Different Mathematical Languages – the Case of Monoidal Categories

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WRITING IN MATHEMATICS WORKSHOP

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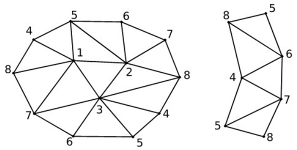
- Introduction

Mathematical Texts

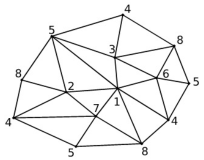
> The distinctive definition-theorem–proof format of professional publications is the single most conspicuous feature of mathematical practice. (Burgess)

- Really?
- Another very conspicuous feature is the use of mathematical notations.
  - Formulas, algebraic formalisms, diagrams, etc.
- What does looking at mathematical texts tell us about the nature of mathematics?
- Different Methods:
  - Quantitative analysis
  - Case Studies

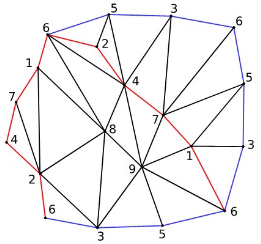
torsion	f-vector
$\mathbb{Z}_3$	$(8, 24, 17)$
$\mathbb{Z}_4$	$(8, 26, 19)$
$\mathbb{Z}_5$	$(9, 32, 24)$
$\mathbb{Z}_6$	$(9, 33, 25)$
$\mathbb{Z}_7$	$(9, 34, 26)$
$\mathbb{Z}_8$	$(9, 35, 27)$
$\mathbb{Z}_9$	$(9, 36, 28)$
$\mathbb{Z}_{10}$	$(9, 36, 28)$
$\mathbb{Z}_{11}$	$(10, 42, 33)$
$\mathbb{Z}_{12}$	$(10, 42, 33)$
$\mathbb{Z}_{13}$	$(10, 43, 34)$



(a) Simplicial complex with 4-torsion.



(b) Simplicial complex with 3-torsion.



(c) Simplicial complex with 5-torsion.

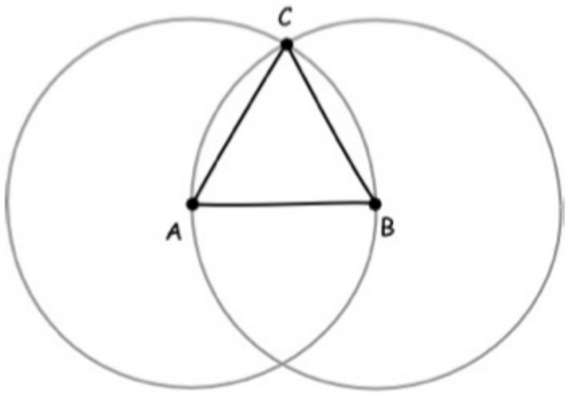
Figure 6: Small substructures with p-torsion of the lens spaces $L(p, 1)$.

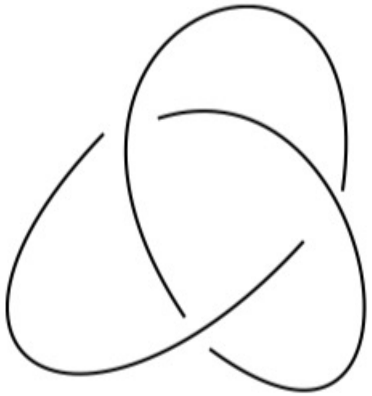
two circles S^1 that are glued together at a point. In general, if we remove a facet from a triangulation of a sphere product, the resulting complex is simple-homotopy equivalent to the wedge product of the constituting spheres. In the case of $S^2 \times S^1$, the wedge product $S^2 \vee S^1$ is of mixed dimension. Since in the implementation of RSHT our focus is on the top-dimensional faces, RSHT is not further touching lower-dimensional parts once these are reached via collapses. Thus, the resulting triangulations of $S^2 \vee S^1$ are of the form $\partial\Delta_3 \cup K^1$, consisting of the vertex-minimal triangulation of S^2 as the boundary complex $\partial\Delta_3$ of a 3-simplex Δ_3 union a 1-dimensional complex K^1.

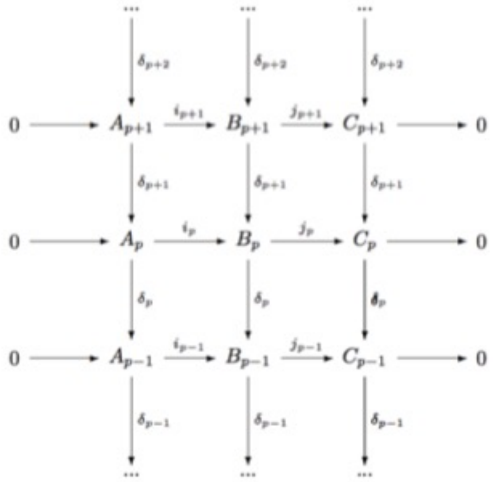
Depending on the intersection of K^1 with $\partial\Delta_3$, K^1 either is a path (a 1-dimensional ball) or a loop (a 1-sphere S^1). For a unified description in Table 4, we write $K^1(4.5382)$ to point out that K^1 has (in 10^4 runs of RSHT) on average 4.5382 edges. Table 4 gives results for further sphere products, where for the lower-dimensional parts the average number of facets are listed. The initial triangulations of the sphere products in Table 4 are produced via product triangulations of boundaries of simplices [Lut03b].

