Controlled/Certifiable Natural Mathematics (CNM)

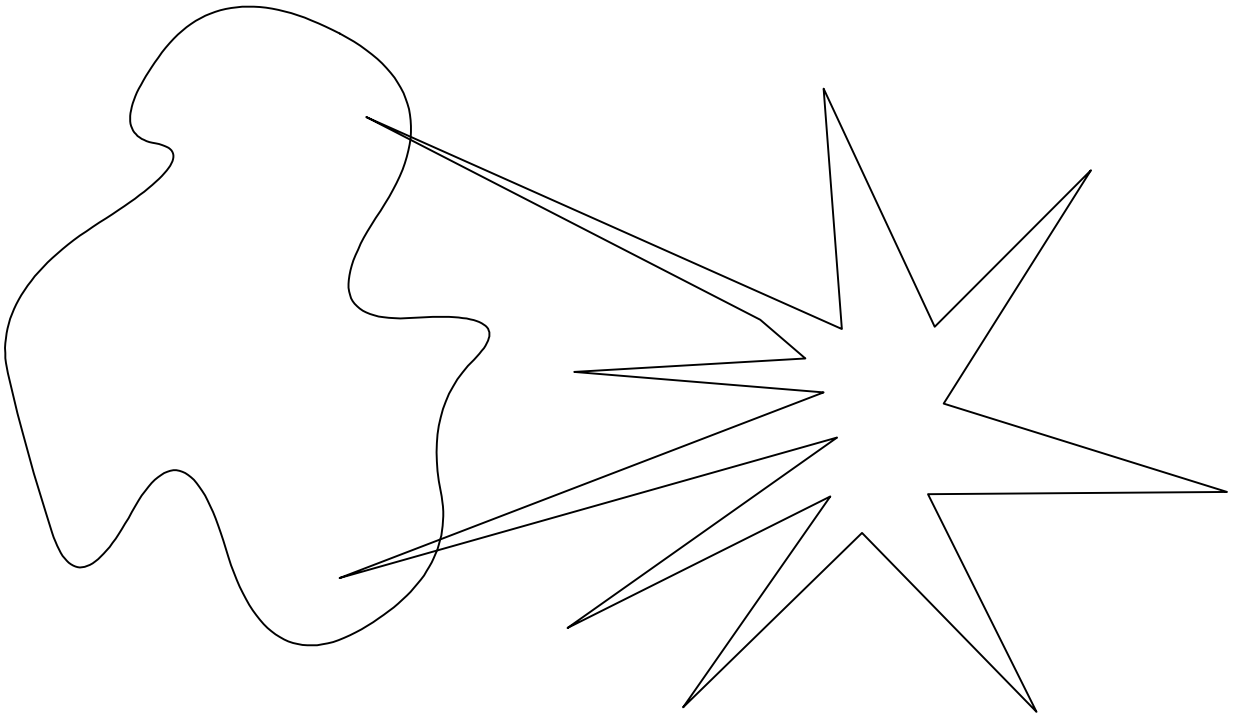
by Peter Koepke

University of Bonn, Germany, https://www.math.uni-bonn.de/ag/logik/

Text-Driven Approaches to the Philosophy of Mathematics (TDPhiMa 2)
Online, 1–3 September 2021

Can one restrict/regulate/control natural mathematics to a fully formal subset which still covers wide parts of pure mathematics?

