Thetic Phrases in Mathematics

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Outline

The phenomenon: thetic phrases (1).

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Let n be prime.

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Introduction of an *entity* together with *properties* ("being a group" or "being prime") and a *name* for it ("G" or "n").

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Rather unusual communication situation: an entity that is not previously named and whose properties are (in a sense) under the control of the speaker.

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Standard exercises for law students in Germany:

> *T stiehlt aus dem Haus des E Objekte.*
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> *T hat O gewaltsam eine Tasche entrissen, um sich das in ihr vermutete Geld zuzueignen.*
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> *Arzt A gibt der Krankenschwester K eine Spritze mit tödlichem Gift, damit diese das Opfer O tötet.*

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Mathematical equivalents:

Every natural number n satisfies the inequality $n^2 \geq 0$.

If a natural number k bounds the values of a function f from below, then integral $\int_0^1 f(x)\mathrm{d}x \geq k$.

The phenomenon: thetic phrases (3).

We call these phrases *thetic phrases*.

(Greek θετικός from τίθεμι: here *to establish*.)

1. Introducing an entity in combination with a means of referring to it (name / notation); possibly also with properties or restrictions.
2. Typically local: once the local scope is finished, the name/notation can be reused with a different meaning.
3. Often but not always (highly) marked constructions. English construction typically: *let x INF*.

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   "Ponatur quod '*a*' solum significet hominem in propositione falsa in qua ponitur, et solum asinum in propositione vera in qua ponitur. Deinde proponitur hoc 'Homo est *a*'." (Swyneshed, *Obligationes*, 39).

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3. *Fiction.*

   "Once upon a time, there was a tailor whose only worldly possession was a silver ring."

Our claims.

1. Thetic phrases constitute a family of linguistic constructions quite specific for mathematics.
2. The main construction type is quite unique in most languages.
3. These constructions are among the oldest specific for the language of mathematics.
4. The construction type has branched into a number of subtypes which differ pragmatically but not semantically.

Typical uses of thetic phrases (1).

Theorem. Every group whose number of elements is even, contains a nontrivial involution.

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1. The notation *n* is already introduced before the word "Let".
2. The thetic phrase introduces a new assumption, e.g. in a proof by contradiction, or a new case in a case distinction.

Typical uses of thetic phrases (2).

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We observe that

1. the two types of uses of thetic phrases show different pragmatic features,
2. but they rely on a common semantics of this construction.

Outline

The origins of thetic phrases.

Greek. Thetic phrases can be traced back to Euclid's Elements (ca. 300 BC).[1]

ἔστω ἡ δοθεῖσα εὐθεία πεπερασμένηρή AB. (Elements, prop.1)

Let AB be the given finite straight line.

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Latin.

Esto data recta terminata AB. (Paris, Bibliothèque Nationale, Fond 7373, 12th century, from Greek)

Verbi gratia: Ponatur linea recta AB definite quantitatis, (Gerard de Cremona, 12th century from Arabic)

Sit linea assignata AB. (Adelard of Bath, ca. 1140 from Arabic)

[Latin translations: Imperative (3rd person) or subjunctive.]

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Thetic constructions with other verbs.

καὶ πάλιν κέντρῳ τῷ Δ καὶ διαστήματι τῷ ΔΗ κύκλος γεγράφθω ὁ ΗΚΛ. (Elements, prop. 2)

and again, with centre D and distance DG let the circle GKL be described.

Et rursus centro D et spatio DI circulus scribatur IKL. (transl. from Greek)

et etiam describam super centrum D secundum quantitatem longitudinis DZ circulum ZIE. [...] (Gerard)

Item ponatur supra centrum D occupeturque spacium inter D et H circulo supra quem HKL. [...] (Adelard)

[Greek: imperative 3rd person perfect tense passive (other tenses in Greek: present tense, aorist);
Latin: subjunctive present tense active or passive.]

Thetic phrases in German.

Sei G eine Gruppe.

Es sey demnach a+b die Wurzel, [...] (Euler, 1770)

Nehmen wir an, dass G eine Gruppe ist ...

Angenommen, dass G eine Gruppe ist ...

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"Sei *G* eine Gruppe" is a contamination of several constructions:

- *sei* is a subjunctive, resembles the imperative of the 2nd person sg.
- V1 construction is known from imperatives and conditional clauses.

Thetic phrases in French.

Soit G un groupe.
Soient r et s deux nombres ... (Jean III Bernoulli, 1774)

Supposons que la quantité a soit une fraction ordinaire ... (Jean III Bernoulli, 1774)

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1. Note the V1 word order.
2. Grévisse, *Le bon usage*.

