**Exercises $p$-adic Galois representations** **summer term 2015**
**Prof. Dr. J. Kohlhaase** **problem sheet 3**
**Due date: April 28, 2015**

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**Problem 1 (8 points):** Let $(F, |\cdot|)$ be a nonarchimedean valued field. If $V$ is an $F$-vector space then a *norm* on $V$ is a map $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ such that for all $v, w \in V$ and $x \in F$ we have

(i) $||v|| = 0$ if and only if $v = 0$,

(ii) $||x \cdot v|| = |x| \cdot ||v||$, and

(iii) $||v + w|| \leq \max\{||v||, ||w||\}$.

A norm defines a metric topology on $V$ by declaring a subset $U$ to be open if and only if for any $v \in U$ there is $\varepsilon \in \mathbb{R}_{>0}$ such that $||v - w|| < \varepsilon$ implies $w \in U$. This allows us to define the notions of Cauchy sequences, limits and completeness in the usual way.

Two norms $||\cdot||$ and $||\cdot||'$ on $V$ are called *equivalent* if there are positive real numbers $C$ and $D$ such that $C||v|| \leq ||v||' \leq D||v||$ for all $v \in V$. Consider the following statements:

*Proposition. Assume that $F$ is complete. If $V$ is finite dimensional over $F$ then all norms on $V$ are equivalent and $V$ is complete with respect to each of them. In particular, if $E|F$ is a finite field extension then there is at most one absolute value on $E$ extending $|\cdot|$. If such an extension exists then it makes $E$ a complete nonarchimedean valued field.*

Give a detailed proof of the proposition by following the strategy outlined below.

*Proof.* The proof is by induction on $n := \dim(V)$, the case $n = 1$ being obvious. Choose an $F$-basis $e_1, \ldots, e_n$ of $V$ and define the norm $||\cdot||$ on $V$ by $||\sum_i x_i e_i|| := \max_{1 \leq i \leq n} |x_i|$. Since $F$ is complete it follows that $V$ is complete with respect to $||\cdot||$ and hence with respect to any equivalent norm. Therefore, it suffices to see that any norm $||\cdot||'$ on $V$ is equivalent to $||\cdot||$. Note that $||\cdot||' \leq D||\cdot||$ with $D := \max_{1 \leq i \leq n} ||e_i||'$.

Set $W := \sum_{i=1}^{n-1} F e_i$. By induction hypothesis there is $A \in \mathbb{R}_{>0}$ such that $||w||' \geq A||w||$ for all $w \in W$. Further, $W$ is complete with respect to $||\cdot||'$, hence is a closed subset of $V$ for the $||\cdot||'$-topology. Therefore, $1 \geq B := (||e_n||')^{-1} \inf\{||e_n + w||' \,:\, w \in W\} > 0$. Now set $C := \min\{AB, B||e_n||'\}$, let $v = \sum_{i=1}^{n} x_i e_i \in V$ and set $w := \sum_{i=1}^{n-1} x_i e_i$. If $x_n \neq 0$ then $||v||' = |x_n| \cdot ||e_n + w/x_n||' \geq B||x_n e_n||'$ and hence $B||w||' = B||v - x_n e_n||' \leq ||v||'$. For $x_n = 0$ both inequalities hold trivially. Altogether, we obtain

$$\begin{aligned} ||v||' &\geq B\max\{||w||', ||x_n e_n||'\} \geq B\max\{A||w||, ||e_n||'|x_n|\} \\ &\geq C\max\{||w||, |x_n|\} = C||v||. \end{aligned}$$

As for the second assertion, consider an inequality of the form $C|x| \leq |x|' \leq D|x|$, replace $x$ by $x^n$, take $n$-th roots and let $n$ tend to infinity. □

**Problem 2 (4 points):** Let $(F, v)$ be a complete discretely valued field. We denote by $\mathfrak{o}$ the valuation ring of $F$ and by $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$. Let $f(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0 \in \mathfrak{o}[t]$ be an *Eisenstein polynomial*, i.e. $a_0, \ldots, a_{n-1} \in \mathfrak{m}$ and $a_0 \notin \mathfrak{m}^2$. Show that if $\alpha \in \overline{F}$ is a root of $f$ in an algebraic closure $\overline{F}$ of $F$ then the field extension $F[\alpha]|F$ is *totally ramified*, i.e. $e(F[\alpha]|F) = [F[\alpha] : F]$.