Natural mathematics and formal mathematics



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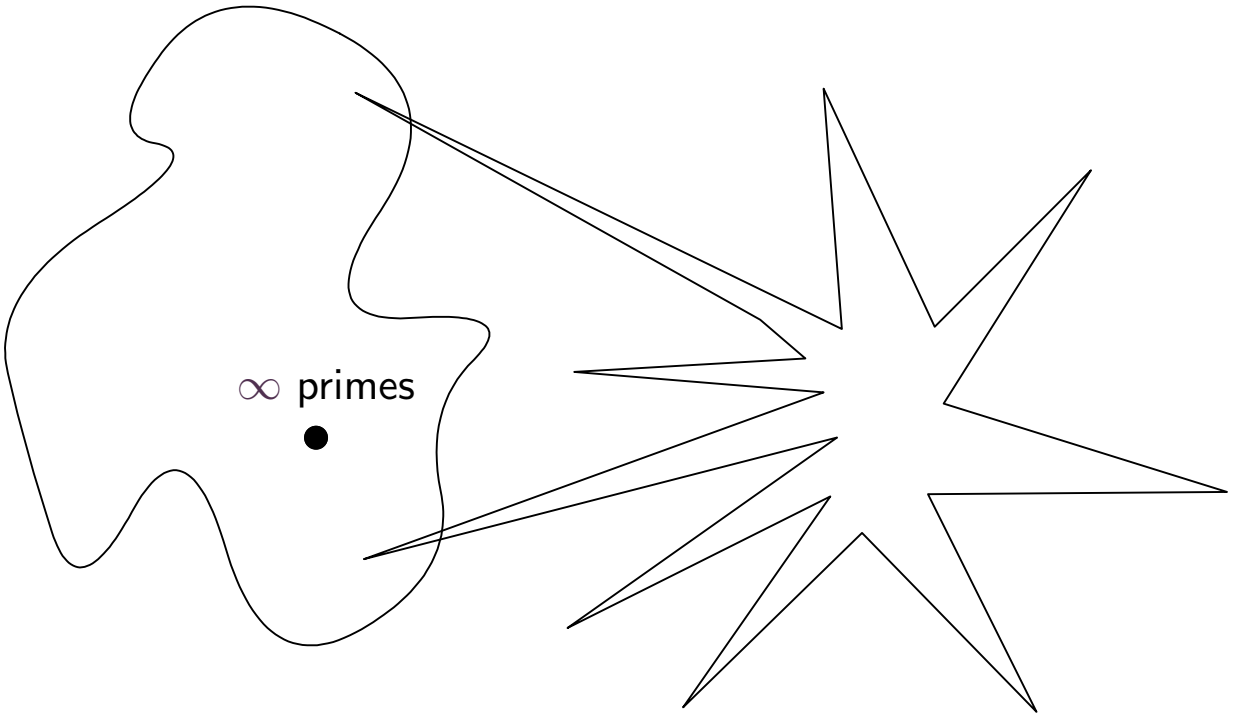

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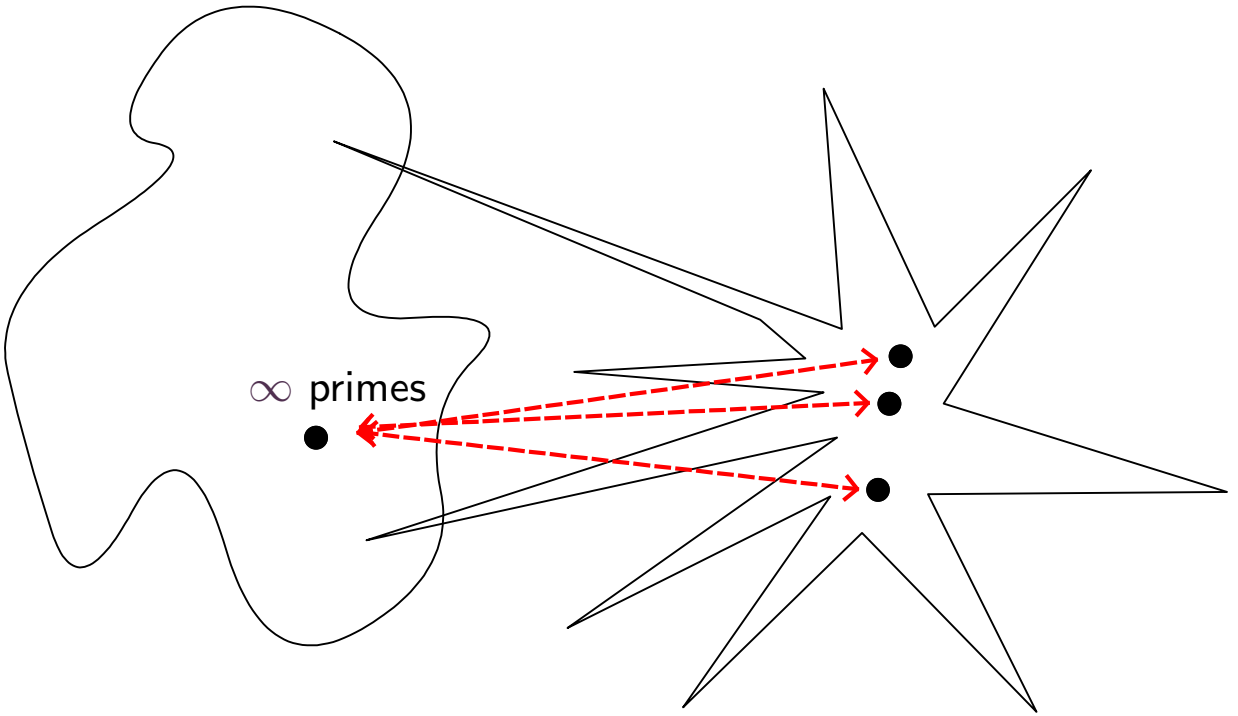

