_1, φ_1). In addition, the function $g' : L^r(\Omega) \to L^r(\Omega)$ is Lipschitz continuous according to Lemma A.1. Thus applying Hölder's inequality with $1/r + 1/s + 2/p = 1$ for the first term on the right-hand side in (3.22) gives

$$\int_\Omega \big|(g'(\varphi_1) - g'(\varphi_2))\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1)\psi\big|\, dx \leq C_1\|\varphi_1 - \varphi_2\|_r\, \|\psi\|_s. \tag{3.24}$$

(ii) Define π and ω through $1/\pi = 1/p + 1/r$ and $1/\omega = 1/p + 1/\pi$. Then (3.12), Corollary 3.7, and Proposition 3.8 result in

$$\begin{aligned} \|\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1) - \mathbb{C}\varepsilon(\boldsymbol{u}_2) : \varepsilon(\boldsymbol{u}_2)\|_\omega &= \|\mathbb{C}[\varepsilon(\boldsymbol{u}_1) + \varepsilon(\boldsymbol{u}_2)] : [\varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2)]\|_\omega \\ &\leq C\|\boldsymbol{u}_1 + \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}_D(\Omega)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,\pi}_D(\Omega)} \\ &\leq C(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|). \end{aligned} \tag{3.25}$$

Then Hölder's inequality with $1/\omega + 1/s = 1$, together with Assumption 2.9, yields

$$\begin{aligned} &\int_\Omega \big|g'(\varphi_2)[\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1) - \mathbb{C}\varepsilon(\boldsymbol{u}_2) : \varepsilon(\boldsymbol{u}_2)]\psi\big|\, dx \\ &\qquad\qquad \leq C(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|)\|\psi\|_s. \end{aligned} \tag{3.26}$$

Inserting (3.24) and (3.26) in (3.22) finally gives the assertion. □

From $p > N$ it follows that

$$r := \frac{2p}{p-2} \in \Big(2, \frac{2N}{N-2}\Big), \tag{3.27}$$

and therefore Sobolev embeddings give $H^1(\Omega) \hookrightarrow L^r(\Omega)$. Moreover, by construction, this r satisfies $2/r + 2/p = 1$. Thus Lemma 3.17 is applicable with $r = s$ yielding the estimate

$$|\langle F(t_1,\varphi_1) - F(t_2,\varphi_2), \psi\rangle| \le C\big(\|\varphi_1 - \varphi_2\|_{\frac{2p}{p-2}} + |t_1 - t_2|\big)\|\psi\|_{\frac{2p}{p-2}} \quad \forall\, \varphi_1, \varphi_2, \psi \in H^1(\Omega). \tag{3.28}$$

Lemma 3.18. *Under Assumption 3.16.1 it holds*

$$\|\varphi\|^2_{\frac{2p}{p-2}} \le C_k\, \|\varphi\|_2^2 + k\, \|\varphi\|^2_{H^1(\Omega)} \quad \forall\, \varphi \in H^1(\Omega) \text{ and } \forall\, k > 0,$$

where $C_k > 0$ is a constant, which depends only on k and the given data.

Proof. For convenience we again set $r := 2p/(p-2)$. First note that, because of Assumption 3.16, there is an index ϱ such that $r \in (2, \varrho)$ and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$. For instance take $\varrho = (2p+1)/(p-2)$ for $N = 2$ and $\varrho = 6$ in case of $N = 3$, cf. (3.27). Therefore there exists $\theta \in (0,1)$ such that $1/r = \theta/2 + (1-\theta)/\varrho$ so that Lyapunov's inequality leads to

$$\|\varphi\|_r^2 \le \|\varphi\|_2^{2\theta}\|\varphi\|_\varrho^{2-2\theta} \le C\|\varphi\|_2^{2\theta}\|\varphi\|^{2-2\theta}_{H^1(\Omega)}. \tag{3.29}$$

Thanks to the generalized Young inequality, (3.29) can be continued as

$$\|\varphi\|_r^2 \le C_k\, \|\varphi\|_2^2 + k\, \|\varphi\|^2_{H^1(\Omega)} \quad \forall\, k > 0, \tag{3.30}$$

where $C_k > 0$ is a constant depending only on k and θ (and thus, on k and p). Since p depends only on the given data, cf. [15], the proof is now complete. □

Lemma 3.19 (Strong monotonicity of $B + F$). *Under Assumption 3.16 the following estimate holds for all $t_1, t_2 \in [0,T]$ and all $\varphi_1, \varphi_2 \in H^1(\Omega)$, $\varphi_1 \ne \varphi_2$,*

$$\frac{\langle B(\varphi_1 - \varphi_2) + F(t_1,\varphi_1) - F(t_2,\varphi_2), \varphi_1 - \varphi_2\rangle_{H^1(\Omega)}}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} \ge C_1\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} - C_2|t_1 - t_2|,$$

where $C_1, C_2 > 0$ are constants independent of $(t_i, \varphi_i)_{i=1,2}$.

Proof. Let $(t_i, \varphi_i)_{i=1,2} \in [0,T] \times H^1(\Omega)$ be arbitrary, but fixed with $\varphi_1 \ne \varphi_2$. Then (3.28) and Lemma 3.18 yield that, for all $k > 0$,

$$\begin{aligned} &|\langle F(t_1,\varphi_1) - F(t_2,\varphi_2), \varphi_1 - \varphi_2\rangle_{H^1(\Omega)}| \\ &\quad \le C\big(C_k\|\varphi_1 - \varphi_2\|_2^2 + k\|\varphi_1 - \varphi_2\|^2_{H^1(\Omega)} + |t_1 - t_2|\, \|\varphi_1 - \varphi_2\|_{H^1(\Omega)}\big). \end{aligned} \tag{3.31}$$

Using the definition of B in (3.16), we infer from (3.31)

$$\begin{aligned} &\frac{\langle B(\varphi_1 - \varphi_2) + F(t_1,\varphi_1) - F(t_2,\varphi_2), \varphi_1 - \varphi_2\rangle}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} \\ &\ge (\alpha - C\, k)\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} - C|t_1 - t_2| + (\beta - \alpha - CC_k)\frac{\|\varphi_1 - \varphi_2\|_2^2}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}}. \end{aligned} \tag{3.32}$$

We now choose $k > 0$ small enough such that $C_1 := \alpha - C\,k > 0$. Furthermore, if

$$\beta > \alpha + C\,C_k, \tag{3.33}$$

cf. Assumption 3.16.2, then (3.32) gives the assertion with $C_1 = \alpha - C\,k$ and $C_2 = C$. Note that value of k, and thus the constant C_1 and the threshold for β, only depends on the given data, see the proof of Lemma 3.18. ◻

THEOREM 3.20 (Unique solvability of (3.1)). *Under Assumption 3.16, the equation (3.18) admits a unique solution, and thus, (3.19) is uniquely solvable as well. Moreover, the optimization problem (3.1) admits a unique minimizer $(\bar{\boldsymbol{u}}, \bar{\varphi}) \in \boldsymbol{W}^{1,p}_D(\Omega) \times H^1(\Omega)$, which is characterized by (3.18) and (3.19), respectively.*

Proof. Let $(t_i, d_i) \in [0,T] \times L^2(\Omega)$, $i = 1,2$, be given and let φ_i denote solutions of (3.18) associated with (t_i, d_i), $i = 1,2$. Note that the existence thereof is ensured by Propositions 3.1 and 3.14. By assuming $\varphi_1 \neq \varphi_2$, we obtain from Lemma 3.19 and Cauchy Schwarz inequality the estimate

$$\begin{aligned}
&\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} \\
&\quad \leq C\Big(\frac{\langle B(\varphi_1 - \varphi_2) + F(t_1,\varphi_1) - F(t_2,\varphi_2), \varphi_1 - \varphi_2\rangle_{H^1(\Omega)}}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} + |t_1 - t_2|\Big) \\
&\quad = C\Big(\beta\frac{(d_1 - d_2, \varphi_1 - \varphi_2)_2}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} + |t_1 - t_2|\Big) \leq C\big(\|d_1 - d_2\|_2 + |t_1 - t_2|\big).
\end{aligned} \tag{3.34}$$

Note that the estimate (3.34) holds trivially also for $\varphi_1 = \varphi_2$. If we set $t_1 = t_2$ and $d_1 = d_2$, then (3.34) implies uniqueness for (3.18), and thus, for (3.19). Moreover, we recall that (3.1) admits solutions, see Proposition 3.1. Since (3.19) constitutes the necessary optimality condition for (3.1), by Proposition 3.14, we deduce that (3.1) is uniquely solvable, too. The regularity and the characterization of the unique solution via (3.18) and (3.19), respectively, is due to Proposition 3.14. The proof is now complete. ◻

The unique solvability of (3.18) leads to the following

DEFINITION 3.21 (Solution operator of (3.18)). *Let Assumption 3.16 be fulfilled. We define the operator $\Phi : [0,T] \times L^2(\Omega) \to H^1(\Omega)$ as*

$$\Phi(t,d) := \big(B + F(t,\cdot)\big)^{-1}(\beta d).$$

As a result of (3.34) we have that, under Assumption 3.16, there exists a constant $K > 0$ such that

$$\|\Phi(t_1,d_1) - \Phi(t_2,d_2)\|_{H^1(\Omega)} \leq K(\|d_1 - d_2\|_2 + |t_1 - t_2|), \tag{3.35}$$

holds true for all $t_1, t_2 \in [0,T]$ and $d_1, d_2 \in L^2(\Omega)$, i.e., the operator Φ is globally Lipschitz continuous.

