Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 5**
Due date: May 19, 2015

Problem 1 (4 points): Let k be a field of positive characteristic p and let $k((t)) = \mathrm{Quot}(k[\![t]\!])$ denote the field of formal Laurent series with coefficients in k. Assume that k admits a Cohen ring C and that C admits a Frobenius lift φ. Let C' denote the Cohen ring of $k((t))$ constructed in the lecture, i.e.

$$C' = \{\sum_{n\in\mathbb{Z}} c_n t^n \mid c_n \in C \text{ and } \lim_{n\to-\infty} v(c_n) = \infty\},$$

where v denotes the p-adic valuation of C.

The *weak topology* on C' is defined by declaring the subsets $p^n C' + t^m C[\![t]\!]$ with $n, m \in \mathbb{N}$ to be a fundamental system of open neighborhoods of zero. Show that if $f \in tC[\![t]\!] \setminus pC'$ then there is a unique ring homomorphism $\varphi_f : C' \to C'$ such that $\varphi_f|_C = \varphi$, $\varphi_f(t) = f$ and such that φ_f is continuous for the weak topology. It is given by $\varphi_f(\sum_{n\in\mathbb{Z}} c_n t^n) = \sum_{n\in\mathbb{Z}} \varphi(c_n) f^n$. Consider φ_{t^p} and $\varphi_{(1+t)^p-1}$ to conclude that C' admits Frobenius lifts.

Problem 2 (4 points): Let k be a field of positive characteristic p. Assume that k admits a Cohen ring C and that C admits a Frobenius lift φ. Set $C^{\varphi=1} = \{c \in C \mid \varphi(c) = c\}$. Show that $C^{\varphi=1}$ is a subring of C which is canonically isomorphic to $\mathbb{Z}_p$.

Hint: Show that $C^{\varphi=1}$ is a Cohen ring for $\mathbb{F}_p$. The injective ring homomorphism $\mathbb{Z} \to C^{\varphi=1}$ extends to an injective ring homomorphism $\mathbb{Z}_p \to C^{\varphi=1}$. Use p-adic series expansions to show that the latter is surjective.

Problem 3 (4 points): Let p be a prime number. For $n \in \mathbb{N}$ let S_n, P_n, F_n and I_n be the polynomials from the construction of the ring of Witt vectors.

(i) Compute S_n, P_n, F_n and I_n for $n = 0$ and $n = 1$.

(ii) Show that $S_n - X_n - Y_n \in \mathbb{Z}[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]$ for any $n \in \mathbb{N}$.

(iii) Show that $I_n = -X_n$ for any $n \in \mathbb{N}$ if p is odd.