Exercises p-adic Galois representations **summer term 2015**
Prof. Dr. J. Kohlhaase **problem sheet 9**
Due date: June 23, 2015

Problem 1 (4 points): Let R be a commutative ring with unit, let $\varphi : R \to R$ be a ring homomorphism and let (M, f) and (N, g) be φ-modules over R.

(i) On the R-module $M \otimes_R N$ there is a unique φ-semilinear map h satisfying $h(m \otimes n) = f(m) \otimes g(n)$ for all $m \in M$ and $n \in N$. The φ-module $(M \otimes_R N, h)$ over R is called the *tensor product* of (M, f) and (N, g).

(ii) The R-modules $R \otimes_{R,\varphi} (M \otimes_R N)$ and $(R \otimes_{R,\varphi} M) \otimes_R (R \otimes_{R,\varphi} N)$ are naturally isomorphic. Under this identification we have $h_R = f_R \otimes g_R$. Deduce that if (M, f) and (N, g) are étale φ-modules over R then so is their tensor product.

Problem 2 (4 points): Let $E|F$ be a finite Galois extension of fields with Galois group $G := \mathrm{Gal}(E|F)$. A *semilinear G-representation over E* is a finite dimensional E-vector space V together with a group homomorphism $\rho : G \to \mathrm{Aut}_F(V)$ such that $\rho(\sigma)(\alpha v) = \sigma(\alpha)\rho(\sigma)(v)$ for all $\alpha \in E$ and $v \in V$. Follow the arguments of the lecture to show that the natural E-linear map $E \otimes_F V^G \to V$ is bijective and hence that $\dim_F(V^G) = \dim_E(V)$.

Remark: *This result is usually called "Galois descent for vector spaces".*

Problem 3 (4 points): Let $E|F$ be a finite Galois extension of fields with Galois group $G := \mathrm{Gal}(E|F)$, and let n be a positive integer. If $\sigma \in G$ and $A = (a_{ij})_{i,j} \in \mathrm{GL}_n(E)$ we set $\sigma(A) := (\sigma(a_{ij}))_{i,j}$. Likewise, $\sigma(x) := (\sigma(x_i))_i$ if $x = (x_i)_i \in E^n$.

A $\mathrm{GL}_n(E)$*-valued* 1*-cocycle on* G is a map $\delta : G \to \mathrm{GL}_n(E)$ with the property that $\delta(\sigma\tau) = \delta(\sigma) \cdot \sigma(\delta(\tau))$ for all $\sigma, \tau \in G$. Two 1-cocyles $\delta, \delta' : G \to \mathrm{GL}_n(E)$ are called *cohomologous* if there is a matrix $A \in \mathrm{GL}_n(E)$ such that $A \cdot \delta(\sigma) = \delta'(\sigma) \cdot \sigma(A)$ for all $\sigma \in G$. In this case we write $\delta \sim \delta'$.

(i) Show that $\sim$ is an equivalence relation on the set of all $\mathrm{GL}_n(E)$-valued 1-cocycles on G. We denote by $\mathrm{H}^1(G, \mathrm{GL}_n(E))$ the associated set of equivalence classes.

(ii) Show that any 1-cocycle $\delta : G \to \mathrm{GL}_n(E)$ gives rise to a semilinear G-representation on E^n via $\rho(\sigma)(x) = \delta(\sigma) \cdot \sigma(x)$ for $\sigma \in G$ and $x \in E^n$.

(iii) Use the previous exercise to show that any 1-cocycle $\delta : G \to \mathrm{GL}_n(E)$ is cohomologous to the trivial 1-cocycle 1, i.e. to the constant map which sends any element of G to the $(n \times n)$-identity matrix. In other words, we have $\mathrm{H}^1(G, \mathrm{GL}_n(E)) = \{1\}$.

Remark: *The statement* $\mathrm{H}^1(G, \mathrm{GL}_n(E)) = \{1\}$ *is in fact a reformulation of the result in problem 2. It is a generalization of Hilbert's Theorem 90 to the nonabelian setting.*