Instructions and constructions: An analysis of proof-writing in Kunen's set theory textbook

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Instructions in proofs

- Most accounts of proof view proof as a series of assertions:
  - Proofs are conceptualized as A → B → C → D…
  - Proofs begin with acceptable starting points and truth “flows downwards” to the new assertions.

- Fenner Tanswell’s insight is that many sentences in proofs are not assertions, but imperatives. They are commands for the reader to take some mathematical actions.

Instructions in proofs: Examples from Kunen's textbook

- "**Fix** $\alpha \in C$" (p. 24).
- "For each α, **pick** a 1-1 map f_α from X_α to κ. **Use** these to **define** a 1-1 map from union *X* into $\kappa \times \kappa$" (p. 30).
- "For $\alpha \in X$, **choose** $C_\alpha \in C$ such that $\mu(p_\alpha \triangle C_\alpha) \leq \delta$" (p. 60).
- "**Observe** that $d(C, D) < \min(h(C), h(D))$" (p. 72).
- "Now **take** $K \in J$ with $K \subseteq K_1$" (p. 74).
- "We may **assume** f_n is the Axiom of Extensionality; if not, just **add** it to the list" (p. 140).
- "**Start** with M satisfying GCH and **apply** Lemma 5.14" (p. 210).
- "For (c), **force** three times to **construct** $M \subseteq N_1 \subseteq N_2 \subseteq N_3$" (p. 216).
- "**extend** the order ≤ on Q to $Q \cup S$ **by putting** $p < s$ for all p in Q and all s in S" (p. 279).
- "As in the MA construction in section 6, **build** in *M* an iterated forcing construction" (p. 286)

[1] Proof. Suppose that (x_n) is a bounded sequence of real numbers.

[2] Let $M = \sup_{n\in\mathbf{N}} x_n$, $m = \inf_{\in\mathbf{N}} x_n$,

[3] and define the closed interval $I_0 = [m, M]$.

[4] Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where $L_0 = [m, (m + M)/2]$, $R_0 = [(m + M)/2, M]$.

[5] At least one of the intervals L_0, R_0 contains infinitely many terms of the sequence, meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in N$ (even if the terms themselves are repeated).

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[10] This is always possible because I_2 contains infinitely many terms of the sequence.

[11] Divide I_2 in half, pick a closed half-interval I_3 that contains infinitely many terms, and choose $n_3 > n_2$ such that $x_{n3} \in I_3$.

[12] Continuing in this way, we get a nested sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \ldots I_k \supset \ldots$ of length $|I_k| = 2^{-k} (M - m)$, together with a subsequence (x_{nk}) such that $x_{nk} \in I_k$.

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[15] Furthermore, since $x_{nk} \in I_K$ for all $k > K$ we have $|x_{nj} - x_{nk}| < \varepsilon$ for all $j, k > K$.

[16] This proves that (x_{nk}) is a Cauchy sequence, and therefore it converges by Theorem 3.46. (Hunter, 2014, p. 89)

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Instructions in proofs

But proof in turn involves the idea of an argument, a narrative structure of sentences, and sentences can be in the imperative rather than the indicative. […] Mathematics is so permeated by instructions for actions to be carried out, orders, commands, injunctions to be obeyed — 'prove theorem T', 'subtract from y', 'drop a perpendicular from point P onto line L', 'count the elements of set S', 'reverse the arrows in diagram D'[…] that mathematical texts seem at times to be little more than sequences of instructions written in an entirely operational, exhortatory language." (Rotman, 1988, p. 8)

Instructions in proofs: Why should we care?

- Tanswell (in press) noted that assertions and instructions have different truth semantics.
- The truth value of an assertion can be true or false or indeterminate. It makes sense to say the truth of one assertion necessitates the truth of another. But instructions can neither be true or false.
- Models of proof as a sequence of assertions are, at best, incomplete. If we want to understand how proof works, we need to understand the role that instructions play.
- And given the different truth semantics, analysis of instructions can potentially highlight new ways that proofs operate.

STUDIES IN LOGIC
AND
THE FOUNDATIONS OF MATHEMATICS

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Set Theory

An Introduction to Independence Proofs

KENNETH KUNEN

Why textbooks? Why Kunen?