3.2. Evolutionary Problem as Operator Differential Equation. This section is devoted to proving existence and uniqueness for our complete damage model

(P). Throughout the section Assumption 3.16 is supposed to hold. Then, in view of the results of Section 3.1, problem (P) can be reformulated as

$$-\partial_d \mathcal{E}(t, \boldsymbol{u}(t), \varphi(t), d)\Big|_{d=d(t)} \in \partial \mathcal{R}_\delta(\dot{d}(t)) \quad \text{f.a.a. } t \in (0,T), \quad d(0) = d_0, \tag{3.36}$$

where $\boldsymbol{u}(t) = \mathcal{U}(t, \varphi(t))$ and $\varphi(t) = \Phi(t, d(t))$. Due to (3.7), the evolutionary equation (3.36) reads

$$-\beta(d(t) - \varphi(t)) \in \partial \mathcal{R}_\delta(\dot{d}(t)) \quad \text{f.a.a. } t \in (0,T), \quad d(0) = d_0. \tag{3.37}$$

We approach (3.37) by showing that it is equivalent to the following operator differential equation, which can be solved by standard arguments.

Lemma 3.22 (Operator differential equation). *The evolutionary equation (3.37) is equivalent to*

$$\dot{d}(t) = \frac{1}{\delta} \max\{-\beta(d(t) - \varphi(t)) - r, 0\} \quad \text{f.a.a. } t \in (0,T), \quad d(0) = d_0. \tag{3.38}$$

Proof. We begin by observing that $\mathcal{R}_\delta(v) = \int_\Omega R_\delta(v(x))\, dx$ for all $v \in L^2(\Omega)$, where the mapping $R_\delta : \mathbb{R} \to [0, \infty]$ is defined as

$$R_\delta(\eta) := \begin{cases} r\eta + \frac{\delta}{2}\eta^2, & \text{if } \eta \geq 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Thus, in view of [28, Corollary 3E], (3.37) is equivalent to $\xi(t,x) \in \partial R_\delta(\dot{d}(t,x))$ f.a.a. $(t,x) \in (0,T) \times \Omega$, where we abbreviate $\xi := -\beta(d - \varphi)$. Due the convexity and lower semicontinuity of R_δ, we can apply a well known convex analysis result, which gives in turn

$$\xi(t,x) \in \partial R_\delta(\dot{d}(t,x)) \Longleftrightarrow \dot{d}(t,x) \in \partial R^*_\delta(\xi(t,x)) \quad \text{f.a.a. } (t,x) \in (0,T) \times \Omega.$$

On the other hand, straightforward computation yields that the conjugate functional $R^*_\delta : \mathbb{R} \to \mathbb{R}$ satisfies $R^*_\delta(\zeta) = \frac{1}{2\delta} \max\{\zeta - r, 0\}^2$ for all $\zeta \in \mathbb{R}$. Moreover, it is differentiable with derivative $(R^*_\delta)'(\zeta) = \frac{1}{\delta} \max\{\zeta - r, 0\}$ for all $\zeta \in \mathbb{R}$. This completes the proof.
□

Since the max-function is a well known complementarity function, one can deduce from Lemma 3.22 that (3.36) is equivalent to the complementarity system

$$0 \leq \delta \dot{d}(t) \perp -\beta(d(t) - \varphi(t)) - r - \delta \dot{d}(t) \leq 0 \quad \text{a.e. in } \Omega, \text{ f.a.a. } t \in (0,T). \tag{3.39}$$

In a completely analogous way, one can show that the evolution equation (2.2) is equivalent to the complementarity system in [6, (18)], as already mentioned at the end of Section 2.1. For this purpose we refer to [6, (13), (19) and (20)].

Theorem 3.23 (Existence and uniqueness for the evolutionary equation). *Under Assumption 3.16 there exists a unique function $d \in C^1([0,T]; L^2(\Omega))$ satisfying (3.36).*

Proof. Lemma 3.22 tells us that (3.36) is equivalent to the operator differential equation given by (3.38). We intend to solve the latter one by means of the Picard-Lindelöf theorem. For this purpose, we define the function $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ as

$$f(t,d) := \frac{1}{\delta} \max\{-\beta(d - \Phi(t,d)) - r, 0\}. \tag{3.40}$$

Due to the Lipschitz continuity of $\max : L^2(\Omega) \to L^2(\Omega)$ with constant 1 and (3.35), it holds for all $(t_1, d_1), (t_2, d_2) \in [0,T] \times L^2(\Omega)$ that

$$\begin{aligned}\|f(t_1,d_1) - f(t_2,d_2)\|_2 &\le \frac{\beta}{\delta}\big(\|\Phi(t_1,d_1) - \Phi(t_2,d_2)\|_{H^1(\Omega)} + \|d_1 - d_2\|_2\big)\\ &\le \frac{\beta}{\delta}\,(K+1)\,\|d_1 - d_2\|_2 + \frac{\beta}{\delta}\,K\,|t_1 - t_2|,\end{aligned}$$

where K is the Lipschitz constant of Φ. Therefore, f is globally Lipschitz continuous, and we can conclude with [8, Theorem 7.2.6] that there exists a unique $d \in C^1([0,T]; L^2(\Omega))$ satisfying

$$\dot{d}(t) = f(t, d(t)) \quad \forall\, t \in [0,T], \quad d(0) = d_0,$$

which in view of (3.40) gives the assertion. □

Note that the continuity of $\dot{d}$ w.r.t. time implies Lipschitz continuity of d w.r.t. time. The latter one readily transfers to φ and $\boldsymbol{u}$, as explained in the sequel. First of all, (3.35) and the Lipschitz continuity of d imply the Lipschitz continuity of φ. Due to $H^1(\Omega) \hookrightarrow L^r(\Omega)$ with $r < \infty$ and $r = 6$ for $N = 2$, respectively $N = 3$, the Lipschitz continuity of $\boldsymbol{u}$ then follows from Proposition 3.8 with $\pi \in (2,p)$ for $N = 2$ and $\pi = 6p/(p+6) > 2$ for $N = 3$ so that $\boldsymbol{u} \in C^{0,1}([0,T]; \boldsymbol{W}_D^{1,\pi}(\Omega))$. The time-regularity of φ and $\boldsymbol{u}$ can be further improved, as we will see in Section 5.

We summarize our results so far in the next theorem.

THEOREM 3.24. *Under Assumption 3.16, there exists a unique solution $(\boldsymbol{u}, \varphi, d)$ of our viscous two-field gradient damage model in (P) satisfying $d \in C^1([0,T]; L^2(\Omega))$, $\varphi \in C^{0,1}([0,T]; H^1(\Omega))$, $\boldsymbol{u} \in C^{0,1}([0,T]; \boldsymbol{W}_D^{1,\pi}(\Omega))$, $\boldsymbol{u}(t) \in \boldsymbol{W}_D^{1,p}(\Omega)$ f.a.a. $t \in [0,T]$, and the following system of differential equations:*

$$-\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) = \ell(t) \qquad \text{in } \boldsymbol{W}_D^{-1,p}(\Omega) \tag{3.41a}$$

$$-\alpha\Delta\varphi(t) + \beta\,\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) = \beta d(t) \quad \text{in } H^1(\Omega)^* \tag{3.41b}$$

$$\dot{d}(t) - \frac{1}{\delta}\max\{-\beta(d(t) - \varphi(t)) - r, 0\} = 0, \qquad d(0) = d_0. \tag{3.41c}$$

4. Improved Spatial Regularity and Lipschitz Continuity of the Nonlocal Damage.

In this section, we show that the nonlocal damage variable possesses higher regularity and satisfies a corresponding Lipschitz condition. We start with the following result on the spatial regularity of φ.

4.1. Improved Spatial Regularity.

Throughout this section we work with an arbitrary, but fixed $(t,d) \in [0,T] \times L^2(\Omega)$ and use for simplicity the notations $\varphi := \Phi(t,d)$ and

$$f := \beta(d - \varphi) + \alpha\,\varphi - F(t,\varphi) \in H^1(\Omega)^*. \tag{4.1}$$

Definition 4.1. *We define the operator $-\Delta + I : H^1(\Omega) \to H^1(\Omega)^*$ by*

$$\langle(-\Delta + I)v, w\rangle_{H^1(\Omega)} := \int_\Omega (\nabla v \cdot \nabla w + v\, w)dx, \quad v, w \in H^1(\Omega).$$

The operator $-\Delta + I$ considered with different domains and ranges will be denoted by the same symbol for the sake of simplicity.

We employ a classical boot-strapping argument to verify the improved regularity. For this purpose consider the equation

$$(-\Delta + I)v = \frac{1}{\alpha} f \quad \text{in } H^1(\Omega)^*. \tag{4.2}$$

By construction of f and Theorem 3.20, φ is the unique solution of this equation. Then, taking advantage of the fact that the linear form f possesses higher regularity than $H^1(\Omega)^*$, we show by means of [13, Theorem 3] that $\varphi \in W^{1,q}(\Omega)$ with some $q > 2$.

