Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 8**
Due date: June 16, 2015

Problem 1 (4 points): Let R be a complete discrete valuation ring with quotient field F. Let V be a finite dimensional F-vector space endowed with its natural norm topology. Given a subset $\Lambda \subseteq V$ show that the following statements are equivalent.

(i) Λ is an R-lattice of V, i.e. a finitely generated R-submodule of V which generates V as an F-vector space.

(ii) Λ is an open and finitely generated R-submodule of V.

(iii) Λ is a free R-submodule of finite rank $\dim_F(V)$ of V.

(iv) Λ is a finitely generated R-submodule of V and the natural F-linear map $F \otimes_R \Lambda \to V$ is bijective.

Hint: You may use without proof that according to the elementary divisor theorem finitely generated torsion free R-modules are free.

Problem 2 (4 points): Let R be a commutative ring with unit, and let $\varphi : R \to R$ be a ring homomorphism. Let M be a finitely generated free R-module, and let $m_1, \ldots, m_r$ be an R-basis of M. Given a φ-semilinear map $f : M \to M$ define the matrix $A_f = (a_{ij})_{1 \le i,j \le r} \in R^{r \times r}$ by $f(m_j) = \sum_{i=1}^{r} a_{ij} m_i$. Finally, given a matrix $B = (b_{ij})_{i,j} \in R^{r \times r}$ set $\varphi(B) := (\varphi(b_{ij}))_{i,j} \in R^{r \times r}$.

(i) Show that the map $(f \mapsto A_f)$ is a bijection between the set of φ-semilinear maps $M \to M$ and the set $R^{r \times r}$. Show that two φ-modules (M, f) and (M, g) over R are isomorphic if and only if there is an invertible matrix $B \in R^{r \times r}$ such that $A_f = B^{-1} A_g \varphi(B)$.

(ii) The φ-module (M, f) is étale if and only if A_f is invertible. If the φ-module (M, f) is étale and if φ is injective then f is injective.

(iii) If R is a field and if φ is bijective then (M, f) is étale if and only if A_f is invertible, if and only if f is surjective, if and only if f is injective.

Problem 3 (4 points): Let R be a commutative ring with unit, and let N be an R-module. Show that if $M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is an exact sequence of R-modules then the sequence $N \otimes_R M' \xrightarrow{\mathrm{id}_N \otimes f} N \otimes_R M \xrightarrow{\mathrm{id}_N \otimes g} N \otimes_R M'' \longrightarrow 0$ of abelian groups is exact.

Hint: Set $Q := (N \otimes_R M)/\mathrm{im}(\mathrm{id}_N \otimes f)$. Given $n \in N$ and $m'' \in M''$ choose $m \in M$ with $g(m) = m''$. Show that mapping (n, m'') to the class of $n \otimes m$ in Q gives a well-defined R-balanced map $N \times M'' \to Q$. The induced group homomorphism $N \otimes_R M'' \to Q$ is inverse to the homomorphism $Q \to N \otimes_R M''$ induced by $\mathrm{id}_N \otimes g$.