In a separate experiment, we started with a triangulation of S^1 with 10 vertices and with a triangulation of S^2 with 100 vertices as the boundary complex of a random sim-

Varieties of Mathematical Diagrams



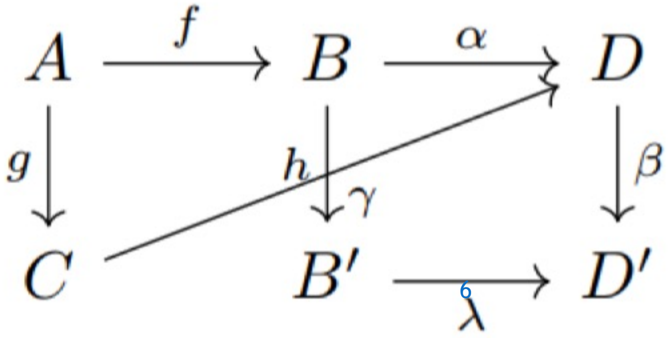




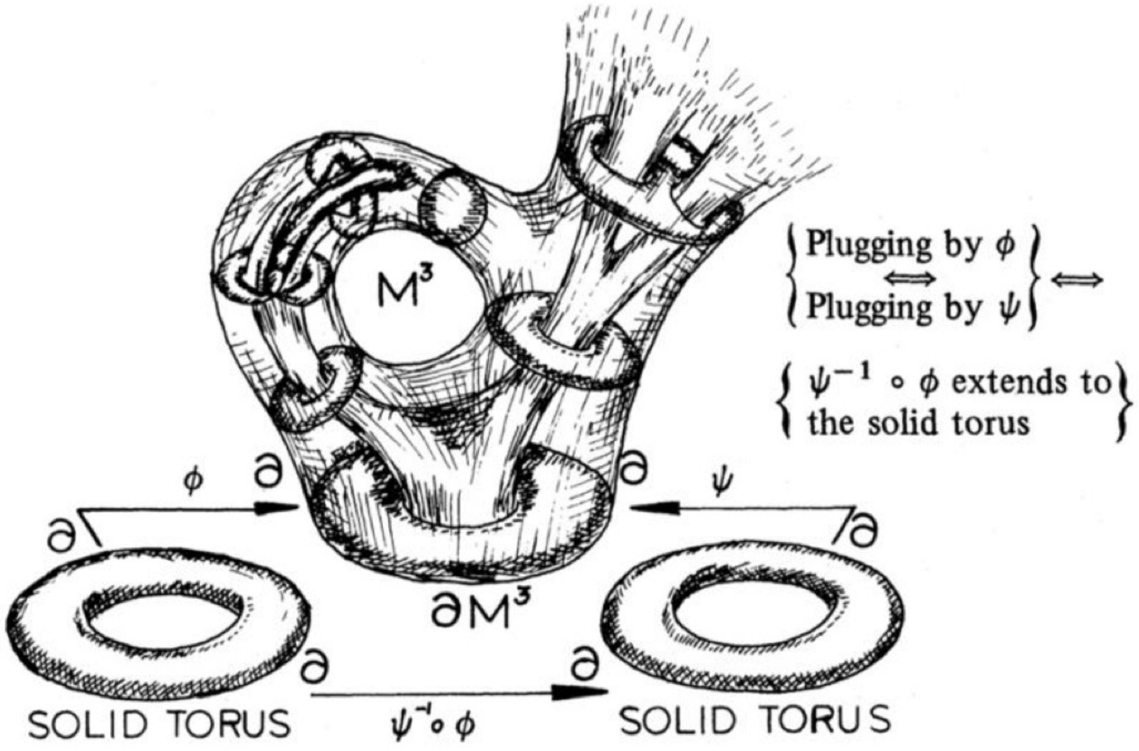
What are Mathematical Diagrams?

Geometric-Topological features

Two-Dimensionality



Diagrams vs. Illustrations



These diagrams are not just there to illustrate, they are used to calculate and to prove results rigorously. (Corfield)

→ Diagrams are SYSTEMATIC

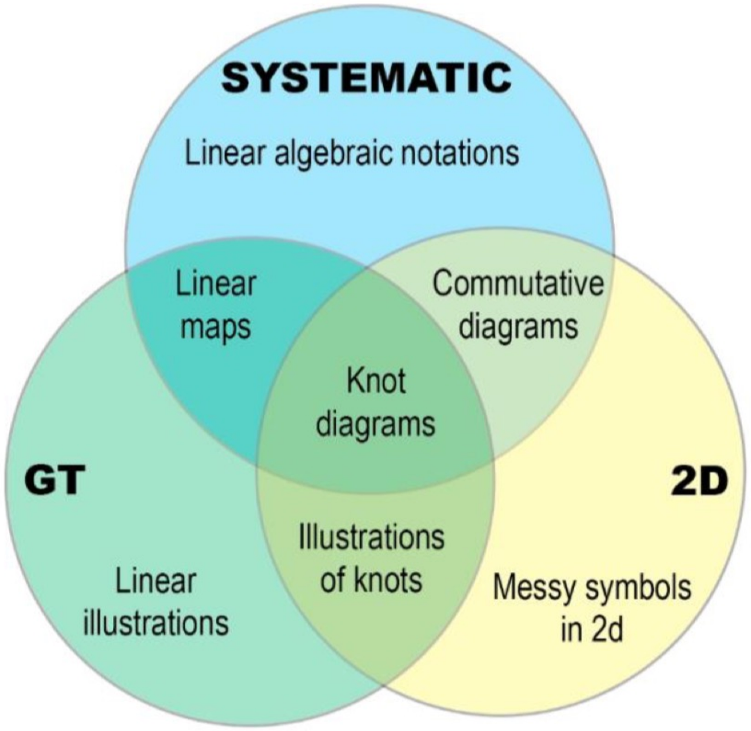
GT Diagrams

- **Definition 1.** A notational item is geometric-topological (GT) if geometric or topological perceptual features are relevant and enable the use of spatio-temporal intuition.