Lemma 4.2. *Under Assumption 3.16 it holds $f \in W^{1,\varrho'}(\Omega)^*$, where*

$$\frac{1}{\varrho} := \max\left\{\frac{2}{p} - \frac{1}{N}, \frac{1}{2} - \frac{1}{N}\right\} < \frac{1}{N}. \tag{4.3}$$

Proof. By means of Sobolev embeddings we have $W^{1,\varrho'}(\Omega) \hookrightarrow L^{\frac{N\varrho'}{N-\varrho'}}(\Omega)$. In view of (4.1), (3.17), and (3.23), one obtains

$$\begin{aligned}|\langle f, \psi\rangle| &\le \big(\|\beta(d-\varphi) + \alpha\,\varphi\|_2 + \|g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))\|_{\frac{p}{2}}\big)\|\psi\|_{\frac{N\varrho'}{N-\varrho'}} \\ &\le C\|\psi\|_{W^{1,\varrho'}} \quad \forall\, \psi \in W^{1,\varrho'}(\Omega),\end{aligned}$$

which implies $f \in W^{1,\varrho'}(\Omega)^*$, provided that Hölder's inequality is applicable. The latter is ensured, if

$$\frac{2}{p} + \frac{N-\varrho'}{N\varrho'} \le 1 \iff \frac{2}{p} - \frac{1}{N} \le \frac{1}{\varrho} \quad \text{and} \quad \frac{1}{2} + \frac{N-\varrho'}{N\varrho'} \le 1 \iff \frac{1}{2} - \frac{1}{N} \le \frac{1}{\varrho},$$

which is guaranteed by (4.3). From Assumption 3.16.1 and $N < 4$ we finally deduce $\varrho > N$. □

Theorem 4.3 (Improved regularity of $\Phi(t,d)$). *Suppose that Assumption 3.16 holds true. Then, there exists a $q > 2$ such that $\Phi(t,d) \in W^{1,q}(\Omega)$ for every $(t,d) \in [0,T] \times L^2(\Omega)$.*

Proof. Since φ solves (4.2), the assertion is a direct consequence of [13, Theorem 3] (in combination with [14, Theorem 5.2, 5.4]) and Lemma 4.2. □

4.2. Improved Lipschitz Continuity. As a consequence of the higher spatial regularity of the solution of (3.18) one expects that Φ satisfies a corresponding Lipschitz condition. For this reason, we now focus in the following on proving $W^{1,q}(\Omega)$-Lipschitz continuity for the solution map of (3.18). For the rest of this section, we suppose that Assumption 3.16 holds and we let $(t_i, d_i) \in [0,T] \times L^2(\Omega)$ be arbitrary, but fixed and $\varphi_i := \Phi(t_i, d_i) \in W^{1,q}(\Omega)$, where $i = 1, 2$. Similarly to (4.1), we introduce the following abbreviation

$$\iota := \frac{1}{\alpha}\big(\beta(d_1 - d_2) - (\beta - \alpha)(\varphi_1 - \varphi_2) - (F(t_1,\varphi_1) - F(t_2,\varphi_2))\big). \tag{4.4}$$

Note that $\iota \in W^{1,\varrho'}(\Omega)^*$ on account of Lemma 4.2. By construction, the difference $\varphi_1 - \varphi_2$ solves

$$(-\Delta + I)(\varphi_1 - \varphi_2) = \iota \quad \text{in } H^1(\Omega)^*$$

and analogously to the preceding section, it follows

$$\|\varphi_1 - \varphi_2\|_{W^{1,\omega}(\Omega)} \leq \|(-\Delta + I)^{-1}\|_{\mathcal{L}(W^{1,\omega'}(\Omega)^*, W^{1,\omega}(\Omega))} \|\iota\|_{W^{1,\omega'}(\Omega)^*} \qquad \forall\, 2 \leq \omega \leq q = \min\{q_{\bar{\Omega}}, \varrho\}, \tag{4.5}$$

where $q_{\bar{\Omega}}$ is the number given by [13, Theorem 3], see the proof of Theorem 4.3, and ϱ is given by (4.3).

However, the desired Lipschitz continuity condition cannot be directly proven by setting $\omega = q$ in (4.5), as one cannot directly derive an estimate of the form $\|\iota\|_{W^{1,q'}(\Omega)^*} \leq C(\|d_1 - d_2\|_2 + |t_1 - t_2|)$. Instead we will apply a finite number of boot-strapping steps to prove the result. Let us shortly outline the rather technical proof. The main idea in each of these steps is as follows: Given the Lipschitz continuity of Φ in $W^{1,\mu}(\Omega)$ with some $\mu \in [2, q]$, we search for ν as large as possible such that

$$\|\iota\|_{W^{1,\nu'}(\Omega)^*} \leq L(\|\varphi_1 - \varphi_2\|_{W^{1,\mu}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|),$$

where $\nu > \mu$. Then we employ (4.5) with $\omega = \nu$ and use the Lipschitz continuity in $W^{1,\mu}(\Omega)$ to verify the result for ν. This procedure is repeated until q is reached. The precise relation between ν and μ is characterized by the following

Lemma 4.4. *Let $\mu \in [2, \varrho]$ be given, where ϱ is defined as in (4.3). Then there exists a constant $C > 0$ such that*

$$\|\iota\|_{W^{1,\nu'}(\Omega)^*} \leq C(\|\varphi_1 - \varphi_2\|_{W^{1,\mu}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|) \quad \forall\, \varphi_1, \varphi_2 \in W^{1,\mu}(\Omega), \tag{4.6}$$

where $\nu > 0$ satisfies

$$\frac{1}{\nu} = \begin{cases} \max\left\{\frac{1}{\mu} + \frac{2}{p} - \frac{2}{N}, \frac{1}{2} - \frac{1}{N}\right\}, & \text{if } \mu < N, \\ \frac{1}{\varrho}, & \text{if } \mu > N, \end{cases} \tag{4.7}$$

and

$$\nu > N, \quad \text{if } \mu = N. \tag{4.8}$$

Proof. We first apply Lemma 3.17 in combination with Sobolev embeddings, which yields

$$W^{1,\mu}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{with} \quad r = \begin{cases} \frac{N\mu}{N-\mu}, & \text{if } \mu < N, \\ < \infty, & \text{if } \mu = N, \\ \infty, & \text{if } \mu > N. \end{cases} \tag{4.9}$$

Due to $\mu \geq 2$, there holds $r \geq 2p/(p-2)$, see (3.27), so that Lemma 3.17 is applicable. For this purpose define ν via

$$\frac{1}{\nu} := \max\left\{\frac{1}{r} + \frac{2}{p} - \frac{1}{N}, \frac{1}{2} - \frac{1}{N}\right\}. \tag{4.10}$$

Since $r \geq 2p/(p-2)$ and $N < p$, there holds $\frac{1}{r} + \frac{2}{p} - \frac{1}{N} < \frac{1}{2}$ such that $\nu > 2$, giving in turn that the corresponding conjugate exponent satisfies $\nu' < 2 \leq N$, which will be important in the sequel. From (4.10) it follows

$$\frac{1}{r} + \frac{2}{p} + \frac{N - \nu'}{N\nu'} \leq 1 \quad \text{and} \quad \frac{1}{2} + \frac{N - \nu'}{N\nu'} \leq 1, \tag{4.11}$$

and consequently, Lemma 3.17 is applicable with $s = (N\nu')/(N-\nu') > 0$. Together with Hölder's inequality with $1/s + 1/s' = 1$ for the first two addends in ι, this gives

$$|\langle \iota, \psi\rangle| \le C\big(\|d_1 - d_2\|_{s'} + \|\varphi_1 - \varphi_2\|_{s'} + |t_1 - t_2| + \|\varphi_1 - \varphi_2\|_r\big)\|\psi\|_s.$$

By virtue of (4.11), it follows that $s \ge 2$ and thus $s' \le 2 \le r$. Hence, we arrive at

$$|\langle \iota, \psi\rangle| \le C\big(\|d_1 - d_2\|_2 + |t_1 - t_2| + \|\varphi_1 - \varphi_2\|_r\big)\|\psi\|_{\frac{N\nu'}{N-\nu'}}. \tag{4.12}$$

Since $\nu' < N$ as seen above, Sobolev embeddings give $W^{1,\nu'}(\Omega) \hookrightarrow L^{\frac{N\nu'}{N-\nu'}}(\Omega)$ and thus, (4.12) and (4.9) imply

$$\|\iota\|_{W^{1,\nu'}(\Omega)^*} \le C(\|\varphi_1 - \varphi_2\|_{W^{1,\mu}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|), \tag{4.13}$$

which is already (4.6). It remains to verify (4.7) and (4.8). If $\mu < N$, then (4.9) and (4.10) yield

$$\frac{1}{\nu} = \max\Big\{\frac{1}{\mu} + \frac{2}{p} - \frac{2}{N}, \frac{1}{2} - \frac{1}{N}\Big\}, \tag{4.14}$$

which gives the first case in (4.7). On the other hand, if $\mu > N$, then (4.9) implies

$$\frac{1}{\nu} = \max\Big\{\frac{2}{p} - \frac{1}{N}, \frac{1}{2} - \frac{1}{N}\Big\} = \frac{1}{\varrho}, \tag{4.15}$$

i.e., the second equation in (4.7). In case of $\mu = N$, the situation is more delicate. If $\frac{1}{r} + \frac{2}{p} - \frac{1}{N} \le \frac{1}{2} - \frac{1}{N}$, then $\frac{1}{\nu} = \frac{1}{2} - \frac{1}{N}$ and, since $N = 2, 3$, this gives $\nu > N$ as claimed. In the second case, we have

$$\frac{1}{\nu} = \frac{1}{r} + \frac{2}{p} - \frac{1}{N}, \tag{4.16}$$

where $r > 0$ can be chosen arbitrarily large, cf. (4.9). If we choose $r = Np/(p-N) > 0$, then (4.16) results in $\frac{1}{\nu} = \frac{1}{p}$, which implies $\nu = p > N$. This concludes the proof. □