- Graduate textbooks represent serious mathematics, but mathematics that is intelligible to a non-trivial portion of the philosophical community.
- Quality textbooks are generally regarded as sufficiently rigorous and are sometimes models of good writing.
- While the styles of writing may not generalize (not even to other textbook writers in that discipline), these textbooks probably have a considerable effect on how mathematicians practice their discipline.

Methods

- I recorded every instance of an instruction inside a proof, and in the prose of a chapter (the “outer text”).
- I did not code for the preface, appendices, homework exercises, or “numbered” statements (theorems/definitions/lemmas/etc.)
- Instructions were when the reader was told to take an action, commonly as an imperative or with the use of “we”.

Methods

- The analysis was qualitative and interpretive and required judgment.

"Since κ is regular, we can pick $g(\zeta)$ so that $\zeta < g(\zeta) < \kappa$ and $G(\zeta) \subset g(\zeta)$. As in the proof of Lemma 6.8, let g^n be the nth iterate of g and $g^\omega(\zeta) = \sup_n g^n(\zeta)$" (p. 80, proof of Lemma II.6.13).

"We show how to find such a $\beta > \alpha$. Let $\beta_0 = \alpha$ and let β_{p+1} be the largest of…" (p. 173, proof of Theorem.IV.7.5).

"We conclude this section by making a case that all reasonable mathematics takes place in WF" (p. 98)

Results:
Instructions were common

- There were 262 proofs analyzed. 196 (74.8%) contained at least one instruction.
  - Some that did not were trivial. "(1) is by induction on *n*. (2) follows from Lemma 10.5" (p. 28, one line proof of Lemma I.10.5)

- There were 151 proof that were five or more lines long. 143 (94.7%) of those contained at least one instruction.

- There were 877 instructions in proofs. (517 in non-proof text).

- Instructions were more frequent in proof text (36.2 per every 100 lines of text) than in non-proof text (13.3 per every 100 lines of text)

Results: There was a variety of instructions

- Eight instruction verbs accounted for over 80% of the instructions in the proofs
  - Let (333), Fix(122), Assume (75), See (75), Define (31), Note (27), Suppose (26), Apply (25). (out of 877 total instructions)

- Nonetheless, there was variety and domain-specific instructions
  - We may start with M satisfying ZFC + GCH
  - Again, suppress mention of $\tau_1 \ldots \tau_n$.
  - We may well-order the elements of A.
  - For (c), force three times.
  - extend the order ≤ on Q to $Q \cup S$
  - We totally order X

Results:
Five types of instructions

Type of instruction	Proof (N = 877)	Non-proof prose (N = 517)
Inclusive	232 (26%)	137 (26%)
Exclusive	455 (52%)	46 (9%)
Observe	53 (6%)	61 (11%)
Derive	48 (5%)	3 (0.6%)
Reference	52 (6%)	244 (47%)
Other	37 (4%)	26 (5%)

Inclusive instructions

- Rotman (1988) defined *inclusive imperatives* as those that involved creating shared mathematical discourse by introducing shared standards, notations, and referents.

- Inclusive instructions in my coding involved any stipulation that established a convention *OR* posited the existence of an object (a set or formula) that was not based on immediate previous objects in the surrounding discourse.

- Common inclusive instructions were: Assume, suppose, let, and fix

Exclusive instructions

- Rotman (1988) defined *exclusive imperatives* as those used to operate within a fixed discourse by manipulating mathematical objects

- Exclusive instructions in my coding involved transforming objects or building new objects from previous objects that have been recently referenced.

- Common inclusive instructions were: Choose, pick, let, fix, and define.

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Constructions

Many of Kunen's set theory proofs are of the form:

"If you give me an object X with property P, then I will show you how to use X to construct a Y with property Q"

Lemma II.6.8 (a) The intersection of any family of less than cf(μ) club sets on μ is club.

If you give me a collection $\{C_\alpha | \alpha < \lambda < cf(\mu)\}$ and a $\zeta < \mu$, I will show you how to construct a β such that $\zeta < \beta < \mu$ and β is a member of each C_α.

Constructions

L does not satisfy Suslin's Hypothesis.

(a) If you give me a model of L, I will show you how to construct a diamond sequence in L.

(b) If you give me a diamond sequence, I will show you how to build a Suslin tree.

CH is independent of the ZFC axioms.