- **Definition 2.** A *GT diagram* is a systematic notational item whose constitutive perceptual features include geometric or topological elements that enable the use of EMI (*Enhanced Manipulative Imagination*).
  - EMI refers to our intuition of space and accounts for the possibility of imagining precise topological and geometric transformations on GT diagrams. It is enhanced by training (and therefore, it is not innate), and it involves kinesthetic sense as well, and thus cannot be reduced to visual imagination alone.

→ Diagrams are not necessarily two-dimensional:

Indianapolis——Bloomington————————Louisville—Michigan City



Diagrams

1) systematic vs. non-systematic,

2) GT vs. non-GT,

3) two-dimensional vs. non-two-dimensional.

Definition 3. A *mathematical diagram* is a systematic notational item that is either GT or two-dimension, or both.

Monoidal Categories

- Category theory is at the same time an abstract mathematical field, where the reasoning is algebraic in nature, and a field in which it is possible to exploit topological intuition, through well-defined GT diagrams.
- This is because of the presence of two different diagrammatic notations in category theory.
- The graphical language for monoidal categories lets us connect algebraic to geometric reasoning:
  - Commutative diagrams: non-GT diagrams.
  - GT diagrams: representation in which we can exploit enhanced manipulative imagination to reason about abstract algebraic structures.
  - Other GT diagrams in which conventions like the one used for knot diagrams.

Categories, Monoidal Categories, and Braided Monoidal Categories

→ See (Selinger 2010)

Categories

Definition. A category $\mathcal{C}$ consists of objects, A, B, C,… and morphisms between them. For each pair of objects A and B, the collection of morphisms between them forms a set: $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Moreover, the following conditions must be satisfied:

1. For any object A, the set $\mathrm{Hom}_{\mathcal{C}}(A, A)$ contains id_A, an identity morphism.
2. A composition, $\circ$, in $\mathcal{C}$ is well-defined. That is, for any three objects A, B, C, and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, the composition $g \circ f$ belongs to the set $\mathrm{Hom}_{\mathcal{C}}(A, C)$.
3. The following relations hold:

   $f \circ id_A = id_B \circ f = f,$

   $(h \circ g) \circ f = h \circ (g \circ f),$

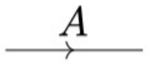
   whenever $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ for objects A, B, C, D in $\mathcal{C}$.

Examples

- $\mathcal{C}$ = Set, the category of sets.
  - Objects are sets and morphisms are maps between sets. For any object A in $\mathcal{C}$, $id_A : A \to A$ is 1_A. Composition of morphisms is just the usual composition of set-theoretic maps.
- $\mathcal{C}$ = $Vect_k$, the category of k-vector spaces, where k is a field.
  - For example, k could simply be R or C. The objects are k-vector spaces and the morphisms are k-linear maps between them. For any object V in $\mathcal{C}$, $id_V : V \to V$ is 1_V. Composition of morphisms is the composition of k-linear maps.
- $\mathcal{C}$ = Grp, the category of groups: objects are groups and morphisms are group homomorphisms between groups.

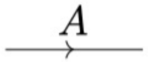
Algebraic and Graphical Languages

Object A

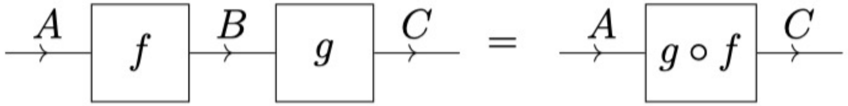


Morphism $f : A \to B$

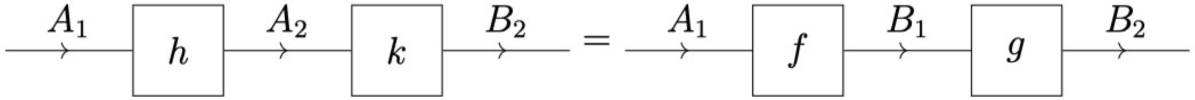
Identity $\mathrm{id}_A : A \to A$

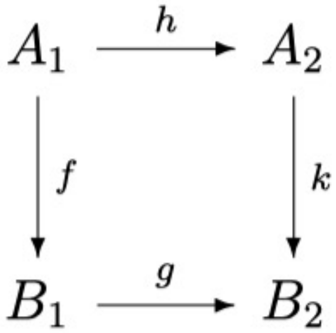


Composition $g \circ f : A \to C$



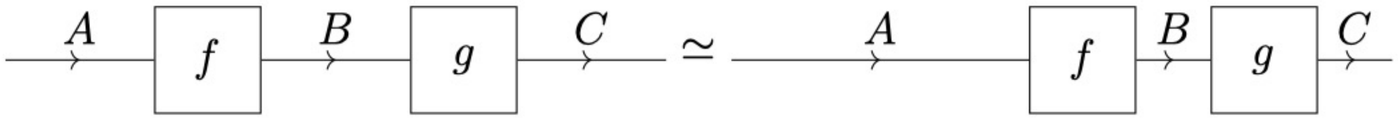
Whereas the graphical language is always linear (for categories without additional structure), the algebraic notation can be either linear or two-dimensional.