Lemma 4.5. *The explicit representation of the recursively defined sequence*

$$\nu_0 = 2, \quad \nu_n = \frac{1}{\frac{1}{\nu_{n-1}} + \frac{2}{p} - \frac{2}{N}},\ n \ge 1, \tag{4.17}$$

is given by

$$\nu_n = \frac{2\,N\,p}{4(N-p)n + N\,p}, \quad n \in \mathbb{N}_0. \tag{4.18}$$

Proof. For $n = 0$ the assertion is obviously true. For $n \ge 1$ the claim follows by induction and straightforward computation. Note that the assertion is also correct for $\nu_n = \infty$, which might happen, since $p > N$. □

Theorem 4.6 (Improved Lipschitz continuity of Φ). *Under Assumption 3.16 there exists $L > 0$ such that for all $(t_i, d_i)_{i=1,2} \in [0,T] \times L^2(\Omega)$ the following estimate holds*

$$\|\Phi(t_1, d_1) - \Phi(t_2, d_2)\|_{W^{1,q}(\Omega)} \le L(\|d_1 - d_2\|_2 + |t_1 - t_2|)$$

with $q > 2$ given by Theorem 4.3.

Proof. As before we abbreviate $\varphi_i = \Phi(t_i, d_i)$, $i = 1, 2$. We apply an iterated bootstrapping procedure as indicated above. As already seen in (3.35), the assertion is correct with $q = 2$. Let us set $\nu_0 = 2$. We distinguish between the cases $N = 2$ and $N = 3$.

(i) $N = 2$
Setting $\mu := \nu_0 = 2 = N$ in Lemma 4.4 yields an estimate of the form (4.6) with $\nu = \nu_1 > N$ because of (4.8). If $\nu_1 \geq q$, then just apply (4.5) with $\omega = q$, which gives the assertion. Otherwise we employ (4.5) with $\omega = \nu_1$ to obtain

$$\begin{aligned}\|\varphi_1 - \varphi_2\|_{W^{1,\nu_1}(\Omega)} &\leq C\,\|\iota\|_{W^{1,\nu_1'}(\Omega)^*} \\ &\leq C(\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|) \quad \text{by (4.6)} \\ &\leq C(\|d_1 - d_2\|_2 + |t_1 - t_2|) \quad \text{by (3.35)}.\end{aligned}$$

Now, we repeat this procedure. Since $\nu_1 > N$, a second application of Lemma 4.4, this time with $\mu = \nu_1$, gives (4.6) with $\nu = \nu_2 := \varrho \geq q$. Then we again apply (4.5) with $\omega = q$, giving the claim for $N = 2$.

(ii) $N = 3$
In the three-dimensional case the situation is slighty more involved. In the first boot-strapping step, we have $\mu = \nu_0 = 2$ so that the first case in (4.7) applies. If the maximum is attained by $\frac{1}{2} - \frac{1}{N}$, then (4.6) holds with $\nu_1 = \frac{2N}{N-2} = 6 > 3 = N$. Now we can argue in exactly the same way as in the second step of the two-dimensional case to show the assertion.

If the maximum in (4.7) is attained by the first argument, then (4.6) is valid with $\nu = \nu_1 := \frac{1}{\frac{1}{\nu_0} + \frac{2}{p} - \frac{2}{N}}$. Now, if $\nu_1 \geq N$, then we argue as in case of $N = 2$ to verify the claim. If not, then, in the second boot-strapping iteration with $\mu = \nu_1$, again the first case in (4.7) applies. If the maximum is attained by $\frac{1}{2} - \frac{1}{N}$, we argue as before to prove the assertion. If this is not the case, we obtain (4.6) with $\nu = \nu_2 := \frac{1}{\frac{1}{\nu_1} + \frac{2}{p} - \frac{2}{N}}$. In this way, we either obtain an index $n \in \mathbb{N}$, where $\nu_n \geq N$ or the maximum in (4.7) is attained by the second argument, so that we can terminate the boot-strapping iteration with the previous arguments, or we create sequence of the form (4.17). For such a sequence however, Lemma 4.5 gives the explicit representation in (4.18). Since $N < p$, the denominator in this representation is decreasing for growing n. Therefore, for some finite $n \in \mathbb{N}$, ν_n will either satisfy $\nu_n \geq N$ or even be negative, which means that the maximum in (4.7) will be attained by the second argument. In both cases, the previous arguments apply, which finally gives the assertion. ∎

5. Differentiability of the Elliptic System. This section is dedicated to the derivatives of the solution operators $\mathcal{U}$ and Φ, as introduced in Definitions 3.6 and 3.21. These results will also be essential for the limit analysis for $\beta \to \infty$ in the companion paper [21].

Differentiability of $\mathcal{U}$. In the light of (3.19), the time dependence of $\mathcal{U}$ and Φ is only due to the time dependence of ℓ. Therefore, to show that the displacement and nonlocal damage are continuously differentiable, we require the following additional

Assumption 5.1. *From now on we assume that the applied volume and boundary load satisfies* $\ell \in C^1([0,T]; \boldsymbol{W}_D^{-1,p}(\Omega))$.

Lemma 5.2 (Partial differentiability of $\mathcal{U}$ w.r.t. time). *Under Assumption 5.1, the operator $\mathcal{U}$ is partially differentiable w.r.t. time. Its partial derivative $\partial_t\mathcal{U}$ belongs to $C([0,T] \times H^1(\Omega), V)$ and satisfies the elliptic equation*

$$A_\varphi\big(\partial_t\mathcal{U}(t,\varphi)\big) = \dot{\ell}(t) \quad \textit{for all } (t,\varphi) \in [0,T] \times H^1(\Omega). \tag{5.1}$$

Proof. Let $\varphi \in H^1(\Omega)$ be arbitrary, but fixed. From Lemma 3.3 we know that $A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}_D^{-1,p}(\Omega), \boldsymbol{W}_D^{1,p}(\Omega))$ and therefore continuously Fréchet-differentiable. By employing Definition 3.6, Assumption 5.1, and the chain rule, we thus obtain that $\mathcal{U}(\cdot,\varphi)$ is differentiable and the derivative fulfills (5.1). Completely analougously to the proof of Lemma 3.9 one deduces in view of Assumption 5.1 that

$$\partial_t \mathcal{U}(t_n,\varphi_n) \to \partial_t \mathcal{U}(t,\varphi) \text{ in } V$$

as $(t_n,\varphi_n) \to (t,\varphi)$ in $\mathbb{R} \times H^1(\Omega)$. □

Note that as a consequence of (3.4) and (5.1), one obtains on account of Assumption 5.1 the following estimate

$$\|\partial_t \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \leq c \quad \forall\,(t,\varphi) \in [0,T] \times H^1(\Omega), \tag{5.2}$$

where $c > 0$ is independent of t and φ.

Lemma 5.3 (Partial differentiability of $\mathcal{U}$ w.r.t. φ). *Let Assumption 3.16.1 be fulfilled. Then there exists an index $\nu \in (2,p)$ such that, for every $t \in [0,T]$, the map $\mathcal{U}(t,\cdot) : H^1(\Omega) \to \boldsymbol{W}_D^{1,\nu}(\Omega)$ is Fréchet differentiable and, for all $\varphi, \delta\varphi \in H^1(\Omega)$, the partial derivative fulfills*

$$A_\varphi\big(\partial_\varphi \mathcal{U}(t,\varphi)(\delta\varphi)\big) = \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)\big) \quad \text{in } \boldsymbol{W}_D^{-1,\nu}(\Omega), \tag{5.3}$$

where div *again denotes the distributional vector valued divergence, cf.* (3.20).

