Invariant distributions on $p$-adic analytic groups

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2000 Mathematics Subject Classification. Primary 11S80, 16S30, 16U70, 22E50.

Abstract. Let $p$ be a prime number, $L$ a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers, $K$ a spherically complete extension field of $L$ and $G$ the group of $L$-rational points of a split reductive group over $L$. We derive several explicit descriptions of the center of the algebra $D(G, K)$ of locally analytic distributions on $G$ with values in $K$. The main result is a generalization of an isomorphism of Harish-Chandra which connects the center of $D(G, K)$ with the algebra of Weyl-invariant, centrally supported distributions on a maximal torus of $G$. This isomorphism is supposed to play a role in the theory of locally analytic representations of $G$ as studied by P. Schneider and J. Teitelbaum.

Introduction

Let $p$ be a prime number, $L$ a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers, $K$ a spherically complete extension field of $L$ and $G$ a locally $L$-analytic group of finite dimension with center $Z$ and Lie algebra $\mathfrak{g}$.

The $K$-algebra $D(G, K)$ of locally analytic distributions on $G$ plays a central role in the theory of locally analytic representations of $G$ on locally convex $K$-vector spaces which was given a systematic treatment by P. Schneider and J. Teitelbaum (cf. [25] and [26]). Such representations appear in the cohomology of $p$-adic symmetric spaces (cf. [27]), as an important tool of M. Emerton’s construction of the Eigencurve of Coleman-Mazur (cf. [13]) and, most recently, in C. Breuil’s hypothetical $p$-adic Langlands program (cf. [5]).

This paper is devoted to the study of the center of the ring $D(G, K)$. Our approach relies on the observation that for locally analytic distributions on $G$ there is a well-defined notion of support and that the support $\text{supp}(\delta)$ is a compact subset of $G$ for any distribution $\delta \in D(G, K)$. It follows from the definition of the convolution product in $D(G, K)$ that any invariant distribution, i.e. any element of $D(G, K)^G$, is supported on a union of relatively compact conjugacy classes of $G$. If $G$ is the group of $L$-rational points of a connected, reductive, linear algebraic group over $L$ all of whose simple factors are $L$-isotropic (e.g. an $L$-split group) then the only such classes of $G$ are the trivial ones, i.e. those belonging to the elements of $Z$ (Sit’s theorem). Therefore, we are led to the investigation of the $K$-algebra $D(G, K)_Z$ of centrally supported distributions on $G$.

If $\mathfrak{z}$ denotes the Lie algebra of $Z$ then we let $U(\mathfrak{z}, K)$ (resp. $U(\mathfrak{g}, K)$) be the
subalgebra of $D(Z, K)$ (resp. $D(G, K)$) consisting of distributions supported in the unit element. There is a natural continuous $K$-linear map

$$D(Z, K) \hat{\otimes}_{U(k, K)} U(g, K) \rightarrow D(G, K)_Z$$

of locally convex $D(Z, K)$-$U(g, K)^{op}$-bimodules (here $\hat{\otimes}$ indicates the inductive tensor product topology). It is the main technical result of our work that under the assumption that $K$ is discretely valued this map is a topological isomorphism (cf. Proposition 1.2.12). Its proof relies for one thing on certain compatibility conditions for global charts of small open subgroups of $G$ and $Z$, respectively (cf. Proposition 1.3.5 and Corollary 1.3.6). On the other hand, we make extensive use of the fact that $D(G, K)$ is a $K$-Fréchet-Stein algebra (a notion introduced by P. Schneider and J. Teitelbaum) and a structure theorem of $D(G, K)$ as a module over $U(g, K)$ after a certain completion process. The latter is due to H. Frommer who proved it for $\mathbb{Q}_p$ as a ground field. We generalize it to any finite extension $L|\mathbb{Q}_p$ (cf. Theorem 1.4.2).

$G$ acts on $U(g, K)$ and $D(G, K)_Z$. If $G$ is an open subgroup of the group of $L$-rational points of a connected, algebraic group over $L$ then we obtain a topological isomorphism

$$D(Z, K) \hat{\otimes}_{U(l, K)} U(g, K)_G \rightarrow D(G, K)_Z^G$$

of $K$-algebras (cf. Theorem 2.2.1). If moreover $G$ satisfies the hypotheses of Sit’s theorem then $D(G, K)_Z^G = D(G, K)_Z$ and it remains to examine the infinitesimal center $U(g, K)$.