Isomorphisms

- Two expressions in the graphical language are *isomorphic* if the boxes and wires of the one expression can be put in a bijection with the boxes and wires of the other expression in such a way as to preserve the connections between boxes and wires.
- In general, we consider expressions in the graphical language up to isomorphism.



Coherence Theorem for the Graphical Language of Categories

A (well-formed) equation is a consequence of the category axioms if and only if it holds in the graphical language up to isomorphism.

Monoids

A monoid (A,e) consists of a set A, a binary operation $*: A \times A \rightarrow A$ and a distinguished element $e \in A$ (the unit element) such that:

1. $e * a = a * e = a$, for any $a \in A$.

2. For any three $a, b, c \in A$ the following holds: $(a * b) * c = a * (b * c)$.

- A monoid is a weaker algebraic structure than a group since the existence of an inverse for each element is not required.
  - For example, take A to be $N \setminus \{ 0 \}$. Taking the binary operation $*$ to be the standard multiplication, one can easily verify that $(N \setminus \{ 0 \}, *)$ with 1 as the unit element is a monoid.

Monoidal Categories

Definition. A *monoidal category* is a category $\mathcal{C}$ with the following additional structure:

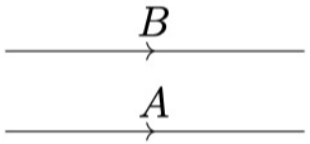
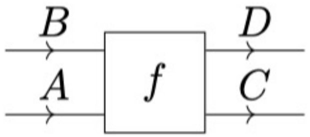
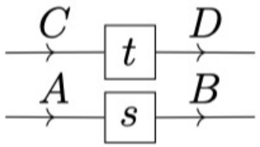
1. For any two objects A, B in $\mathcal{C}$, one can form a new object, $A \otimes B$. $\otimes$ is the tensor product for $\mathcal{C}$.
2. For any two morphisms $f : A \to C$, $g : B \to D$, one can form a new morphism $f \otimes g : A \otimes B \to C \otimes D$, , whenever A, B, C, D are objects in $\mathcal{C}$.
3. Whenever A, B, C are objects in $\mathcal{C}$, one has an isomorphism, the associativity constraint: $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.
4. There exists a distinguished object, I, the unit object, such that, whenever A is an object in $\mathcal{C}$, one has the following isomorphisms: $l_A: I \otimes A \to A$ and $r_A: A \otimes I \to A$.

Moreover, these ingredients are required to be *compatible*. That is, they have to fulfill seven additional equations.

Example

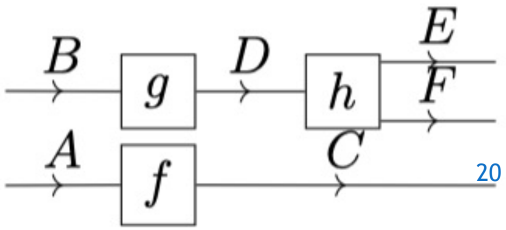
- $\mathcal{C}$ = Set. For any two objects X, Y in C, we can take $X \otimes Y$ to be the Cartesian product $X \times Y$, which also works on maps.
  - Taking any three objects X, Y, Z, the isomorphism $a_{X,Y,Z}$ can be taken to be the canonical isomorphism in Set (the bijection) between the sets $(X \times Y) \times Z$ and $X \times (Y \times Z)$ such that $((x, y), z) \mapsto \rightarrow (x, (y, z))$. As unit object, it is possible to take the singleton $\{ * \}$, which is unique up to bijection.
  - For any object X, l_X is then the obvious bijection between $\{ * \} \times X$ and X (and similarly for r_X). It is then possible to verify that all the additional compatibility conditions are fulfilled, turning $(\text{Set}, \times, \{ * \}, a, l, r)$ into a monoidal category.

Algebraic and Graphical Languages

Tensor product	$A \otimes B$	
Unit object	I	(empty)
Morphism	$f : A \otimes B \to C \otimes D$	
Tensor product	$s \otimes t : A \otimes C \to B \otimes D$	

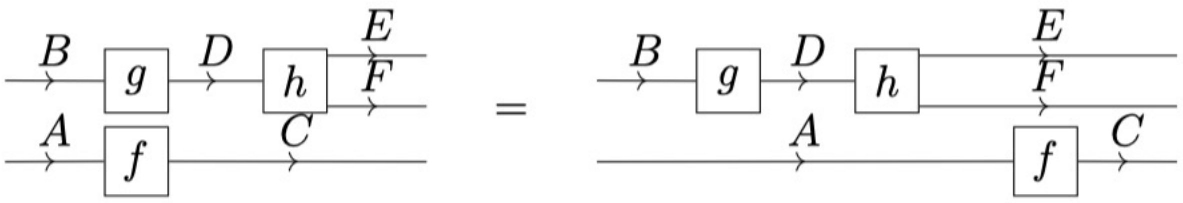
Now both languages can be two-dimensional

$$(\mathrm{id}_C \otimes h) \circ (f \otimes g) : A \otimes B \xrightarrow{f \otimes g} C \otimes D \xrightarrow{\mathrm{id}_C \otimes h} C \otimes (F \otimes E)$$



Equivalent Expressions

Two diagrams, drawn in a rectangle in the plane with incoming and outgoing wires attached to the boundaries of the rectangle, are *planary isotopic* if it is possible to transform one to the other by continuously moving around boxes in the rectangle, without allowing boxes or wires to cross each other or to be detached from the boundary of the rectangle during the moving.