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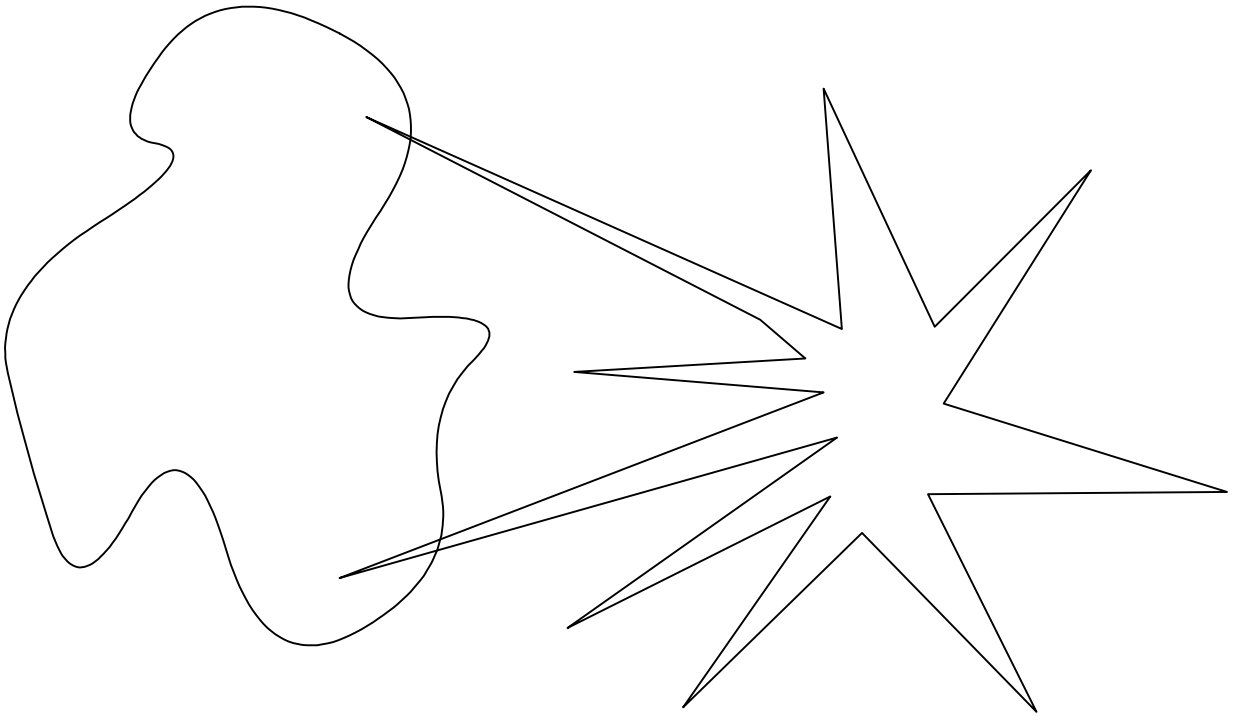