   Compare: "La réunion a duré 130 minutes, soit plus de deux heures."

   Grévisse: "*Soit* est souvent invariable ... Pourtant, bien des mathématiciens continuent à écrire: SOIENT *deux triangles*..."

Grammatical constructions for thetic phrases.

(Grammatical) construction: a pairing of form and content which cannot be fully derived compositionally

> *Thetic phrases seem to constitute special constructions in various languages due to their syntactic pecularities.*

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> *Thetic phrases seem to constitute special constructions in various languages due to their syntactic pecularities.*

Nice project to study the grammatical form of thetic phrases in other languages and check whether they a marked by unusual or old-fashioned constructions.

Outline

Semantics of thetic phrases

- conditional clauses with indefinite scope (Euclid: scope is proof of a proposition)
- existential quantifiers (indefinite NPs) are interpreted in the usual dynamic sense of conditionals

  If we discover a planet bearing life, it will fascinate people.

$$(\exists^{\delta} x\ p(x) \wedge BL[x] \wedge D[x]) \rightarrow FP[x]$$

$$\forall x(p(x) \wedge BL[x] \wedge D[x] \rightarrow FP[x])$$

Dynamic Predicate Logic (DPL)

$$(\exists^{\delta} x\ P[x]) \wedge Q[x] \Leftrightarrow (\exists x\ P[x] \wedge Q[x])$$

$$(\exists^{\delta} x\ P[x]) \rightarrow Q[x] \Leftrightarrow \forall x(P[x] \rightarrow Q[x])$$

- Special semantics for $\exists^{\delta}, \wedge, \rightarrow$, cf. (Groenendijk and Stokhof, 1991)

In order to analyse the interaction between dynamic quantifiers and variables we have to consider the linguistic status of variables (and complex notation).

Semantics of DPL

- Dynamic Logic, describes transition between state, here: variable assignments
- $\exists^{\delta} x$: there is a path changing the current variable assignment wrt x which leads to an successful interpretation
- $p \rightarrow q$: For every variable assignment compatible with p, there is a successful path through q, after $p \rightarrow q$ the variable assignment is reset to that before.

Variables in mathematics

Variables and proper names

- uniqueness presupposition (+)
- mostly without determiner (+)
- local reference (-)
- reference can be dependent on the reference of other variables (-)
- rarely pronominalized (-)
- act of naming part of the text (-)

  For any finite set $\{p_1, ..., p_r\}$ of primes, consider the number $n = p_1 p_2, \cdots p_r + 1$. (Aigner/Ziegler 2010, p.3)

Variables and determiners

> *Also existiert [...] ein* $n \in Kern(h) \setminus \{0\}$ *[...] (Müller: Lecture Notes, S. 172)*
> *Um das zu sehen, untersucht man ein großes k [...] (Aigner/Ziegler 2018, 8)*

Variables can be treated as quantifiable appellatives in such cases.

The semantics of thetic phrases (1)

Case: Newly introduced or newly set variable, no indefinite NP

Let n be prime.

$$\begin{aligned} &\exists^{\delta} n \; n \in \mathbb{N} \wedge \text{prime}(n) \rightarrow \cdots \\ \Leftrightarrow\; &\forall n(n \in \mathbb{N} \wedge \text{prime}(n) \rightarrow \cdots) \end{aligned} \tag{1}$$

The semantics of thetic phrases (2)

Case: Newly introduced or newly set variable, indefinite NP

Let G be an abelian group.

$$\begin{aligned} & \exists^{\delta} G\ \mathrm{group}(G) \wedge \exists^{\delta}\gamma\ \mathrm{abelian\_group}(\gamma) \wedge G = \gamma \rightarrow \cdots \\ \Leftrightarrow\ & \exists^{\delta} G\ \mathrm{group}(G) \wedge \mathrm{abelian\_group}(G) \rightarrow \cdots \qquad (2) \\ \Leftrightarrow\ & \forall G(\mathrm{abelian\_group}(G) \rightarrow \cdots) \end{aligned}$$

The semantics of thetic phrases (3)

Case: Given variable

Now, assume that n is even.

$$\exists^{\delta} n \; n \in \mathbb{N} \cdots$$
$$(\text{even}(n) \rightarrow \cdots) \tag{3}$$
$$\cdots$$

The semantics of thetic phrases (4)

Let $P(p + a)$ be the greatest prime divisor of $p + a$.

If p and a are given, the greatest prime divisor is implicitly given (definite NP), only the variable / notation $P(...)$ is new.