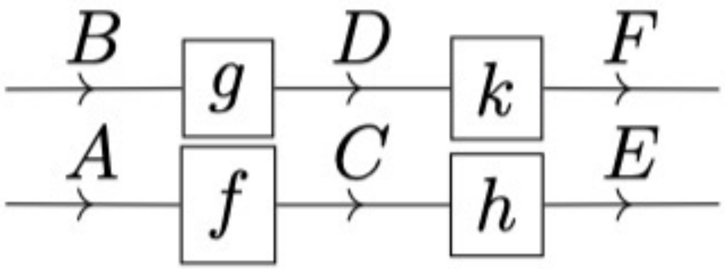


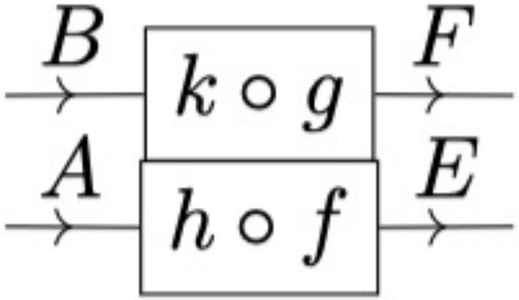
$$(\mathrm{id}_C \otimes h) \circ (f \otimes g) = (f \otimes \mathrm{id}_F \otimes \mathrm{id}_E) \circ (\mathrm{id}_A \otimes h) \circ (\mathrm{id}_A \otimes g).$$

Why it Matters?

Compatibility of the Two Operations:

$$(h \otimes k) \circ (f \otimes g) = (h \circ f) \otimes (k \circ g)$$



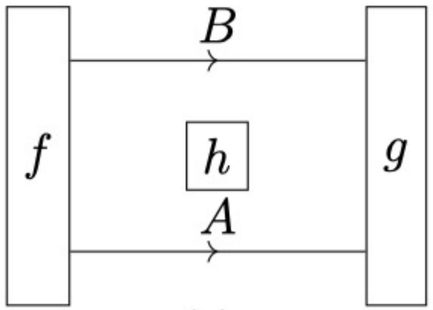


→ FREE RIDES

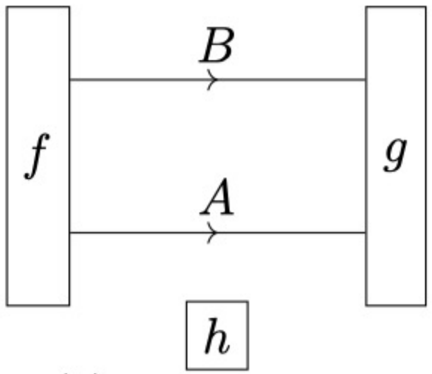
Checking Equivalences

$$\alpha := g \circ ((r_A \circ (\mathrm{id}_A \otimes h) \circ {r_A}^{-1}) \otimes \mathrm{id}_B) \circ f$$

$$\beta := g \circ ((l_A \circ (h \otimes \mathrm{id}_A) \circ {l_A}^{-1}) \otimes \mathrm{id}_B) \circ f$$



(a)



(b)

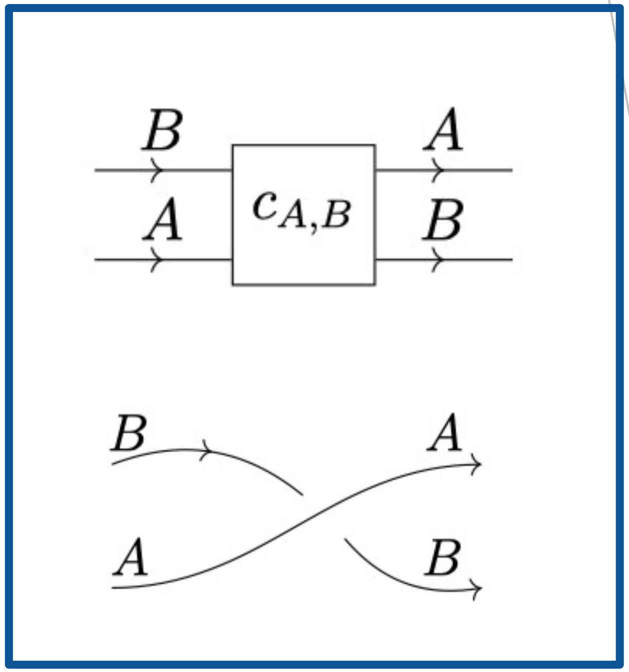
Coherence Theorem for the Graphical Language of Monoidal Categories

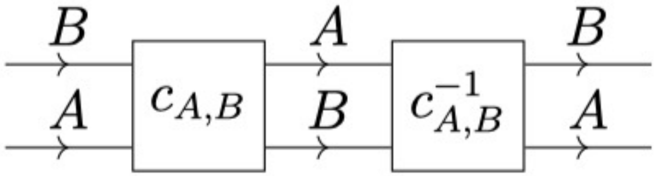
A (well-formed) equation is a consequence of the monoidal category axioms if and only if it holds in the graphical language up to planar isotopy.