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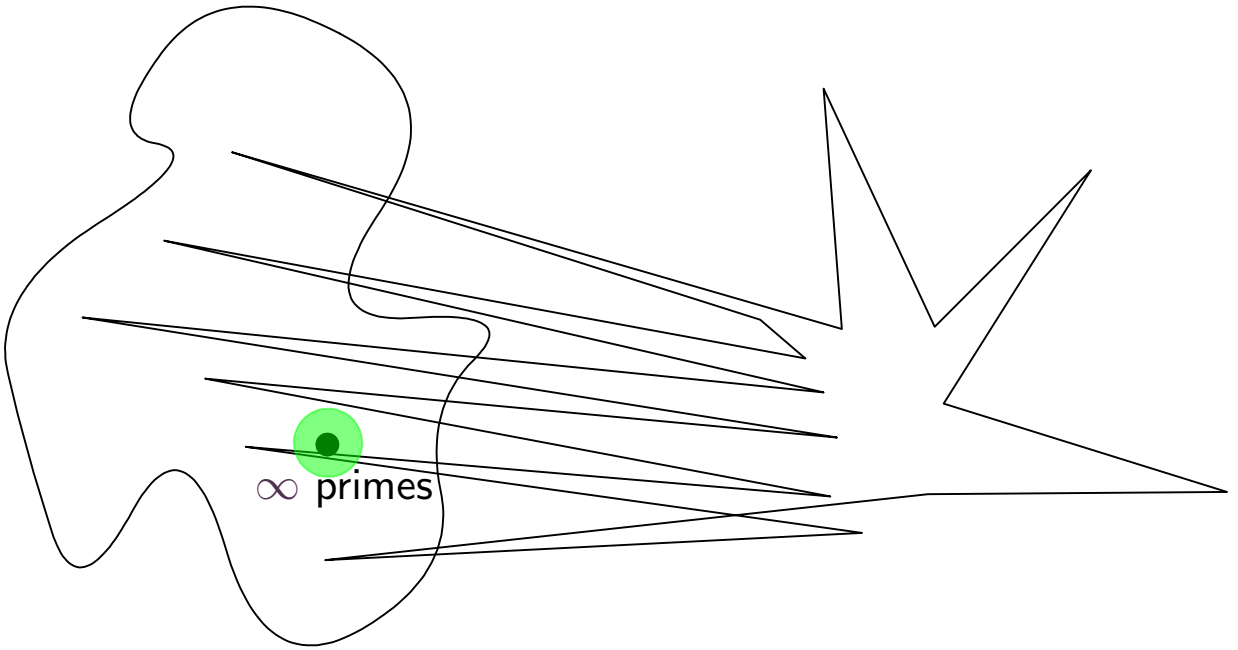

Overview

- Formalizing intuitive mathematical statements

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- $\mathbb{N}$aproche and CNM
- Towards current mathematical research