Proof. Let $t \in [0,T]$ and $\varphi, \delta\varphi \in H^1(\Omega)$ be arbitrary, but fixed, and set $r := 2p/(p-2)$. As shown at the beginning of the proof of Lemma 3.18, Assumption 3.16.1 guarantees the existence of an index ϱ such that $r \in (2,\varrho)$ and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$. For $\varrho > r$, there is another index κ with $r < \kappa < \varrho$, say $\kappa = (r+\varrho)/2$. Then we define ν through

$$\frac{1}{\nu'} = 1 - \frac{1}{\kappa} - \frac{1}{p}. \tag{5.4}$$

Since $\kappa > r$, this implies $\nu' < 2$, whence $\nu > 2$. Moreover, (5.4) yields $1/\nu' < 1 - 1/p = 1/p'$ so that $\nu' > p'$ and thus

$$\nu \in (2,p). \tag{5.5}$$

For the right-hand side in (5.3), Hölder's inequality with $1/\nu' + 1/\kappa + 1/p = 1$ and Corollary 3.7 imply

$$\begin{aligned} \|\operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)\big)\|_{\boldsymbol{W}_D^{-1,\nu}(\Omega)} &\leq \|g'(\varphi)\|_\infty \|\delta\varphi\|_\kappa \|\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\|_p \\ &\leq C\|\delta\varphi\|_\kappa. \end{aligned} \tag{5.6}$$

Due to (5.5), Lemma 3.3 is applicable with the exponent ν such that (5.6) implies that the linear operator, defined by

$$\mathcal{W}(\delta\varphi) := A_\varphi^{-1} \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi),$$

is bounded and hence, continuous from $L^\kappa(\Omega)$ to $\boldsymbol{W}_D^{1,\nu}(\Omega)$ so that, by virtue of $H^1(\Omega) \hookrightarrow L^\kappa(\Omega)$,

$$\mathcal{W} \in \mathcal{L}(H^1(\Omega), \boldsymbol{W}_D^{1,\nu}(\Omega)) \tag{5.7}$$

follows. As this operator is the candidate for the derivative, consider now the remainder term

$$R_\varphi(\delta\varphi) := \mathcal{U}(t, \varphi + \delta\varphi) - \mathcal{U}(t, \varphi) - \mathcal{W}(\delta\varphi). \tag{5.8}$$

By employing Definitions 3.2 and 3.6, the above definition of $\mathcal{W}$, a straightforward computation yields

$$\begin{aligned} A_\varphi(R_\varphi(\delta\varphi)) &= \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon\big(\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\big)\big) \\ &\quad + \operatorname{div}\Big(\big(\underbrace{g(\varphi+\delta\varphi) - g(\varphi) - g'(\varphi)(\delta\varphi)}_{=:\, r_\varphi(\delta\varphi)}\big)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi+\delta\varphi))\Big). \end{aligned} \tag{5.9}$$

Next define s via $1/s = 1 - 1/\varrho - 1/\nu'$. Since $\nu' < 2$ as seen above, we obtain $s > 2$. Moreover, because of $\kappa < \varrho$, (5.4) yields

$$\frac{1}{s} = 1 - \frac{1}{\varrho} - \frac{1}{\nu'} > 1 - \frac{1}{\kappa} - \frac{1}{\nu'} = \frac{1}{p} \quad \Longrightarrow \quad 2 < s < p.$$

Applying Hölder's inequality with these exponents in combination with Corollary 3.7 and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$ then gives

$$\begin{aligned} \|A_\varphi(R_\varphi(\delta\varphi))\|_{\boldsymbol{W}_D^{-1,\nu}(\Omega)} &\le C\|r_\varphi(\delta\varphi)\|_\kappa \|\mathcal{U}(t,\varphi+\delta\varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \\ &\quad + C\|g'(\varphi)\|_\infty \|\delta\varphi\|_\varrho \|\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,s}(\Omega)} \\ &\le C\big(\|r_\varphi(\delta\varphi)\|_\kappa + \|\delta\varphi\|_{H^1(\Omega)} \|\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,s}(\Omega)}\big), \end{aligned}$$

which together with (3.4) implies

$$\|R_\varphi(\delta\varphi)\|_{\boldsymbol{W}_D^{1,\nu}(\Omega)} \le C\big(\|r_\varphi(\delta\varphi)\|_\kappa + \|\delta\varphi\|_{H^1(\Omega)} \|\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,s}(\Omega)}\big). \tag{5.10}$$

We recall that $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$ with $\varrho > \kappa$, which allows us to deduce from Lemma A.1 that $g : H^1(\Omega) \to L^\kappa(\Omega)$ is Fréchet differentiable. Together with Lemma 3.9 and (5.10), this leads to

$$\frac{\|R_\varphi(\delta\varphi)\|_{\boldsymbol{W}_D^{1,\nu}(\Omega)}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0, \quad \text{as } \|\delta\varphi\|_{H^1(\Omega)} \to 0,$$

i.e., the Fréchet differentiability of $\mathcal{U}(t,\cdot) : H^1(\Omega) \to \boldsymbol{W}_D^{1,\nu}(\Omega)$. The derivative is given by the operator $\mathcal{W}$, whence equation (5.3). □

Clearly, Lemma 5.3 implies that $\mathcal{U}(t,\cdot)$ is also Fréchet-differentiable from $H^1(\Omega)$ to $V = \boldsymbol{W}_D^{1,2}(\Omega)$, and the corresponding derivative satisfies (5.3) as an equation in V^*. Furthermore, analogously to (5.6), Hölder's inequality with $1/2 + 1/p + 1/r = 1$, where again $r = 2p/(p-2)$, leads to

$$\|\operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\big)\|_{V^*} \le C\|\delta\varphi\|_r.$$

Therefore, we deduce from (5.3) and (3.4) the following estimate, which turns out to be useful in the next section, see the proof of Lemma 5.11 below:

LEMMA 5.4. *Let Assumption 3.16.1 hold. Then, for all $\varphi, \delta\varphi \in H^1(\Omega)$, there holds*

$$\|\partial_\varphi \mathcal{U}(t,\varphi)(\delta\varphi)\|_V \le C\,\|\delta\varphi\|_r$$

with $r = 2p/(p-2)$.

LEMMA 5.5 (Continuity of $\partial_\varphi\mathcal{U}$). *Under Assumption 3.16.1 the operator* $\partial_\varphi\mathcal{U} : [0,T]\times H^1(\Omega) \to \mathcal{L}(H^1(\Omega), V)$ *is continuous.*

Proof. Let $(t_i, \varphi_i)_{i=1,2} \in [0,T]\times H^1(\Omega)$ and $\delta\varphi \in H^1(\Omega)$ be arbitrary, but fixed with $\delta\varphi \neq 0$. Further, let us abbreviate $\boldsymbol{u}_i' := \partial_\varphi\mathcal{U}(t_i,\varphi_i)\delta\varphi$ and $\boldsymbol{u}_i := \mathcal{U}(t_i,\varphi_i)$ for $i = 1, 2$. Moreover, define $f_1 := A_{\varphi_2}\boldsymbol{u}_2' - A_{\varphi_1}\boldsymbol{u}_2' \in V^*$ and $f_2 := A_{\varphi_1}\boldsymbol{u}_1' - A_{\varphi_2}\boldsymbol{u}_2' \in V^*$ such that

$$A_{\varphi_1}(\boldsymbol{u}_1' - \boldsymbol{u}_2') = f_1 + f_2. \tag{5.11}$$

Thanks to Lemma 5.3 there is an index $\nu \in (2,p)$ such that $\mathcal{U}(t_2,\cdot) : H^1(\Omega) \to \boldsymbol{W}_D^{1,\nu}(\Omega)$ is Fréchet differentiable. We set $\kappa = 2\nu/(\nu-2) \in (2p/(p-2), \infty)$ such that $1/\kappa + 1/\nu + 1/2 = 1$. Note that, in view of Assumption 3.16.1, we have the embedding $H^1(\Omega) \hookrightarrow L^\kappa(\Omega)$. Then, Hölder's inequality yields

$$\|f_1\|_{V^*} \leq C_1\|g(\varphi_2) - g(\varphi_1)\|_\kappa \|\boldsymbol{u}_2'\|_{\boldsymbol{W}_D^{1,\nu}(\Omega)} \leq C_1\|g(\varphi_2) - g(\varphi_1)\|_\kappa \, \|\delta\varphi\|_{H^1(\Omega)},$$

where we used (5.6) in combination with $H^1(\Omega) \hookrightarrow L^\kappa(\Omega)$, (5.3), and Lemma 3.3 with $\bar{p} := \nu$ for the last inequality. Thanks to Lemma A.1, this gives

$$\sup_{\substack{\delta\varphi\in H^1(\Omega)\\ \delta\varphi\neq 0}} \frac{\|f_1\|_{V^*}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0, \qquad \text{as } \varphi_1 \to \varphi_2 \text{ in } H^1(\Omega). \tag{5.12}$$

From the definition of $\boldsymbol{u}_i$ and $\boldsymbol{u}_i'$ and equation (5.3) if follows that

$$A_{\varphi_i}\boldsymbol{u}_i' = \operatorname{div}\big(g'(\varphi_i)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_i)\big) \qquad \text{for } i = 1, 2.$$

This allows us to rewrite f_2 as

$$\begin{aligned} f_2 = &\operatorname{div}\big(g'(\varphi_1)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_1)\big) - \operatorname{div}\big(g'(\varphi_1)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_2)\big) \\ &+ \operatorname{div}\big(g'(\varphi_1)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_2)\big) - \operatorname{div}\big(g'(\varphi_2)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_2)\big) \end{aligned}$$