(a) If you give me a model V of ZFC, I will show you how to construct a model L that satisfies ZFC + CH.

(b) If you give me a model V of ZFC, I will show you how to construct a model M[G] that satisfies ZFC + not CH.

Constructions

(1) Constructions begin with **inclusive instructions** stipulating the object(s) X with properties P that you are starting with.

(2) The construction then has a series of **exclusive instructions**, showing how to build a succession of objects (often by manipulating X or using X as a tool) to obtain a new object Y.

(3) The new object Y is then shown to have the desired properties.

(4) For (2) and (3), there is often an **observation inference** that an instruction is possible to implement an instruction or that an object has a desired property. This in turn is justified by a sub-proof, or less often, with a **derive instruction** or a **reference** to a previous result.

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The proof starts with an inclusive instruction stipulating the object with the property of being a bounded sequence.

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[6] Choose I_1 to be one of the intervals L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbf{N}$ such that $x_{n1} \in I_1$.

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The proof then continues with a series of exclusive instructions on how to build a convergent subsequence.

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The proof concludes by verifying that the sequence of terms that was produced was, in fact, a convergent subsequence.

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Lines 5, 8, and 10 are observations that justify why it is possible to perform the instructions in lines 6 and 9.

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Theorem II.4.4. If there is a Suslin line, then there is a Suslin line X such that

(1) X is dense in itself. (i.e., $a<b \to (a,b) \neq 0$)

(2) no non-empty open subset of X is separable.

Proof., Let Y be any Suslin line. Define an equivalence relation ~ on Y by setting x ~ y iff the interval between them is separable. Let X be the set of ~-equivalence classes. If $I \in X$, then I is convex; i.e., $x, y \in I$ and $x < y \to (x, y) \subset I$. We totally order X by setting $I < J$ iff some (any) element of I is less than some (any) element of J.

Note that each $I \in X$ is separable. To see this, let M be a maximal disjoint collection of non-0 open intervals of the form (x, y) with $x, y \in I$. M is countable since Y has the c.c.c., so let $M = \{(x_n, y_n) | n \text{ in } w\}$ […]

To see that X is dense in itself, suppose $I < J$ and $(I, J) = 0$. Pick $x \in I$ and $y \in J$

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The proof is of the form, “If you give me a Suslin line, I will show you how to use it to construct a nice Suslin line”.

So the proof starts with the inclusive instruction of fixing a Suslin line.

Theorem II.4.4. If there is a Suslin line, then there is a Suslin line X such that

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There is a sequence of exclusive inferences where new objects are defined in terms of Y (and each other) to produce a set and a total order that will be the nice Suslin line.

The convexity of I is noted to show the total ordering is well-defined.

(X, <) and its elements are then shown to have properties.

A reference instruction claims that each element of X is separable. This is then proven with a sub-construction (with exclusive instruction).

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1. Standard accounts of proof describe proofs as sequence of assertions. Instructions in proofs are interesting because they are not assertions and have different truth semantics.

2. At least in Kunen's *Set theory*, instructions are common and appear in the majority of proofs.

3. In Kunen's text, the types of instructions in proofs differs from expository text. Notably, proofs have far more exclusive instructions in which new objects are defined in terms of previous objects.

4. Many proofs are constructions. They show the reader how an object X with property P can be used to construct an object Y with property Q via a sequence of explicit instructions.

Can't we just translate instructions to assertions?

Can't we just translate instructions to assertions?

Observational instructions:

As in instruction: Note that A

As an assertion: A

Inclusive instruction:

As in instruction: Assume that A. (Followed by B and C and D).

As an assertion: $A \rightarrow B$. $A \rightarrow C$. $A \rightarrow D$.

Exclusive instruction:

As an instruction: Choose object X with property P.

As assertions: There exists an object with property P and

Assume that X is an object with property P.

Metaphorical talk on acting on Suslin lines and trees

Our definition of Suslin lines allowed for lines which could be very bad; for example, X could have gaps in it or isolated points. We show now how to manipulate a line into nicer form

Theorem II. 4.4 [If there is a Suslin line, there is a nice Suslin line]

Further manipulation on X will make it nicer still; see Exercise 30.

(p. 67-68).

Metaphorical talk on acting on Suslin lines and trees

- Similar talk is used about “pruning” Suslin trees (i.e., removing sickly branches).