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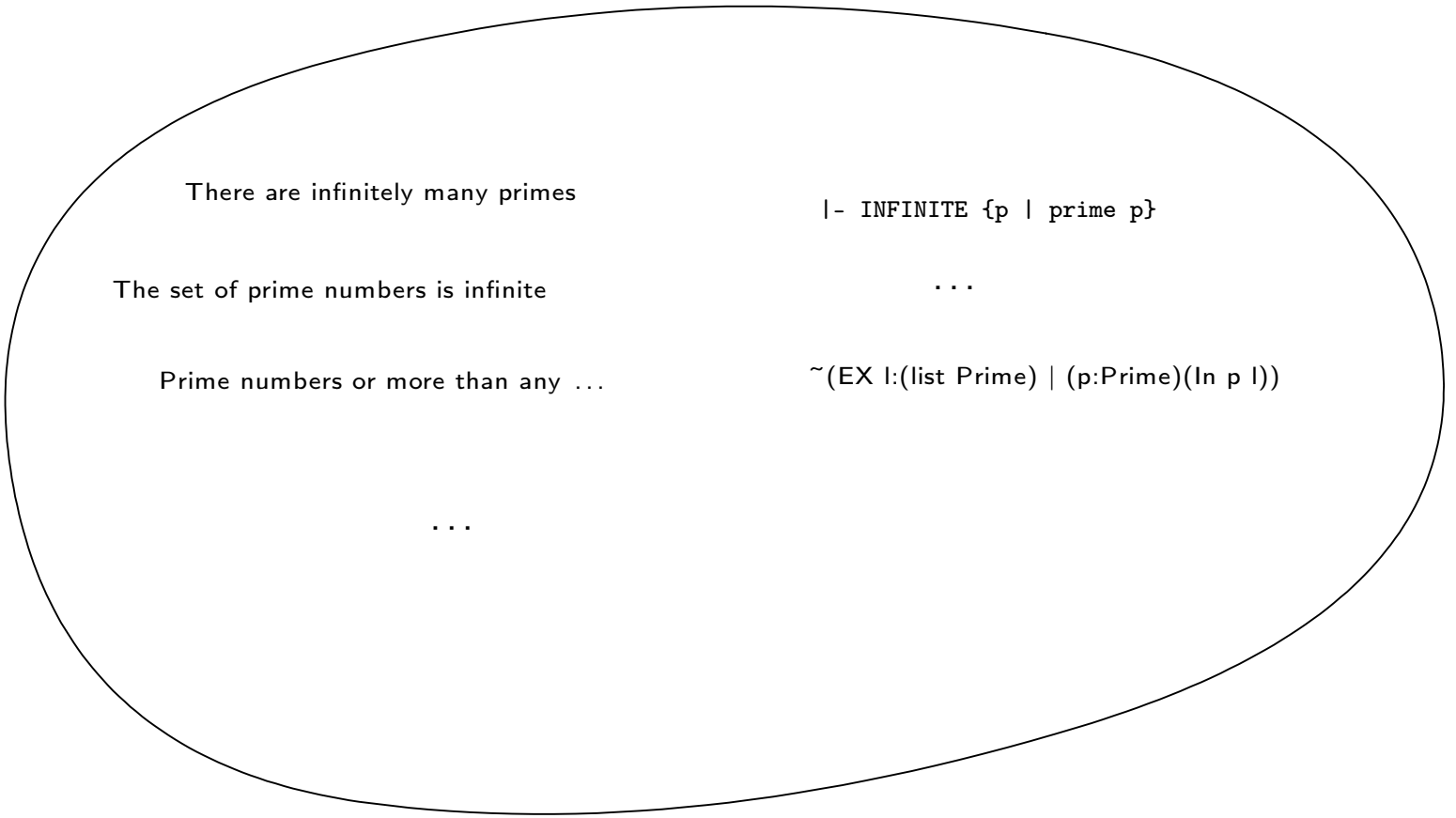

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The (natural) language of mathematics

- Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomophic, and let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth closed curve. Then:

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- M. Ganesalingam. *The Language of Mathematics.* 2013.
- Ganesalingam: *. . . long-term project to construct a computer language for expressing pure mathematics in a way that was as close as possible to real mathematics. . . .*
  . . . half of sentences drawn from textbooks can be expressed without any changes.

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- `A customer enters a card X and a code Y.`
  `If the code Y is valid then the ATM accepts the card X.`

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- Naproche Project, 1st Phase (Bernhard Schröder, PK, $\sim$ 2002), Naproche System (Marcos Cramer, 2013)

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- ℕaproche as a proof language for Isabelle/HOL

The infinitude of primes in Euclid and ℕaproche

M. Aigner, G. Ziegler, *Proofs from the Book*:

For any finite set $\{p_1, \ldots, p_r\}$ of primes,

consider the number $n = p_1\, p_2 \cdots p_r + 1$.
This n has a prime divisor p.
But p is not one of the p_i:

otherwise

p would be a divisor of n and of the product $p_1\, p_2 \cdots p_r$ and thus also the the difference
$n - p_1\, p_2 \cdots p_r = 1$,
which is impossible.
So a finite set $\{p_1, \ldots, p_r\}$ cannot be the collection of *all* prime numbers. □

ℕaproche formalization:

Signature. $\mathbb{P}$ is the class of prime natural numbers.

Theorem (Euclid). $\mathbb{P}$ is infinite.

Proof.

Assume that r is a natural number and p is a sequence of length r and $\{p_1, \ldots, p_r\}$ is a subclass of $\mathbb{P}$.
(1) p_i is a nonzero natural number for every i such that $1 \leqslant i \leqslant r$.
Consider $n = p_1 \cdots p_r + 1$.
Take a prime divisor q of n.
Let us show that $q \neq p_i$ for all i such that $1 \leqslant i \leqslant r$.
Proof by contradiction. Assume that $q = p_i$ for some natural number i such that $1 \leqslant i \leqslant r$.
q is a divisor of n and q is a divisor of $p_1 \cdots p_r$ (by factor property, 1).
Thus q divides 1.
Contradiction. qed.
Hence $\{p_1, \ldots, p_r\}$ is not the class of prime natural numbers. □

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- A formal syntax and semantics that can be mapped to existing formal languages and axiomatic systems.
- Identify natural mathematical texts that can be transformed into each other by truth-preserving linguistic equivalences, e.g., "infinite" $\leftrightarrow$ "not finite", or "A and B." $\leftrightarrow$ "A. B."