Consider $g$ as an abelian locally $L$-analytic group and let $S(g, K)$ be the subalgebra of $D(g, K)$ consisting of distributions supported in $0 \in g$. $S(g, K)$ and $U(g, K)$ carry actions of $G$ and $g$. We show that Duflo’s famous isomorphism $S(g)^g \rightarrow U(g)^g$ extends to a topological isomorphism $S(g, K)^g \rightarrow U(g, K)^g$ of $K$-Fréchet algebras (cf. Proposition 2.1.5; $S(g)$ and $U(g)$ denote the symmetric and the universal enveloping algebra of $g$, respectively). If $g$ is split semisimple with split maximal toral subalgebra $t$ and corresponding Weyl group $W$ then $W$ naturally acts on the algebra $S(t, K)$ of locally analytic distributions on $t$ supported in $0 \in t$. We show that the classical isomorphism $S(g)^g \rightarrow S(t)^{W}$ extends to a topological isomorphism $S(g, K)^g \simeq S(t, K)^{W}$ of $K$-algebras (cf. Theorem 2.1.6). It follows that

$$U(g, K) \simeq S(t, K)^{W}.$$  

Even more is true: Just as $S(t)^{W}$ is a polynomial ring in $n := \dim_L (t)$ variables, $S(t, K)^{W}$ is the algebra of holomorphic functions on the rigid analytic affine space $(k^n_K)^{an}$ of dimension $n$ over $K$ (loc.cit.).

If $G$ is the group of $L$-rational points of a connected, split reductive $L$-group $G$ then the above results enable us to give two different, explicit descriptions of $D(G, K)^G$. Using results on the Fourier transform of $Z$ obtained by M. Emerton, P. Schneider and J. Teitelbaum we deduce the existence of an explicitly computable quasi-Stein rigid analytic $K$-variety $X_K$ and a continuous, injective homomorphism of $K$-algebras

$$D(G, K)^G \rightarrow \mathcal{O}(X_K)$$

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with dense image (cf. Corollary 2.3.3 and Remark 2.3.4). If $T$ is a maximal $L$-split torus of $G$, $T := \mathfrak{T}(L)$ and $W := N_G(T)/T$ the corresponding Weyl group then we also construct a topological isomorphism

$$D(G, K)^G \simeq D(T, K)^W$$

of separately continuous $K$-algebras extending Harish-Chandra’s isomorphism $U(g)^G \simeq S(t)^W$ (cf. Theorem 2.4.2). Since the latter plays a fundamental role in the representation theory of the Lie algebra $g$ our extension is hoped to be of importance for the theory of locally analytic representations of the group $G$. We point out that in the theory of smooth representations – subsumed by the locally analytic theory – such an isomorphism does not exist.

The present work comprises parts of the author’s thesis. He is deeply indebted to Prof. Dr. P. Schneider without whose guidance it would not have come into existence. He is also grateful to Prof. Dr. S. Bosch and Dr. M. Strauch for many helpful discussions, as well as to two anonymous referees for helping to improve an earlier version of this article.

Conventions and notation. Throughout this paper $p$ denotes a prime number and $L$ a finite extension of $\mathbb{Q}_p$. Let $\mathfrak{o}_L$ be the ring of integers of $L$ with maximal ideal $m_L$ and uniformizer $\pi_L$. We assume the valuation $\omega$ on $L$ to be normalized such that $\omega(\pi_L) = 1$. Let further $e := \omega(p)$ be the ramification index of the extension $L|\mathbb{Q}_p$ and $m$ its degree. The absolute value $|\cdot|$ of $L$ corresponding to $\omega$ is assumed to be normalized through $|p| = p^{-1}$. We let $K$ be a fixed spherically complete extension of $L$ which for many results will have to be assumed to be discretely valued (cf. subsection 1.4, in particular). Let $\mathfrak{o}_K$ denote its ring of integers. We assume the absolute value $|\cdot|$ on $K$ to extend the one on $L$. If $V$ is a locally convex vector space over $K$ then we let $V' := \text{Hom}_K^\text{cont}(V, K)$ denote the space of continuous functionals on $V$. We write $V'_c$ for the locally convex $K$-vector space $V'$ endowed with the topology of strong convergence. $G$ will always be a locally $L$-analytic group of finite dimension $d$ with center $Z$. The Lie algebra of $Z$ will be denoted by $\mathfrak{z}$. We also fix an exponential map $\exp : \mathfrak{g} \longrightarrow G$ defined locally around zero on the Lie algebra $\mathfrak{g}$ of $G$.

