Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 11**
Due date: July 7, 2015

Problem 1 (8 points):

(i) We endow the abelian group $\mathbb{Z}_p^\times$ with the subspace topology induced from $\mathbb{Z}_p$. A continuous group homomorphism $\chi : G_E \to \mathbb{Z}_p^\times$ is also called a *continuous character*. Show that the set $\mathrm{Hom}^{\mathrm{cont}}(G_E, \mathbb{Z}_p^\times)$ of continuous characters forms an abelian group under the pointwise multiplication of group homomorphisms.

(ii) If $V \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$ and if the underlying $\mathbb{Z}_p$-module of V is free of rank one then there is a continuous character $\chi : G_E \to \mathbb{Z}_p^\times$ such that $\sigma v = \chi(\sigma)v$ for all $\sigma \in G_E$, $v \in V$. If $V' \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$ is also free of rank one over $\mathbb{Z}_p$ and if $V \cong V'$ in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$ then the associated continuous characters of V and V' coincide.

(iii) Let $\chi : G_E \to \mathbb{Z}_p^\times$ be a continuous character and let $V = \mathbb{Z}_p$ be the free $\mathbb{Z}_p$-module of rank one. We let G_E act on V by $\sigma a = \chi(\sigma)a$ for all $\sigma \in G_E$, $a \in \mathbb{Z}_p$. Show that this gives $V = V_\chi$ the structure of a continuous $\mathbb{Z}_p$-linear representation of G_E. If also $\chi' : G_E \to \mathbb{Z}_p^\times$ is a continuous character then there is an isomorphism $V_\chi \otimes_{\mathbb{Z}_p} V_{\chi'} \cong V_{\chi\chi'}$ in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$.

(iv) If $a \in \mathfrak{o}_{\mathcal{E}}^\times$ then $M_a := (\mathfrak{o}_{\mathcal{E}}, f_a)$ with $f_a(b) := \varphi(b)a$ is an étale φ-module over $\mathfrak{o}_{\mathcal{E}}$. If also $b \in \mathfrak{o}_{\mathcal{E}}^\times$ then there is an isomorphism $M_a \otimes_{\mathfrak{o}_{\mathcal{E}}} M_b \cong M_{ab}$ of φ-modules over $\mathfrak{o}_{\mathcal{E}}$.

(v) Consider the group homomorphism $(a \mapsto \varphi(a)a^{-1}) : \mathfrak{o}_{\mathcal{E}}^\times \to \mathfrak{o}_{\mathcal{E}}^\times$ and denote by $\mathfrak{o}_{\mathcal{E}}^\times/(\varphi-1)\mathfrak{o}_{\mathcal{E}}^\times$ its cokernel. Use problem 8.2 and problem 10.3 to show that if $\chi : G_E \to \mathbb{Z}_p^\times$ is a continuous character then there is a unique class $[a] = [a(\chi)] \in \mathfrak{o}_{\mathcal{E}}^\times/(\varphi-1)\mathfrak{o}_{\mathcal{E}}^\times$ such that $\mathbb{D}(V_\chi) \cong M_a$ as φ-modules over $\mathfrak{o}_{\mathcal{E}}$.

(vi) Use problem 10.2 and the equivalence of categories $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E) \cong \Phi_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét}}$ to show that the map $(\chi \mapsto [a(\chi)]) : \mathrm{Hom}^{\mathrm{cont}}(G_E, \mathbb{Z}_p^\times) \to \mathfrak{o}_{\mathcal{E}}^\times/(\varphi-1)\mathfrak{o}_{\mathcal{E}}^\times$ is an isomorphism of groups.

Problem 2 (4 points): Consider the φ-module $(M, f) := (\mathcal{E}, p^{-1}\varphi)$ over $\mathcal{E}$. Show that the $\mathcal{E}$-linearization $f_{\mathcal{E}}$ of f is bijective. Show that φ preserves p-adic valuations and deduce that $\frac{\varphi(x)}{x} \in \mathfrak{o}_{\mathcal{E}}^\times$ for any nonzero $x \in M$. Now consider the equation $f(x) = p^{-1}\frac{\varphi(x)}{x}x$ and conclude that M does not admit any f-stable $\mathfrak{o}_{\mathcal{E}}$-lattices. In particular, the φ-module (M, f) cannot be of slope zero.