Outline

Pragmatic types of thetic phrases

V new variable or redefined variable (+/-)

O new object(s) introduced existentially (+/-)

V+/O+ is the prototypical case; V-/O- introduces a new condition, only. In the case of O+ extistential introduction can be explicit or implicit.

Outline

Thetic phrases as constructions?

- (Grammatical) construction: a pairing of form and content which cannot be fully derived compositionally
- common content core
- diverse syntaktic realisations
- $\Rightarrow$ rather a family of constructions
- syntactic realisations of a certain frame, cf. (Fisseni, Sarikaya, Schmitt, and Schröder, 2019), (Carl, Cramer, Fisseni, Sarikaya, and Schröder, 2021)

What are Frames?

Properties

- a concept in knowledge representation
  $\rightsquigarrow$ Fillmore (1968) and Minsky (1974)
- represent conceptual structure or prototypical situations
  e.g. *birthday celebration*, *restaurant*.
- *roles* and *participants* (slots and fillers)
  e.g. *waiter*, *diners*, *food*, ...
- organized in an *inheritance hierarchy*
  typed feature structures (Carpenter, 1992)

Usage

- e.g., in cognitive linguistics and artificial intelligence
- explain how receiver completes information conveyed by sender

$\rightsquigarrow$ linguistic project: FrameNet database (1,200 semantic frames)

Framing and Frames

One event can be framed differently, e.g. as *buying* and as *selling*

Frame: BUYING

[Buyer John] **bought** [Goods a beautiful medieval book] [Time yesterday] .

Frame: SELLING

[Seller Petra] **sold** [Goods a beautiful medieval book] to [Buyer John] for [Money twenty Euros] .

Frames as feature structures

$$\begin{bmatrix} \textit{buy} & \\ \textsc{Buyer}! & \textit{John} \\ \textsc{Goods}! & \textit{a beautiful medieval book} \\ \textsc{Time} & \textit{yesterday} \\ \textsc{Seller} & \textit{person} \\ \textsc{Money} & \textit{money} \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix} = \begin{bmatrix} \textit{buy} & \\ \textsc{Buyer}! & j \\ \textsc{Goods}! & b \\ \textsc{Time} & \begin{bmatrix} \textit{point-in-time} & \\ \textsc{year} & \textit{2021} \\ \textsc{month} & \textit{09} \\ \textsc{day} & \textit{03} \\ \textsc{hour} & \{1, \ldots, 24\} \\ \textsc{minute} & \{0, \ldots, 60\} \\ \ldots & \end{bmatrix} \\ \textsc{Seller} & \textit{person} \\ \textsc{Money} & \textit{money} \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix}$$

The thetic frame

$$\left[\begin{array}{ll} \textit{thetic} & \\ \textsc{antecedent} & \textit{condition} \\ \textsc{new-objects} & \textit{list-of}\left(\left[\begin{array}{ll} \textit{object} & \\ \textsc{object-id} & \textit{object-id} \\ \textsc{sort} & \textit{sort} \end{array}\right]\right) \\ \textsc{new-var-assign} & \textit{list-of}\left(\left[\begin{array}{ll} \textit{var} & \\ \textsc{var-name} & \textit{var-name} \\ \textsc{sort} & \textit{sort} \\ \textsc{mapping} & \textit{object} \end{array}\right]\right) \\ \textsc{consequent} & \textit{proved-under-hypothes} \end{array}\right]$$

The thetic frame: the protypical case V+/O+

$$
\begin{bmatrix}
\textit{thetic} & \\
\textsc{antecedent} & \textit{condition} \\
\textsc{new-objects} & \textit{ne-list-of}\left(\begin{bmatrix} \textit{object} & \\ \textsc{object-id} & \textit{object-id} \\ \textsc{sort} & \textit{sort} \end{bmatrix}\right) \\
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\textsc{consequent} & \textit{proved-under-hypothes}
\end{bmatrix}
$$