Formal Abstracts

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- ground every statement and definition in the system in some foundational system for doing mathematics

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- libraries of ForTheL documents with natural import and export mechanisms

Formal mathematics and current mathematical research

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CNM and current research

From P. Scholze, *Étale cohomology of diamonds*:

Definition. *A Tate ring R is perfectoid if R is complete, uniform, i.e. $R^o \subset R$ is bounded, and there exists a pseudo-uniformizer $\bar{\omega} \in R$ such that $\bar{\omega}^p | \, p$ in R^o and the Frobenius map*

$$\Phi \colon R^o / \bar{\omega} \to R^o / \bar{\omega}^p \colon x \mapsto x^p$$

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From the Lean formalization of perfectoid spaces:

```
structure perfectoid_ring (R : Type) [Huber_ring R] extends Tate_ring R :
(complete  : is_complete_hausdorff R)
(uniform   : is_uniform R)
(ramified : ∃ω : pseudo-uniformizer R, ω^p | p in R°)
(Frobenius : surjective (Frob R°/p))
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From a $\mathbb{N}$aproche formalization of perfectoid rings:

Definition. *R is perfectoid iff R is complete and uniform and there exists a pseudouniformizer ω in R such that $\omega^{p,R} | p$ in R^o within R and*

$$\Phi^R \colon R^o / \omega \cong R^o / \omega^{p,R}.$$

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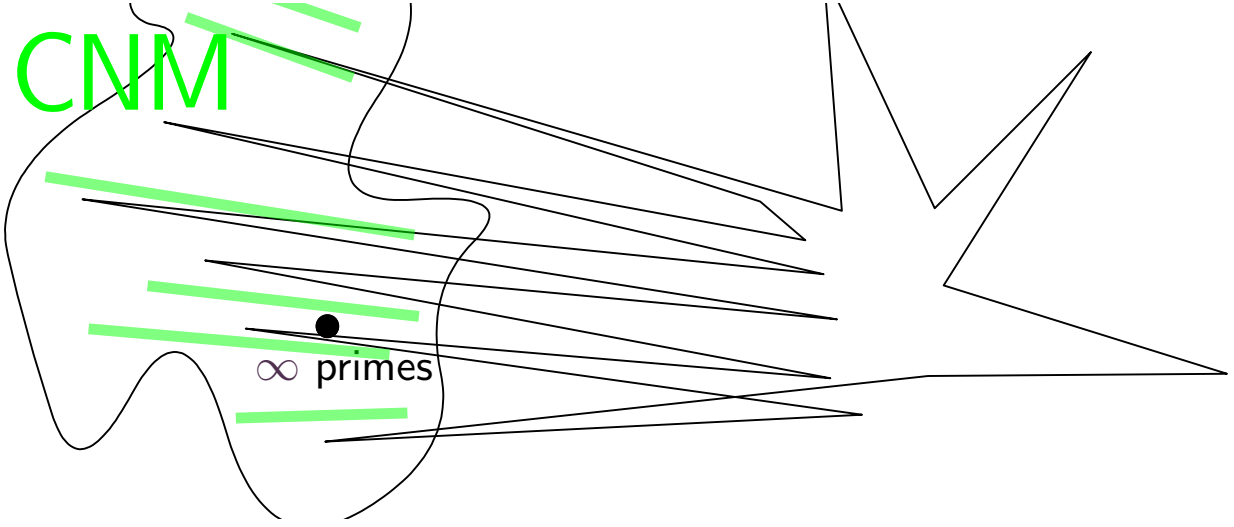
Formal mathematics should be "dense" in natural mathematics,

i.e., formal mathematics should intersect every equivalence class of natural statements and proofs modulo truth-preserving natural language modifications and reformulations.



Thank you