- Similar talk is used about forcing constructions where you iteratively “destroy” Suslin trees by “adding” branches through them.

- As Kanomori described the historical development of the proof showing the consistency of SH, “Tennenbaum saw ‘how to kill a Suslin tree’ […] But what now? Once a Suslin tree is thus killed, it will stay dead. But other Suslin trees may have sprung up and they must be killed by iterating the process”

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[6] Choose I_1 to be one of the intervals L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbf{N}$ such that $x_{n1} \in I_1$.

[7] Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals.

[8] One or both of the intervals L_1, R_1 contains infinitely many terms of the sequence.

[9] Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such that $x_{n2} \in I_2$.

[10] This is always possible because I_2 contains infinitely many terms of the sequence.

[11] Divide I_2 in half, pick a closed half-interval I_3 that contains infinitely many terms, and choose $n_3 > n_2$ such that $x_{n3} \in I_3$.

[12] Continuing in this way, we get a nested sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \ldots I_k \supset \ldots$ of length $|I_k| = 2^{-k} (M - m)$, together with a subsequence (x_{nk}) such that $x_{nk} \in I_k$.

[13] Let $\varepsilon > 0$ be given.

[14] Since $|I_k| \to 0$ as $k \to \infty$, there exists $K \in N$ such that $|I_k| < \varepsilon$ for all $k > K$.

[15] Furthermore, since $x_{nk} \in I_K$ for all $k > K$ we have $|x_{nj} - x_{nk}| < \varepsilon$ for all $j, k > K$.

[16] This proves that (x_{nk}) is a Cauchy sequence, and therefore it converges by Theorem 3.46. (Hunter, 2014, p. 89)

[1] Proof. Suppose that (x_n) is a bounded sequence of real numbers.

[2] Let $M = \sup_{n\in\mathbf{N}} x_n$, $m = \inf_{\in\mathbf{N}} x_n$,

[3] and define the closed interval $I_0 = [m, M]$.

[4] Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where $L_0 = [m, (m + M)/2]$, $R_0 = [(m + M)/2, M]$.

[5] At least one of the intervals L_0, R_0 contains infinitely many terms of the sequence, meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in N$ (even if the terms themselves are repeated).

[6] Choose I_1 to be one of the intervals L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbf{N}$ such that $x_{n1} \in I_1$.

[7] Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals.

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[16] This proves that (x_{nk}) is a Cauchy sequence, and therefore it converges by Theorem 3.46. (Hunter, 2014, p. 89)

The use of the axiom of choice in recursive constructions

- The previous proof was a warrant to believe for:
  - French mathematicians who did not believe in AC
  - Mathematicians who have no interest in AC
  - Undergraduates who have never heard of AC

- We warrant the legitimacy of constructions if we can imagine a Platonic agent who is always able to carry out each individual instruction.

Platonic choice and AC come apart: The case of the Reflection Theorem

"Can one, arguing in ZFC, produce a set model of ZFC? This is almost the case. Given any finite list of axioms ϕ_1, ϕ_2 … ϕ_n, one can prove in ZFC that there is a transitive set in which ' ϕ_1 AND ϕ_2 AND ϕ_n' is true. These M will be very important in our discussion of forcing in VII. Our construction of M will indicate that it is 'true' Platonistically that one can make M satisfy all of ZFC, but by results due to Gödel's Incompleteness Theorem, this platonistic argument cannot be formalized within ZFC (see section 10)"

(p. 134).

Platonic choice and AC come apart: The case of the Reflection Theorem

- Let S_n be the first n axioms of an enumeration of ZFC.
- Using the (Tarski-Vaught-based) construction methods from Kunen, we can make a series of choices to find an α_1 such that $V_{\alpha 1}$ satisfies S_1.
- Given α_k, in a similar fashion, we can construct an α_{k+1} such that $V_{\alpha(k+1)}$ satisfies S_{k+1}.
- If we let $\beta = \sup\{\alpha_n\}$, V_β will satisfy all of ZFC.
- While we can "Platonically choose" our (α_n) sequence, this is not definable in ZFC. (In fact, (α_n) might be cofinal in our model so $\sup\{\alpha_n\}$ may not exist in the model).

