Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 10**
Due date: June 30, 2015

Problem 1 (4 points): Let R be a commutative ring with unit, let M and N be R-modules, and let $r \in R$ be an arbitrary element. Show that the following abelian groups are canonically isomorphic:

(i) $(M \otimes_R N)/r(M \otimes_R N)$

(ii) $M \otimes_R (N/rN)$

(iii) $(M/rM) \otimes_R (N/rN)$

(iv) $(M/rM) \otimes_{(R/rR)} (N/rN)$

Hint: The groups in (i), (ii) and (iii) are isomorphic by the right exactness of the tensor product. Use the universal property of the tensor product to construct inverse homomorphisms between the groups in (iii) and (iv).

Problem 2 (4 points): Let $V, W \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$ be two continuous $\mathbb{Z}_p$-linear G_E-representations with associated étale φ-modules $\mathbb{D}(V), \mathbb{D}(W) \in \Phi_{\mathfrak{o}_\mathcal{E}}^{\text{ét}}$. We let $\mathbb{D}(V) \otimes_{\mathfrak{o}_\mathcal{E}} \mathbb{D}(W) \in \Phi_{\mathfrak{o}_\mathcal{E}}^{\text{ét}}$ be the tensor product of $\mathbb{D}(V)$ and $\mathbb{D}(W)$ as on problem sheet 9.

(i) Show that if the $\mathbb{Z}_p$-module $V \otimes_{\mathbb{Z}_p} W$ is endowed with the diagonal action of G_E then $V \otimes_{\mathbb{Z}_p} W \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$.

(ii) Show that the étale φ-modules $\mathbb{D}(V) \otimes_{\mathfrak{o}_\mathcal{E}} \mathbb{D}(W)$ and $\mathbb{D}(V \otimes_{\mathbb{Z}_p} W)$ over $\mathfrak{o}_\mathcal{E}$ are naturally isomorphic.

Hint: As for (i), use problem 1 to check the continuity condition. As for (ii), construct an $\mathfrak{o}_\mathcal{E}$-linear map $\mathbb{D}(V) \otimes_{\mathfrak{o}_\mathcal{E}} \mathbb{D}(W) \to \mathbb{D}(V \otimes_{\mathbb{Z}_p} W)$ which is an isomorphism after the faithfully flat base extension from $\mathfrak{o}_\mathcal{E}$ to $\mathfrak{o}_{\breve{\mathcal{E}}}$. You may use without proof that if $R \to S$ is a homomorphism of commutative rings with unit and if M and N are R-modules then there is a unique isomorphism of abelian groups $(S \otimes_R M) \otimes_S (S \otimes_R N) \to S \otimes_R (M \otimes_R N)$ sending $(s \otimes m) \otimes (s' \otimes n)$ to $ss' \otimes (m \otimes n)$.

Problem 3 (4 points): Let $V \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$ be a continuous $\mathbb{Z}_p$-linear G_E-representation with associated étale φ-module $\mathbb{D}(V) \in \Phi_{\mathfrak{o}_\mathcal{E}}^{\text{ét}}$. Show that the underlying $\mathbb{Z}_p$-module of V is free if and only if the underlying $\mathfrak{o}_\mathcal{E}$-module of $\mathbb{D}(V)$ is free. In this case, we have $\mathrm{rk}_{\mathbb{Z}_p}(V) = \mathrm{rk}_{\mathfrak{o}_\mathcal{E}}(\mathbb{D}(V))$.

Hint: A finitely generated module over a principal ideal domain is free if and only if it is torsion free. Therefore, everything can be checked after the faithfully flat base extension from $\mathfrak{o}_\mathcal{E}$ to $\mathfrak{o}_{\breve{\mathcal{E}}}$.