The thetic frame: the subtype V+/O-

$$\left[\begin{array}{ll}
\textit{thetic} & \\
\textsc{antecedent} & \textit{condition} \\
\textsc{new-objects} & \textit{empty-list} \\
\textsc{new-var-assign} & \textit{ne-list-of}\left(\left[\begin{array}{ll} \textit{var} & \\ \textsc{var-name} & \textit{var-name} \\ \textsc{sort} & \textit{sort} \\ \textsc{mapping} & \textit{object} \end{array}\right]\right) \\
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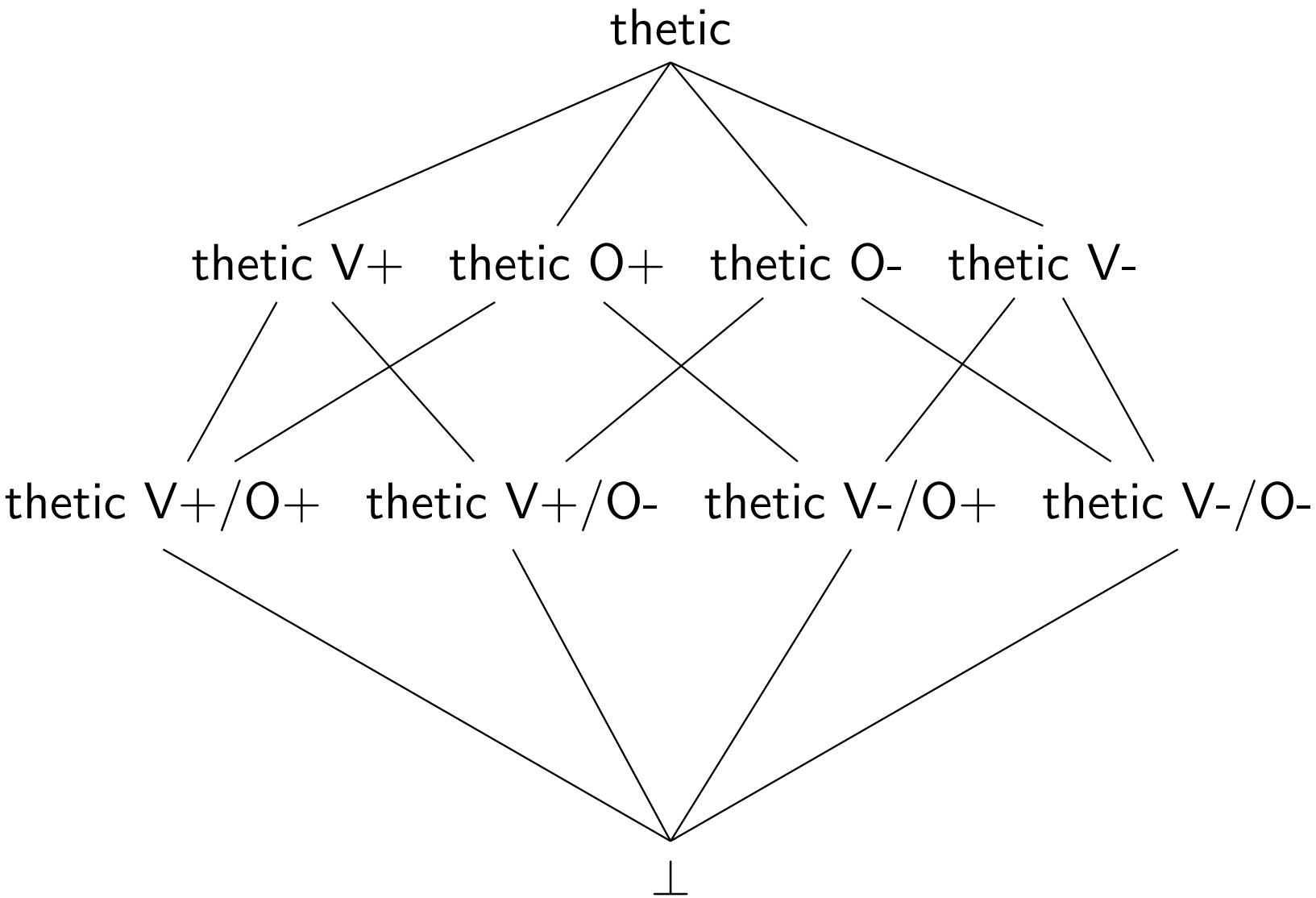
\end{array}\right]$$

Type hierarchy thetic frames



Default inferences in the thetic frames

- indefinite NP $\rightarrow$ O+ $\rightsquigarrow$ V+
- definite NPs $\rightarrow$ O-
- new variable $\rightarrow$ V+
- given variable $\rightsquigarrow$ V-
- proof contexts (e.g. in the constituent parts of an induction proof: V-)
- proof styles, linguistic form
- sortal inferences by interaction of frames

Pragmatic expectations

New objects, new variables (or variable assigments)

- play a major role in the consequent,
- are used in the conclusion of the thetic construction,
- are connected to special slots in the surrounding proof frame, e.g. to a universally quantified variable in a universal claim

Outline

Conclusion

- Thetic phrases can be traced back to Euclid's Elements.
- Thetic phrase are quite marked constructions.
- They have a common conditional semantics with indefinte scope.
- Thetic phrases branch into several subtypes.
- They constitute a family of linguistic constructions which can be described as a type hierarchy of frames.

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