We again abbreviate $r := 2p/(p-2)$, which implies in view of Assumption 3.16.1 that there is an index ϱ such that $r \in (2,\varrho)$ and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$, as shown at the beginning of the proof of Lemma 3.18. By construction we have $1/r + 1/p + 1/2 = 1$ and, in view of $r \in (2,\varrho)$, there exists $s \in (2,p)$ such that $1/\varrho + 1/s + 1/2 = 1$. By applying Hölder's inequality with these exponents and Corollary 3.7 we arrive at

$$\begin{aligned} \|f_2\|_{V^*} \leq\ & C_2\|g'(\varphi_1)(\delta\varphi)\|_\varrho \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,s}(\Omega)} \\ &+ \|(g'(\varphi_2) - g'(\varphi_1))(\delta\varphi)\|_r \|\boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \\ \leq\ & C_2\|\delta\varphi\|_{H^1(\Omega)}\big(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,s}(\Omega)} + \|g'(\varphi_2) - g'(\varphi_1)\|_{\mathcal{L}(H^1(\Omega),L^r(\Omega))}\big). \end{aligned}$$

Note that for the second inequality we used again that $g : H^1(\Omega) \to L^r(\Omega)$ is Fréchet differentiable due to Lemma A.1 and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$ with $\varrho > r$. Lemmas 3.9 and A.1 now ensure that

$$\sup_{\substack{\delta\varphi\in H^1(\Omega)\\ \delta\varphi\neq 0}} \frac{\|f_2\|_{V^*}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0, \qquad \text{as } (t_1,\varphi_1) \to (t_2,\varphi_2) \text{ in } \mathbb{R}\times H^1(\Omega). \tag{5.13}$$

Altogether, it follows from (5.11), (5.12), (5.13) and (3.4) that

$$\sup_{\substack{\delta\varphi\in H^1(\Omega)\\ \delta\varphi\neq 0}} \frac{\|\boldsymbol{u}_1' - \boldsymbol{u}_2'\|_V}{\|\delta\varphi\|_{H^1(\Omega)}} \le C \sup_{\substack{\delta\varphi\in H^1(\Omega)\\ \delta\varphi\neq 0}} \frac{\|f_1 + f_2\|_{V^*}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0$$

for $(t_1, \varphi_1) \to (t_2, \varphi_2)$ in $\mathbb{R} \times H^1(\Omega)$. This completes the proof. □

We are now in the position to state the main result of this section.

Proposition 5.6 (Fréchet differentiability of the operator $\mathcal{U}$). *Under Assumptions 3.16.1 and 5.1 it holds $\mathcal{U} \in C^1([0,T] \times H^1(\Omega); V)$.*

Proof. From Lemma 3.9 we know that $\mathcal{U} \in C([0,T]\times H^1(\Omega); V)$, while Lemmas 5.2, 5.3 and 5.5 state that $\mathcal{U}$ possesses partial derivatives with $\partial_t\mathcal{U} \in C([0,T] \times H^1(\Omega); V)$ and $\partial_\varphi\mathcal{U} \in C([0,T]\times H^1(\Omega); \mathcal{L}(H^1(\Omega), V))$, respectively. Hence, we can apply [4, Theorem 3.7.1.], which gives the assertion. □

Differentiability of Φ. To differentiate the operator Φ from Definition 3.21, we employ the implicit function theorem. For this purpose, let us define the following:

Definition 5.7. *Let Assumption 3.10 be fulfilled. We define the mapping $\Psi : [0,T] \times L^2(\Omega) \times H^1(\Omega) \to H^1(\Omega)^*$ by $\Psi(t, d, \varphi) := B\varphi + F(t, \varphi) - \beta d$.*

Note that $\varphi = \Phi(t, d)$ implies $\Psi(t, d, \varphi) = 0$. First we show that Ψ is continuously Fréchet differentiable. To this end we need the following

Assumption 5.8. *From now on we assume that $g \in C^2(\mathbb{R})$ and $g'' \in L^\infty(\mathbb{R})$.*

Lemma 5.9. *Let Assumptions 3.16.1, 5.1 and 5.8 hold. Then the function $F : [0,T] \times H^1(\Omega) \to H^1(\Omega)^*$ from Definition 3.13 is continuously Fréchet differentiable. Its derivative at $(t,\varphi) \in [0,T] \times H^1(\Omega)$ in direction $(\delta t, \delta\varphi) \in \mathbb{R} \times H^1(\Omega)$ is given by*

$$\begin{aligned}\langle F'(t,\varphi)(\delta t, \delta\varphi), z\rangle_{H^1(\Omega)} &= \frac{1}{2}\int_\Omega g''(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))z\,dx \\ &\quad + \int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon\Big(\mathcal{U}'(t,\varphi)(\delta t, \delta\varphi)\Big)z\,dx, \quad z \in H^1(\Omega),\end{aligned} \tag{5.14}$$

where $\mathcal{U}'$ is the Fréchet-derivative of $\mathcal{U}$ according to Proposition 5.6.

Proof. We prove the result in two steps, by splitting F into two products and applying Lemma B.1 for these. To do so, let us introduce the following mappings:

$$\mathcal{H} : (0,T) \times H^1(\Omega) \to L^{p/2}(\Omega), \quad \mathcal{H}(t,\varphi) := \mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi)) \tag{5.15}$$

and

$$\begin{aligned}&P_1 : L^\infty(\Omega) \times L^{p/2}(\Omega) \to H^1(\Omega)^*, \\ &\langle P_1(y_1, y_2), z\rangle_{H^1(\Omega)} := \frac{1}{2}\int_\Omega y_1 \cdot y_2 \cdot z\,dx, \quad z \in H^1(\Omega)\end{aligned} \tag{5.16}$$

such that

$$F : (t,\varphi) \mapsto P_1\big(g'(\varphi), \mathcal{H}(t,\varphi)\big). \tag{5.17}$$

Notice that these mappings are indeed well defined because of $H^1(\Omega) \hookrightarrow L^{2p/(p-2)}(\Omega)$ by Assumption 3.16.1 and due to the mapping properties of $\mathcal{U}$. We now prove the assertion by applying the product rule from Lemma B.1 to $\mathcal{H}$ and F in the form (5.17). To this end, let $s \in (N,p)$ be arbitrary, but fixed. Note that such an index exists thanks to Assumption 3.16.1. Moreover, define ω and r through

$$\frac{1}{\omega} = \frac{1}{p} + \frac{1}{s} \quad \text{and} \quad \frac{1}{r} = \frac{1}{2} + \frac{1}{s}. \tag{5.18}$$

Due to $p > s > 2$, there holds $r < \omega < p/2$ so that $\mathcal{H}$ is well defined, if considered with $L^\omega(\Omega)$ and $L^r(\Omega)$, respectively, as range.

(i) We first show that $\mathcal{H}$ is continuous as an operator with range in $L^\omega(\Omega)$ and continuously Fréchet differentiable, if considered as an operator with range in $L^r(\Omega)$. Concerning the continuity, we estimate similarly to (3.25) by using (5.18):

$$\|\mathcal{H}(t_1,\varphi_1)-\mathcal{H}(t_2,\varphi_2)\|_\omega \leq C\,\|\mathcal{U}(t_1,\varphi_1)+\mathcal{U}(t_2,\varphi_2)\|_{\boldsymbol{W}^{1,p}_D(\Omega)}\|\mathcal{U}(t_1,\varphi_1)-\mathcal{U}(t_2,\varphi_2)\|_{\boldsymbol{W}^{1,s}_D(\Omega)}$$

for all $(t_i,\varphi_i)_{i=1,2} \in [0,T] \times H^1(\Omega)$. The continuity of $\mathcal{U}$ in $\boldsymbol{W}^{1,s}_D(\Omega)$, $s < p$, by Lemma 3.9 in combination with Corollary 3.7 then gives the desired continuity of $\mathcal{H}$. To prove the differentiability, consider the mapping

$$P_2 : \boldsymbol{W}^{1,s}_D(\Omega) \times V \ni (\boldsymbol{u},\boldsymbol{v}) \mapsto \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \in L^r(\Omega) \tag{5.19}$$

such that

$$\mathcal{H}(t,\varphi) = P_2\big(\mathcal{U}(t,\varphi),\mathcal{U}(t,\varphi)\big). \tag{5.20}$$

In view of (5.18), P_2 is bilinear and continuous. To apply Lemma B.1, we set

$$\begin{aligned} &U := (0,T) \times H^1(\Omega), \quad X := \mathbb{R} \times H^1(\Omega), \quad W := L^r(\Omega), \\ &P = P_2, \quad f_i := \mathcal{U}, \quad Y_i := \boldsymbol{W}^{1,s}_D(\Omega), \quad Z_i := V, \quad i = 1,2. \end{aligned}$$

From Lemma 3.9 and Proposition 5.6 we know that $\mathcal{U} : (0,T) \times H^1(\Omega) \to \boldsymbol{W}^{1,s}_D(\Omega)$ is continuous and $\mathcal{U} : (0,T) \times H^1(\Omega) \to V$ is continuously Fréchet differentiable, respectively. Hence, we can apply Lemma B.1 to (5.20), giving in turn that $\mathcal{H} : (0,T) \times H^1(\Omega) \to L^r(\Omega)$ is continuously Fréchet differentiable with

$$\mathcal{H}'(t,\varphi)(\delta t,\delta\varphi) := 2\,\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon\big(\mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\big) \tag{5.21}$$

for all $(t,\varphi) \in (0,T) \times H^1(\Omega)$ and all $(\delta t,\delta\varphi) \in \mathbb{R} \times H^1(\Omega)$.