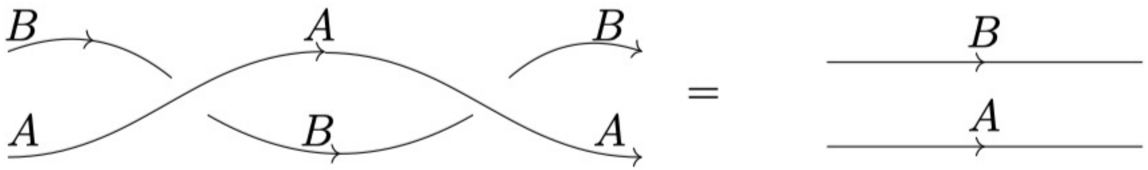
Note that because of the coherence theorem, it is not actually necessary to memorize the axioms of monoidal categories: indeed, one could use the coherence theorem as the definition of monoidal category! (Selinger 2010)

Braided Monoidal Categories

In certain monoidal categories it is possible to define an additional braid structure. Given two arbitrary objects A and B in a monoidal category, a braiding c gives a natural isomorphism: $c_{A,B}$: A ⊗ B → B ⊗ A

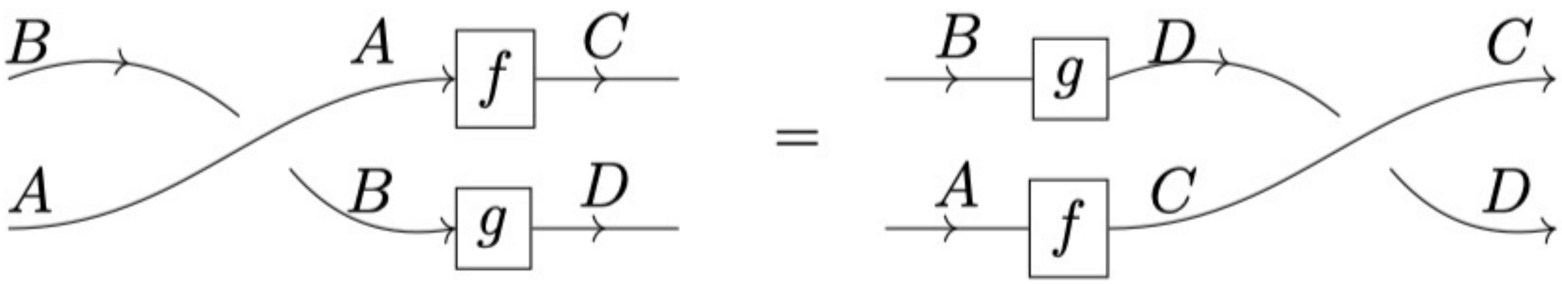






$$c^{-1}_{A,B} \circ c_{A,B} = \mathrm{id}_{A \otimes B}$$

Sliding Boxes



- This convention is easily interpreted and used correctly.
- As with knots, we interpret two-dimensional diagrams as representing curves in three-dimensional space and exploit enhanced manipulative imagination to manipulate them.

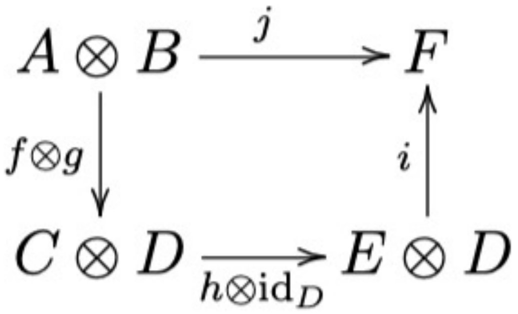
Coherence Theorem for the Graphical Language of Braided Monoidal Categories

A (well-formed) equation is a consequence of the braided monoidal category axioms if and only if it holds in the graphical language up to three-dimensional isotopy.

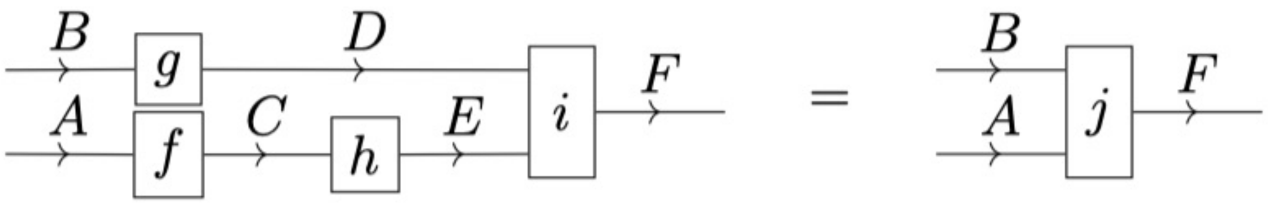
- Discussion

Different "languages"

$$j = i \circ (h \otimes \mathrm{id}_D) \circ (f \otimes g)$$



- Expressiveness
- Calculability
- Transparency



Graphical Language: Expressiveness

- The same information can be expressed via the algebraic and the graphical languages.
  - We extend the graphical language to two dimensions for monoidal categories and we add a special crossing convention for braided monoidal category.
- We can also adopt further conventions to extend it to the case of 2-categories.

Graphical Language: Calculability

- Thanks to the coherence theorems, we know that topological features of the graphical language can be mapped back to properties of the categories at issue.
  - Diagrammatic and algebraic inferences perfectly match each other.
- The fact that we can translate back and forth from the graphical to the algebraic language does not imply that the graphical language is dispensable.
- The graphical notation allows the practitioners to grasp immediately certain relations, avoiding long and tedious calculations.

