Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 2**
Due date: April 21, 2015

Let R be a ring, let M be an R-module, and let $(M_j)_{j\in\mathbb{N}}$ be a descending family of R-submodules of M. Set

$$\hat{M} := \varprojlim_{j\in\mathbb{N}} M/M_j$$

and denote by $\hat{M}_j \subseteq \hat{M}$ the kernel of the projection $\hat{M} \longrightarrow M/M_j$ onto the j-th component. We endow M and $\hat{M}$ with the topology defined by the descending family of R-submodules $(M_j)_{j\in\mathbb{N}}$ and $(\hat{M}_j)_{j\in\mathbb{N}}$, respectively. Further, we denote by $g : M \longrightarrow \hat{M}$ the canonical R-linear map.

Problem 1 (4 points): Let N be an R-module which is Hausdorff and complete with respect to the topology defined by a descending family $(N_j)_{j\in\mathbb{N}}$ of R-submodules. Show that if $\varphi : M \longrightarrow N$ is a continuous R-linear map then there is a unique continuous and R-linear map $\hat{\varphi} : \hat{M} \longrightarrow N$ such that $\hat{\varphi} \circ g = \varphi$.

Hint: By continuity there is an increasing sequence $(i_j)_{j\in\mathbb{N}}$ of natural numbers i_j such that the composition of φ with the natural map $N \longrightarrow N/N_j$ factors through M/M_{i_j}. Use the resulting maps $M/M_{i_j} \longrightarrow N/N_j$ and the universal property of projective limits to construct an R-linear map $\hat{M} \to \varprojlim_j M/M_{i_j} \to \varprojlim_j N/N_j \cong N$.

Problem 2 (4 points): Show that the canonical map $g : M \longrightarrow \hat{M}$ is continuous, i.e. inverse images of open subsets are open. Endow the image of g with the subspace topology of $\hat{M}$ and show that the map $g : M \longrightarrow g(M)$ is open, i.e. maps open sets to open sets. In particular, if M is Hausdorff then g is a homeomorphism onto its image.

Problem 3 (4 points): Endow each of the quotient modules M/M_j with the discrete topology and endow $\prod_{j\in\mathbb{N}} M/M_j$ with the product topology. Show that the topology of $\hat{M}$ defined by the descending family $(\hat{M}_j)_{j\in\mathbb{N}}$ of R-submodules coincides with the subspace topology of $\prod_{j\in\mathbb{N}} M/M_j$ and that $\hat{M}$ is closed in $\prod_{j\in\mathbb{N}} M/M_j$. Use Tychonoff's theorem to deduce that the ring $\mathbb{Z}_p$ of p-adic integers is a compact topological space.