(ii) The result from the previous step allows us now to prove the continuously Fréchet differentiability of F. We again apply the product rule from Lemma B.1, this time to (5.17). To fix the setting, let $\kappa > 0$ satisfy

$$\frac{1}{\kappa} < 1 - \frac{1}{r} = \frac{1}{2} - \frac{1}{s} \quad \text{and} \quad \frac{1}{\kappa} < \frac{1}{2} - \frac{1}{2\omega} = \frac{1}{2} - \frac{1}{2p} - \frac{1}{2s}. \tag{5.22}$$

Since $s > N$ and $p > N$, the right-hand sides in the above inequalities are strictly larger than $(N-2)/(2N)$ and consequently, κ can be chosen such that

$$H^1(\Omega) \hookrightarrow L^\kappa(\Omega), \tag{5.23}$$

which is assumed in the following. Given κ we define τ and ρ via

$$\frac{1}{\tau} + \frac{1}{\omega} + \frac{1}{\kappa} = 1 \quad \text{and} \quad \frac{1}{\rho} + \frac{1}{r} + \frac{1}{\kappa} = 1. \tag{5.24}$$

Because of (5.22), these indices satisfy

$$0 < \rho < \infty \quad \text{and} \quad 0 < \tau < \kappa. \tag{5.25}$$

To apply Lemma B.1, we then choose

$$\begin{aligned} U &:= (0,T) \times H^1(\Omega), \quad X := \mathbb{R} \times H^1(\Omega), \quad W := H^1(\Omega)^*, \\ P = P_1, &\quad f_1 := g', \quad Y_1 := L^\rho(\Omega), \quad Z_1 := L^\tau(\Omega), \\ & f_2 := \mathcal{H}, \quad Y_2 := L^\omega(\Omega), \quad Z_2 := L^r(\Omega), \end{aligned}$$

where we considered g' as a mapping on U with a little abuse of notation. From the previous step, we already know that $f_2 = \mathcal{H}$ fulfills the required continuity and differentiability conditions. Moreover, due to (5.25) and (5.23), Assumption 5.8 together with Lemma A.1 yields that $f_1 = g'$ is continuous from $H^1(\Omega)$ to $L^\rho(\Omega)$ and continuously Fréchet-differentiable from $H^1(\Omega)$ to $L^\tau(\Omega)$. Finally, thanks to (5.24) and (5.23), the bilinear form P_1 from (B.3) satisfies

$$\begin{aligned} \|P(y_1, y_2)\|_{H^1(\Omega)^*} &\leq C\|y_1\|_\tau \|y_2\|_\omega \quad \forall\, (y_1, y_2) \in L^\tau(\Omega) \times L^\omega(\Omega), \\ \|P(y_1, y_2)\|_{H^1(\Omega)^*} &\leq C\|y_1\|_\rho \|y_2\|_r \quad \forall\, (y_1, y_2) \in L^\rho(\Omega) \times L^r(\Omega), \end{aligned}$$

and is therefore continuous in the required spaces. Hence Lemma B.1 yields the continuous Fréchet differentiability of $F : (0,T) \times H^1(\Omega) \to H^1(\Omega)^*$ and (5.14), as a result of (5.16), (5.15) and (5.21). Note that the derivative of F can be continued at $(0, \varphi)$ and (T, φ) for every $\varphi \in H^1(\Omega)$ due to Lemma 3.9 and Proposition 5.6. ∎

As an immediate consequence of Lemma 5.9 we obtain

Corollary 5.10 (Fréchet differentiability of Ψ). *Under Assumptions 3.16.1, 5.1 and 5.8 it holds* $\Psi \in C^1([0,T] \times L^2(\Omega) \times H^1(\Omega), H^1(\Omega)^*)$.

Proof. The result directly follows from Definition 5.7 combined with Lemma 5.9 and the fact that $B \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$. ∎

The last result required for the application of the implicit function theorem is the following

Lemma 5.11. *Under Assumptions 3.16 and 5.8 the operator* $\partial_\varphi \Psi(t, d, \varphi) : H^1(\Omega) \to H^1(\Omega)^*$ *is bijective for all* $(t, d, \varphi) \in [0,T] \times L^2(\Omega) \times H^1(\Omega)$.

Proof. Throughout this proof let $(t, d, \varphi) \in [0,T] \times L^2(\Omega) \times H^1(\Omega)$ be arbitrary, but fixed. On account of Definition 5.7, we have to show that, for every $h \in H^1(\Omega)^*$, the equation

$$B\delta\varphi + \partial_\varphi F(t, \varphi)\delta\varphi = h \tag{5.26}$$

admits a unique solution $\delta\varphi \in H^1(\Omega)$. We prove the result by means of the Lax-Milgram lemma. Thanks to $B, \partial_\varphi F(t,\varphi) \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ one obtains

$$|\langle B\delta\varphi + \partial_\varphi F(t,\varphi)(\delta\varphi), z\rangle_{H^1(\Omega)}| \leq C\|\delta\varphi\|_{H^1(\Omega)} \|z\|_{H^1(\Omega)} \quad \forall\, \delta\varphi, z \in H^1(\Omega),$$

whence the boundedness of $B + \partial_\varphi F(t,\varphi)$. We now address the coercivity thereof. To this end, let $z \in H^1(\Omega)$ be arbitrary, but fixed. By the definition of the (directional) derivative combined with the linearity of B, we have

$$\begin{aligned}\langle Bz + \partial_\varphi F(t,\varphi)z, z\rangle_{H^1(\Omega)} &= \langle Bz, z\rangle_{H^1(\Omega)} + \lim_{\tau\searrow 0} \frac{\langle F(t,\varphi+\tau z) - F(t,\varphi), z\rangle_{H^1(\Omega)}}{\tau} \\ &= \lim_{\tau\searrow 0} \frac{\langle B(\varphi+\tau z - \varphi) + F(t,\varphi+\tau z) - F(t,\varphi), \tau z\rangle_{H^1(\Omega)}}{\tau^2} \\ &\geq c \lim_{\tau\searrow 0} \frac{\|\varphi+\tau z-\varphi\|^2_{H^1(\Omega)}}{\tau^2} = c\|z\|^2_{H^1(\Omega)},\end{aligned}$$

where the last inequality follows from the strong monotonicity of $B+F(t,\cdot)$, cf. Lemma 3.19. Lax-Milgram's Lemma thus gives the unique solvability of (5.26) as claimed. □

Proposition 5.12 (Fréchet differentiability of the operator Φ). *Let Assumptions 3.16, 5.1 and 5.8 hold. Then $\Phi \in C^1([0,T]\times L^2(\Omega), H^1(\Omega))$, and its derivative at $(t,d) \in [0,T]\times L^2(\Omega)$ in direction $(\delta t, \delta d) \in \mathbb{R}\times L^2(\Omega)$ is given by*

$$B\Phi'(t,d)(\delta t,\delta d) + F'(t,\varphi)\big(\delta t, \Phi'(t,d)(\delta t,\delta d)\big) = \beta\delta d, \tag{5.27}$$

with the abbreviation $\varphi := \Phi(t,d)$.

Proof. Let $(t,d) \in (0,T)\times L^2(\Omega)$ be arbitrary, but fixed. We apply the implicit function theorem to Ψ as given in Definition 5.7, cf. e.g. [32, Theorem 4.B(d)]. Due to Corollary 5.10 and Lemma 5.11, Ψ is continuously Fréchet-differentiable and $\partial_\varphi\Psi(t,d)$ is continuously invertible by Banach's inverse theorem. Thus the implicit function theorem is applicable and implies that Φ is as smooth as Ψ, i.e. continuously Fréchet-differentiable from $(0,T)\times L^2(\Omega)$ to $H^1(\Omega)$, and its derivative is given by

$$\Phi'(t,d)(\delta t,\delta d) = -[\partial_\varphi\Psi(t,d,\varphi)]^{-1}\partial_{(t,d)}\Psi(t,d,\varphi)(\delta t,\delta d),$$

which is equivalent to (5.27) in view of Definition 5.7.

It remains to prove that the derivative can be continuously extended to $t=0$ and $t=T$. From Corollary 5.10 we know that $\partial_{(t,d)}\Psi$ and $\partial_\varphi\Psi$ can be continuously extended to $(0,d,\varphi)$ with $\varphi = \Phi(0,d)$. Furthermore, in the light of Lemma 5.11, we are allowed to define

$$\Phi'(0,d)(\delta t,\delta d) := -[\partial_\varphi\Psi(0,d,\varphi)]^{-1}\partial_{(t,d)}\Psi(0,d,\varphi)(\delta t,\delta d).$$

The continuity of the inversion $\mathcal{L}(H^1(\Omega),H^1(\Omega)^*) \ni A \mapsto A^{-1} \in \mathcal{L}(H^1(\Omega)^*,H^1(\Omega))$ on the set of linear isomorphisms, see e.g. [29, Ch. III.8], then yields the continuity of Φ' at $(0,d)$. In the exactly same way one shows the continuity Φ' at (T,d). □

We collect the above findings in our final theorem on the regularity of the solution to our viscous two-field gradient damage model:

Theorem 5.13. *Let Assumptions 3.16, 5.1 and 5.8 be fulfilled. Then there exists a unique solution $(\boldsymbol{u},\varphi,d)$ of the problem (P), satisfying $d \in C^1([0,T];L^2(\Omega))$, $\varphi \in C^{0,1}([0,T];W^{1,q}(\Omega)) \cap C^1([0,T];H^1(\Omega))$, $\boldsymbol{u} \in C([0,T];\boldsymbol{W}^{1,s}_D(\Omega)) \cap C^1([0,T];V)$ with $q>2$ and $s\in(2,p)$, and the system of differential equations in (3.41).*