Graphical Language: Transparency

- The algebraic diagrammatic language allows us to grasp immediately combinatorial properties of the represented structure and to perform meaningful algebraic operations on the representation.
- Being GT, the graphical language enables us to use EMI.
- **Free rides** are possible in the graphical language. Complicated algebraic relations are immediately appreciable from the graphical notation.
- In many cases it would be cumbersome and difficult to express certain relations (e.g., the compatibility conditions) with the algebraic language; the graphical language allows for a much simpler and more mathematically tractable representation.
- The type of intuition enabled by the graphical language is specifically topological.

Between Algebra and Geometry

Higher-dimensional algebra blurs the distinction between topology and algebra. Pieces of algebraic notation are taken as dimensioned topological entities inhabiting a space. Deformation within that space then corresponds to calculation. In this way, higher-dimensional algebra accounts for many uses of diagrams as means with which to calculate and reason. (Corfield)

- By their duality, the two diagrammatic notations analyzed, graphic and algebraic, connect topological and algebraic reasoning in mathematics.
- It is common to appeal to algebra to arrive at topological results, for example with algebraic invariants in algebraic topology: “Now, we work the other way around, using topological objects to allow us to calculate in algebra.”

Reliability

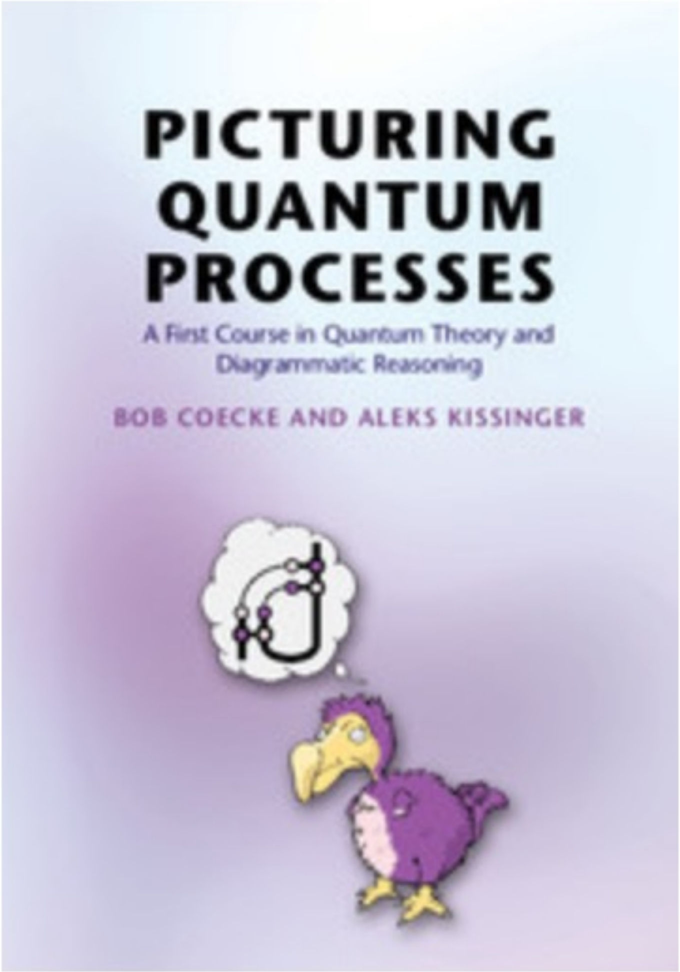
> For practical purposes, reasoning in the graphical notation is almost always easier than reasoning from the axioms. On the other hand, the graphical definition is not very useful when one has to check whether a given category is monoidal; in this case, checking finitely many axioms is easier. (Selinger 2010)

> having followed a representation of the proof written in standard linear notation, one has no reason to be more confident in its correctness than having followed the pictures. (Corfield, 2003)

- The classical challenge to the use of diagrams according to which reasoning with diagrams would lead to the introduction of mistakes does not apply.
- It is true that adopting the graphical notation does not give the guarantee of producing an error-free argument.
- Nothing does!

Picturing Quantum Processes

The diagrammatic language as it currently stands allows for intuitive reasoning about interacting quantum systems, and trivializes many otherwise involved and tedious computations. [...] As a logic, it supports 'automation': it enables a (classical) computer to reason about interacting quantum systems, prove theorems, and design protocols. (Coecke)



By far ***the*** coolest thing about monoidal categories is that they admit a purely pictorial calculus, and these pictures automatically account for the logical mechanisms which we intuitively perform. (Coecke)

