Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 1**
Due date: April 14/15, 2015

Problem 1 (4 points): Let R be a discrete valuation ring. Choose a uniformizer π of R and define the map $v : R\backslash\{0\} \longrightarrow \mathbb{Z}_{\geq 0}$ as in the lecture. Let K denote the quotient field of R. If $z \in K^\times$ write $z = \frac{x}{y}$ with $x, y \in R\backslash\{0\}$ and set $v(z) := v(x) - v(y) \in \mathbb{Z}$.

(i) Show that the map $v : K^\times \longrightarrow \mathbb{Z}$ is well-defined, surjective and independent of the choice of π.

(ii) If $z, z' \in K^\times$ then $v(zz') = v(z) + v(z')$.

(iii) If $z, z' \in K^\times$ with $z + z' \neq 0$ then $v(z + z') \geq \min\{v(z), v(z')\}$.

(iv) We have $\{z \in K^\times \mid v(z) \geq 0\} = R \setminus \{0\}$.

Problem 2 (4 points): Let k be a field and let $k[\![t]\!]$ be the ring of *formal power series* in the variable t over k. The addition of two formal power series is defined coefficientwise. The product of two formal power series is defined by

$$\left(\sum_{m=0}^{\infty} a_m t^m\right) \cdot \left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{k=0}^{\infty} c_k t^k \quad \text{with} \quad c_k = \sum_{\substack{n,m\geq 0\\ n+m=k}} a_m b_n.$$

Show that $k[\![t]\!]$ is a complete discrete valuation ring with uniformizer t.

Hint: Start by showing that a formal power series $\sum_{m=0}^{\infty} a_m t^m$ is a unit in the ring $k[\![t]\!]$ if and only if $a_0 \in k^\times$. Further, $(\sum_{m=0}^{\infty} a_{mn} t^m)_{n\geq 0}$ is a Cauchy sequence of formal power series over k if and only if any of the coefficient series $(a_{mn})_{n\geq 0}$ becomes stationary.

Problem 3 (4 points): Let $(I, \leq)$ be a directed set and let $((M_i)_{i\in I}, (f_{ij})_{i\leq j})$ be a projective system of sets, groups, rings or modules over I. Assume that N is another object of the same type together with a family of homomorphisms $g_i : N \to M_i$, $i \in I$, such that $f_{ij} \circ g_j = g_i$ for all $i, j \in I$ with $i \leq j$. Show that there is a unique homomorphism $g : N \to \varprojlim_{i\in I} M_i$ with the property that for all $j \in I$ the homomorphism

$$N \xrightarrow{\;g\;} \varprojlim_{i\in I} M_i \hookrightarrow \prod_{i\in I} M_i \xrightarrow{\;proj_j\;} M_j$$

is equal to g_j. Here $proj_j$ denotes the projection onto the j-th component.