Proof. In Theorem 3.24 we already established that the unique solution of (P) satisfies (3.41), as well as the regularity of the local damage. Since $\boldsymbol{u}(t) = \mathcal{U}(t,\varphi(t))$ and $\varphi(t) = \Phi(t,d(t))$, the additional regularity results follow from Theorem 4.6, Proposition 5.12, Lemma 3.9, and Proposition 5.6 in combination with the chain rule. □

Remark 5.14. *We point out that in the two-dimensional case one can show, by proceeding as above and by assuming $g'' \in C^{0,1}(\mathbb{R})$, that $\mathcal{U} \in C^1([0,T]\times W^{1,q}(\Omega);\boldsymbol{W}^{1,p}_D(\Omega))$*

and $\Phi \in C^1([0,T] \times L^2(\Omega); W^{1,q}(\Omega))$*, with* $q > 2$ *given by Theorem 4.3. This is mainly due to the Sobolev embedding* $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$*, combined with the fact that* $g, g' : W^{1,q}(\Omega) \to L^\infty(\Omega)$ *are continuously Fréchet differentiable. Therefore, in the two-dimensional case, the unique solution* $(\boldsymbol{u}, \varphi, d)$ *of the problem* (P)*, satisfies* $d \in C^1([0,T]; L^2(\Omega))$*,* $\varphi \in C^1([0,T]; W^{1,q}(\Omega))$ *and* $\boldsymbol{u} \in C^1([0,T]; \boldsymbol{W}^{1,p}_D(\Omega))$*.*

Appendix A. Nemytskii Operators. In this section, we enumerate some useful properties of the Nemytskii-operators associated with g and its derivative g'. Recall that the mapping $g : \mathbb{R} \to [\epsilon, 1]$, with $\epsilon > 0$, is supposed to satisfy Assumption 2.9, i.e., $g \in C^{1,1}(\mathbb{R})$.

Lemma A.1. *The mappings* g *and* g' *possess the following properties:*

1. *For all* $\rho \in [1, \infty]$*, the Nemytskii-operators* $g : L^\rho(\Omega) \to L^\infty(\Omega)$ *and* $g' : L^\rho(\Omega) \to L^\infty(\Omega)$ *are well defined and Lipschitz continuous from* $L^\rho(\Omega)$ *to* $L^\rho(\Omega)$*.*
2. *The operators* $g, g' : L^1(\Omega) \to L^\varrho(\Omega)$ *are continuous for all* $\varrho \in [1, \infty)$*.*
3. *The operator* $g : L^\rho(\Omega) \to L^\tau(\Omega)$ *is continuously Fréchet differentiable for* $1 \le \tau < \rho < \infty$*. If we assume that the map* g *satisfies* $g \in C^2(\mathbb{R})$ *with* $g'' \in L^\infty(\mathbb{R})$*, then the operator* $g' : L^\rho(\Omega) \to L^\tau(\Omega)$ *is continuously Fréchet differentiable as well.*

Proof. The first assertion is due to [11, Theorem 1 (iii),(iv)] and the Lipschitz continuity of $g' : \mathbb{R} \to \mathbb{R}$, while the second statement is a direct result of [11, Theorem 4]. The continuous Fréchet differentiability follows from [11, Theorems 4 and 7]. □

Appendix B. Product Rule.

This appendix is dedicated to a generalization of the well known product rule in the sense that the spaces, where the inner functions are continuous and continuously differentiable, respectively, may differ.

Lemma B.1. *Let* X*,* W *and* Y_i*,* Z_i*,* $i = 1, 2$*, be Banach spaces with* $Y_i \subset Z_i$*. Moreover, let* $U \subset X$ *be an open set and* $f_i : U \to Y_i$*,* $i = 1, 2$*, be continuous mappings, which are continuously Fréchet differentiable, when considered as mappings from* U *to* Z_i*. Additionally, let* $P : Z_1 \times Y_2 \to W$ *be a product, i.e., a continuous bilinear mapping, and assume that* P *possesses the same properties, when considered as a mapping from* $Y_1 \times Z_2$ *to* W*. Then the map*

$$h : x \in U \to P(f_1(x), f_2(x)) \in W$$

is continuously Fréchet differentiable with

$$h'(x)(\delta x) = P(f_1'(x)(\delta x), f_2(x)) + P(f_1(x), f_2'(x)(\delta x)) \quad \forall\, x \in U,\ \forall\, \delta x \in X. \tag{B.1}$$

Proof. Let $x \in U$ be arbitrary, but fixed and $\delta x \in X$ with $\|\delta x\|_X \neq 0$ small enough such that $x + \delta x \in U$. straightforward computation yields

$$\begin{aligned}
&\|R(\delta x)\|_W := \\
&:= \|h(x+\delta x) - h(x) - P(f_1'(x)(\delta x), f_2(x)) - P(f_1(x), f_2'(x)(\delta x))\|_W \\
&\le \|P(f_1(x+\delta x), f_2(x)) - P(f_1(x), f_2(x)) - P(f_1'(x)(\delta x), f_2(x))\|_W \\
&\quad + \|P(f_1(x+\delta x), f_2(x+\delta x)) - P(f_1(x+\delta x), f_2(x)) - P(f_1(x+\delta x), f_2'(x)(\delta x))\|_W \\
&\quad + \|P(f_1(x+\delta x), f_2'(x)(\delta x)) - P(f_1(x), f_2'(x)(\delta x))\|_W.
\end{aligned}$$

Since $P : Z_1 \times Y_2 \to W$, $P : Y_1 \times Z_2 \to W$ are continuous bilinear mappings, we obtain in view of the Fréchet differentiability of $f_i : U \to Z_i$ for every $i \in \{1,2\}$, combined with the continuity of $f_1 : U \to Y_1$ that

$$\frac{\|R(\delta x)\|_W}{\|\delta x\|_X} \leq C\Big(\frac{\|R_{f_1}(\delta x)\|_{Z_1}}{\|\delta x\|_X}\|f_2(x)\|_{Y_2} + \frac{\|R_{f_2}(\delta x)\|_{Z_2}}{\|\delta x\|_X}\|f_1(x+\delta x)\|_{Y_1} + \|f_1(x+\delta x) - f_1(x)\|_{Y_1}\frac{\|f_2'(x)(\delta x)\|_{Z_2}}{\|\delta x\|_X}\Big) \to 0, \quad \text{as } \|\delta x\|_X \to 0,$$

where we denote $R_{f_i}(\delta x) := f_i(x+\delta x) - f_i(x) - f_i'(x)(\delta x)$ for every $i \in \{1,2\}$. Therefore, h is Fréchet differentiable at $x \in U$, with derivative given by (B.1). In order to show the continuity thereof, let $\{x_n\} \subset U$ with $x_n \to x$ in X be given. By employing the properties of P we obtain for all $\delta x \in X$

$$\begin{aligned}
&\|P(f_1'(x_n)(\delta x), f_2(x_n)) - P(f_1'(x)(\delta x), f_2(x))\|_W \\
&\leq \|P(f_1'(x_n)(\delta x) - f_1'(x)(\delta x), f_2(x_n))\|_W + \|P(f_1'(x)(\delta x), f_2(x_n) - f_2(x))\|_W \\
&\leq C\big(\|f_1'(x_n)(\delta x) - f_1'(x)(\delta x)\|_{Z_1}\|f_2(x_n)\|_{Y_2} + \|f_1'(x)(\delta x)\|_{Z_1}\|f_2(x_n) - f_2(x)\|_{Y_2}\big) \\
&\leq C\big(\|f_1'(x_n) - f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|\delta x\|_X\|f_2(x_n)\|_{Y_2} \\
&\qquad + \|f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|\delta x\|_X\|f_2(x_n) - f_2(x)\|_{Y_2}\big).
\end{aligned}$$

The continuity of $f_1' : U \to \mathcal{L}(X, Z_1)$ and $f_2 : U \to Y_2$ thus implies

$$\begin{aligned}
&\sup_{\|\delta x\|_X = 1} \|P(f_1'(x_n)(\delta x), f_2(x_n)) - P(f_1'(x)(\delta x), f_2(x))\|_W \\
&\quad \leq C\big(\|f_1'(x_n) - f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|f_2(x_n)\|_{Y_2} \\
&\qquad + \|f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|f_2(x_n) - f_2(x)\|_{Y_2}\big) \to 0, \quad \text{as } x_n \to x.
\end{aligned} \tag{B.2}$$

Completely analogously we obtain

$$\sup_{\|\delta x\|_X = 1} \|P(f_1(x_n), f_2'(x_n)(\delta x)) - P(f_1(x), f_2'(x)(\delta x))\|_W \to 0, \quad \text{as } x_n \to x. \tag{B.3}$$

Finally, (B.1), (B.2), and (B.3) result in

$$\sup_{\|\delta x\|_X = 1} \|h'(x_n)(\delta x) - h'(x)(\delta x)\|_W \to 0 \text{ as } x_n \to x \text{ in } X,$$

which completes the proof. ∎

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