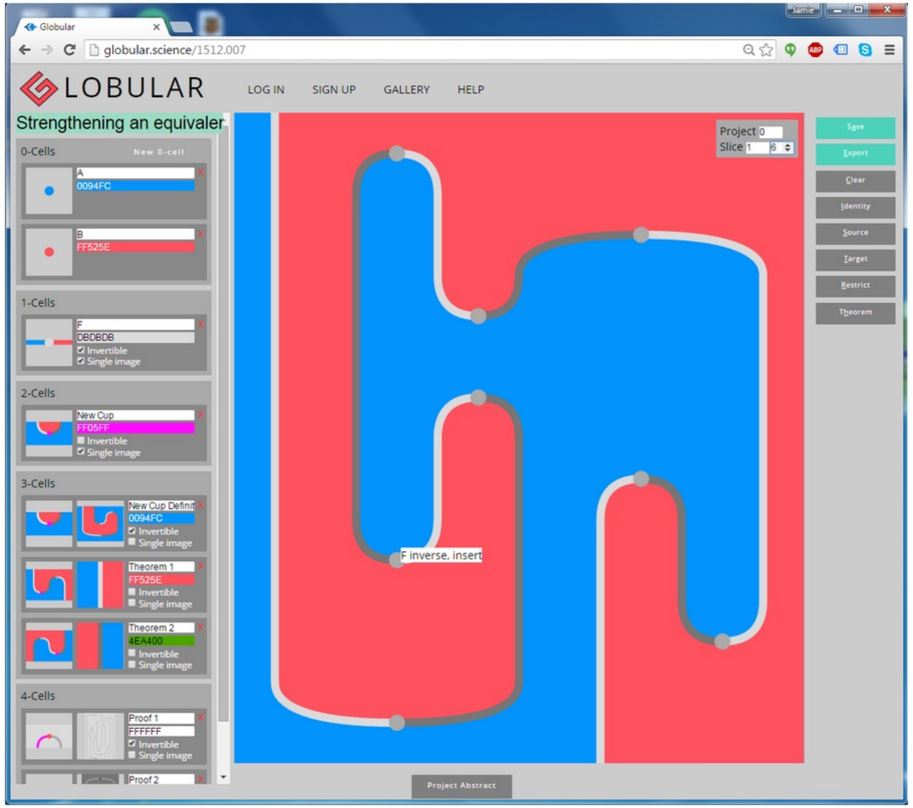
Extensions… ?

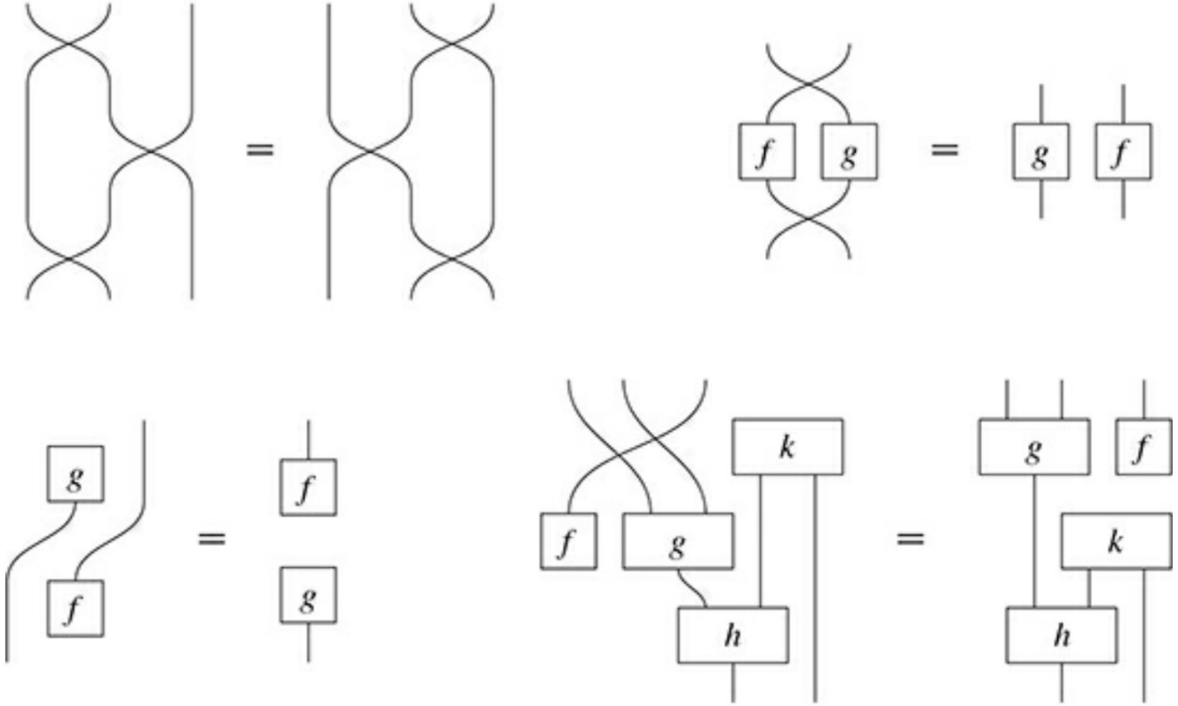
- A natural extension of the case study would be to broaden the domain of the case to 2-categories, and more generally to higher-dimensional categories.
- We need to encode more information into the notations, and this is done by adopting different colorings.
  - For 2-categories, boxes correspond … regions in the plane to objects.
- In the higher-dimensional case we …
- Even without the proof of soundnes… and, as Jamie Vicary (2014) comme…
- This is evidence that the graphical notation, and the graphical calculus it supports, are extremely useful in the practice: even without the proof that the graphical language can be used rigorously, researchers in the field continue to use it extensively.

2021 UPDATE from Vicary: Bicategories are just 'multicoloured' monoidal categories, i.e. where coloured regions are allowed. The proof of correctness of the graphical calculus for monoidal categories passes immediately for bicategories. The details are boring so we didn't include those (and as far as I know no one has published that anywhere either), but it is true.

http://globular.science/

"Globular is a web-based proof assistant for finitely-presented semistrict globular higher categories. It allows one to formalize higher-categorical proofs in finitely-presented n-categories, visualize them as string diagrams, and share them with collaborators, or with the world."





Thank you for your attention!