Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 6**
Due date: June 2^{nd}, 2015

Problem 1 (4 points): Let B denote a commutative ring with unit, let $(W(B), \oplus, \odot)$ denote the ring of Witt vectors with coefficients in B, and let F and V denote Frobenius and Verschiebung, respectively.

(i) The map $F : W(B) \to W(B)$ is a ring homomorphism.

(ii) The map $V : W(B) \to W(B)$ is a homomorphism of additive groups.

(iii) If $b \in W(B)$ then $(F \circ V)(b) = p \cdot b := \underbrace{b \oplus \ldots \oplus b}_{p \text{ times}}$.

(iv) If $a, b \in W(B)$ then $V(a \odot F(b)) = V(a) \odot b$.

(v) If $b \in W(B)$ then $F(b) \equiv b^p := \underbrace{b \odot \ldots \odot b}_{p \text{ times}} \mod pW(B)$.

Hint: Consider the ring $B_1 := \mathbb{Z}[(X_b)_{b \in B}]$. First prove similar statements for the maps f_{B_1} and v_{B_1} on $B_1^{\mathbb{N}}$. Then use that $\Phi_{B_1} : W(B_1) \to B_1^{\mathbb{N}}$ is an injective ring homomorphism. Finally, consider $W(\rho) : W(B_1) \to W(B)$ where $\rho : B_1 \to B$ is defined by $\rho(X_b) = b$.

Problem 2 (4 points): Let B denote a commutative ring with unit and let $W(B)$ denote the ring of Witt vectors with coefficients in B. For $m \in \mathbb{N}$ consider the ideal $V_m(B) = \{(b_n)_{n \in \mathbb{N}} \in W(B) \mid b_0 = \ldots = b_{m-1} = 0\}$ of $W(B)$. Show that $W(B)$ is Hausdorff and complete for the topology defined by the descending family of ideals $(V_m(B))_{m \geq 0}$.

Problem 3 (4 points): Let B be a commutative ring with unit containing an element ζ with $\sum_{i=0}^{p-1} \zeta^i = 0$. Show that in the ring $W(B)$ of Witt vectors with coefficients in B we have the relation

$$\sum_{i=0}^{p-1} \tau(\zeta^i) = (0, 1, 0, 0, 0, \ldots).$$

Here $\tau : B \to W(B)$ denotes the Teichmüller lift.

Hint: First reduce to the case $B = \mathbb{Z}[\zeta] \subset \mathbb{C}$ where ζ is a primitive p-th root of unity. Then apply the ring homomorphism $\Phi_B : W(B) \to B^{\mathbb{N}}$.