Platonic choice and AC come apart: The case of the Reflection Theorem

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What is the right theory of forcing?

- Forcing involves “building” models V[G], but V[G] is difficult (and to many, paradoxical) to conceptualize.

- There are many theories of forcing, all of which yield the same equivalence results. Which is the best one is a philosophical debate.

- What Joel David Hamkins and Neil Barton are trying to do is create formal theories that pulls together the instructions we give with the objects we produce.

Forcing and building models of ZFC: Kunen's approach

- Roughly speaking, Kunen's approach to forcing is to:
  - Start with a countable transitive model M that satisfies ZFC.
  - Find a special "generic" subset G of a partial order P that is not in M (Kunen provides a construction on how to find this G).
  - Build the smallest model M[G] of ZFC that contains G and all the sets in M.

- To prove that ZFC does not imply CH, Kunen starts with a countable transitive model of ZFC and constructs a model M[G] that satisfies ZFC and not CH.

- But there is a hitch. The process used to generate our ctm M relies on the Reflection Theorem, and the Reflection Theorem cannot be used on all the axioms of ZFC.

Forcing and building models of ZFC: Kunen's approach

- What Kunen actually is showing is a scheme to build a model M[G] that satisfies S + not CH for any finite set S of ZFC axioms.

- He writes as if he is building models M[G] of ZFC.
  - We shall show how to construct another ctm, called M[G] for ZFC … [This] will yield a method of obtaining a generic extension N, of M, which is also a model of ZFC. (p. 187),

- But says he doesn't mean it.
  - "when we say 'let M be a countable model for ZFC', the reader may consider this an abbreviation for 'let M be a countable transitive model for enough axioms of ZFC to carry out the arguments at hand" (p. 186).

Forcing and building models of ZFC

- In thinking of the right formal way to frame forcing, both Hamkins (2012) (implicitly) and Barton (2020) advocate for a ‘naturalist interpretation’ of forcing that is faithful to how proofs about forcing our written:
  - … whatever is denoted by ‘V[G]’ really is obtainable by the usual forcing idea of the addition of a generic to whatever is denoted by V
  - Steps in proofs that use forcing constructions could be interpreted with minimal amount of change, so additional or different steps do not need to be made to keep the proof in line with this interpretation (Barton, 2020).

- Since most steps in the proof are instructions, a naturalist theory of forcing really should sanction an idealized agent carrying out these instructions.

Forcing and building models of ZFC

- Barton's idea is that in V, take a c.t.m. V' that is elementarily equivalent to V and to force using that as the ground model.
  - The existence of V' is an even stronger assumption than the existence of a c.t.m. M.
  - As Hamkins (2012) noted, there is a "consistency strength tax" in doing this.
  - Barton also says there is little 'ontological tax' to be paid.
  - So he pays the 'formalist tax' by showing how one could license the production of V' by extending the theory of ZFC.

Forcing and building models of ZFC

The Löwenheim-Skolem theorem allows us to pass to countable sub-models of a given model. Now, the "universe" does not form a set and so we cannot, in ZF, prove the existence of a countable sub-model. However, informally we can repeat the proof of the theorem. We recall that the proof merely consisted of choosing successively sets which satisfied certain properties, if such a set existed. In ZF we can do this process finitely often. There is no reason to believe that in the real world this process cannot be done countably many times and thus yield a countable standard model for ZF.

(Cohen, 1966, p. 79)

Forcing and building models of ZFC

- Instructions in forcing proofs are dictating philosophical evaluation of formal theory in two ways:
  - A good formalization of forcing should treat instructions faithfully. We really should be constructing the objects we claim to be constructing
  - If an axiom licenses a construction that we can imagine a Platonic agent carrying out, and it is analogous to one that is used in published proofs, then that is a reason in favor of that axiom.

1. In reading a proof, many mathematicians believes an object exists because they can imagine a Platonic agent following a construction to build this object.

2. In set theory, the objects that a Platonic agent can construct and the objects that can be proven to exist as sets come apart.

3. For philosophical accounts of set theory, including recent accounts of forcing, “naturalness” is a desideratum. This means the instructions that the Platonic agent carries out in a proof can also be carried out formally without distortion.

4. What a Platonic agent ought to be able to do is used to license axioms and formalism.

Please send comments, advice, and (especially) criticisms and further readings to:

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