
Problem 1 (4 points): : Let B denote a commutative ring with unit, let $W(B)$ denote the ring of Witt vectors with coefficients in B, and let F and V denote Frobenius and Verschiebung on $W(B)$, respectively. Show that the following assertions are equivalent:

(i) The ring B has characteristic p, i.e. $p \cdot 1_B = 0_B$.
(ii) We have $V \circ F = F \circ V$.
(iii) We have $V(1_{W(B)}) = p \cdot 1_{W(B)}$.

Hint: The map $F \circ V$ is multiplication with $p \cdot 1_{W(B)}$, whereas $V \circ F$ is multiplication with $V(1_{W(B)})$. Now compare the entries at position 0.

Problem 2 (4 points): Let F be a field, let G_F be the absolute Galois group of F, and let V be a $\mathbb{Z}_p$-linear G_F-representation, i.e. a $\mathbb{Z}_p$-module V together with a map $((g, v) \mapsto gv) : G_F \times V \to V$ satisfying

(a) for all $g \in G_F$ the map $(v \mapsto gv) : V \to V$ is $\mathbb{Z}_p$-linear,
(b) for all $v \in V$ we have $1_{G_F} v = v$,
(c) for all $g, h \in G_F$ and for all $v \in V$ we have $(gh)v = g(hv)$.

Endow V with the p-adic topology, G_F with the Krull topology and $G_F \times V$ with the corresponding product topology. Show that the following statements are equivalent.

(i) The structure map $G_F \times V \to V$ is continuous.
(ii) For any $n \in \mathbb{N}$ the induced map $G_F \times V/p^n V \to V/p^n V$ is continuous.
(iii) For any $n \in \mathbb{N}$ the induced action of G_F on $V/p^n V$ is *smooth* in the sense that for any $v \in V$ there is an open subgroup H of G_F with $h(v + p^n V) := hv + p^n V = v + p^n V$ for all $h \in H$.

Show that if the $\mathbb{Z}_p$-module V is finitely generated then (i)–(iii) are equivalent to the following statement.

(iv) For any $n \in \mathbb{N}$ there is a finite Galois extension $F_n|F$ such that the induced action of G_F on $V/p^n V$ factors through $\mathrm{Gal}(F_n|F)$.

Problem 3 (4 points): Let R be a commutative ring, let M and N be R-modules, and let $M \otimes_R N$ be the tensor product of M and N over R, viewed as an abelian group.

(i) Show that $M \otimes_R N$ can be given the structure of an R-module in a unique way such that $r \cdot (m \otimes n) = rm \otimes n = m \otimes rn$ for all $r \in R$, $m \in M$ and $n \in N$.

(ii) Let S be a commutative ring and let $\varphi : R \to S$ be a ring homomorphism. We view S as an R-module via $r \cdot s := \varphi(r)s$ for all $r \in R$ and $s \in S$. Show that the abelian group $S \otimes_R M$ can be given the structure of an S-module in a unique way such that $s * (s' \otimes m) = ss' \otimes m$ for all $s, s' \in S$ and $m \in M$. Show that this S-module structure commutes with the R-module structure defined in (i), i.e. that $s * (r \cdot x) = r \cdot (s * x)$ for all $r \in R$, $s \in S$ and $x \in S \otimes_R M$.