1 Locally analytic distributions

1.1 Functoriality

Recall that a topological Hausdorff space $M$ is called (strictly) paracompact if any open covering of $M$ admits a locally finite refinement by (pairwise disjoint) open subsets. Let $M$ be a paracompact, locally $L$-analytic manifold of finite dimension $d$. We note that in this situation $M$ is automatically strictly paracompact (cf. [23], p. 35). The locally convex $K$-vector space $C^a(M, K)$ of locally analytic functions on $M$ with values in $K$ is the locally convex inductive limit

$$C^a(M, K) = \lim_{\leftarrow} F_I(K)$$

where $I$ runs through the inductive system of all “indices”. An index $I$ is a family of charts $\{(D_i, \varphi_i)\}_{i \in I}$ of $M$ such that $(D_i)_{i \in I}$ is a covering of $M$ by disjoint
open subsets and such that each $\varphi_i(D_i)$ is an affinoid ball in $L^d$. Further,

$$\mathcal{F}_I(K) := \prod_{i \in I} \mathcal{F}_{\varphi_i}(K)$$

is the locally convex direct product of the $K$-Banach spaces $\mathcal{F}_{\varphi_i}(K)$ of functions $f : D_i \to K$ such that $f \circ \varphi_i^{-1}$ is a $K$-valued rigid analytic function on the affinoid ball $\varphi_i(D_i)$. The space of locally analytic distributions on $M$ is the locally convex $K$-vector space

$$D(M, K) := \prod_{i \in I} D(M_i, K)$$

If $(M_i)_{i \in I}$ is a covering of $M$ by disjoint open subsets $M_i$, then there is a topological isomorphism

$$C^an(M, K) \cong \prod_{i \in I} C^an(M_i, K)$$

dualizing to a topological isomorphism

$$D(M, K) \cong \bigoplus_{i \in I} D(M_i, K)$$

(cf. [14], Korollar 2.2.4). If $M$ is compact, then $C^an(M, K)$ is a $K$-vector space of compact type and, in particular, is reflexive (cf. [22], Proposition 9.10). In this case $D(M, K)$ is a nuclear Fréchet space (cf. [25] Theorem 1.3).

There is an embedding $M \hookrightarrow D(M, K)$, sending $m \in M$ to the Dirac distribution $\delta_m := (f \mapsto f(m))$.

**Lemma 1.1.1.** The subspace $K[M]$ of $D(M, K)$ generated by all Dirac distributions $\delta_m$, $m \in M$, is dense.

Choosing a covering $(M_i)_{i \in I}$ of $M$ by disjoint compact open subsets, (1.1) shows that $C^an(M, K)$ is reflexive (cf. [22], Proposition 9.10 and Proposition 9.11). Hence the proof of Lemma 1.1.1 can be done as in [25], Lemma 3.1.

Let $N, M$ be paracompact, locally $L$-analytic manifolds of finite dimension and $\varphi : N \to M$ be a morphism. $\varphi$ defines a $K$-linear map $\varphi^* : C^an(M, K) \to C^an(N, K)$ via $\varphi^*(f) := f \circ \varphi$ for $f \in C^an(M, K)$. Using the definition of $C^an(M, K)$ and $C^an(N, K)$ via indices one can show that $\varphi^*$ is continuous with respect to the locally convex topologies defined above (cf. [23], p. 65 or [14], Bemerkung 2.1.11). Thus, $\varphi^*$ dualizes to a continuous $K$-linear map $\varphi_* : D(N, K) \to D(M, K)$.

**Proposition 1.1.2.** Let $\varphi : N \to M$ be a closed embedding of paracompact, locally $L$-analytic manifolds of finite dimension. Then $\varphi^* : C^an(M, K) \to C^an(N, K)$ is a strict surjection and $\varphi_* : D(N, K) \to D(M, K)$ is a topological embedding.

Proof: Let $f \in C^an(N, K)$ and $a \in N$. There is an open neighborhood $U_a$ of $a$ in $N$, an open neighborhood $V_a$ of $\varphi(a)$ in $M$ and a locally analytic manifold $Z_a$ with the following properties: $\varphi$ restricts to a morphism

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\( \varphi_a : U_a \to V_a \) and there is an isomorphism \( g : V_a \to U_a \times Z_a \) such that \( pr_{U_a} \circ g \circ \varphi_a = id_{U_a} \) (cf. [7], 5.7.1; here \( pr_{U_a} \) is the projection onto \( U_a \)). It follows that \( f|U_a = \varphi_a^*(pr_{U_a} \circ g)^*(f|U_a)) \in im(\varphi_a^*). \)

Let \( C \) be a closed and open subset of \( M \) with \( \varphi(N) \subseteq C \subseteq \bigcup_{a \in N} V_a \) (cf. [23], p. 37). Choose a refinement \( (V_i)_{i \in I} \) of the open covering \( (C \cap V_a)_{a \in N} \) of \( C \) consisting of disjoint open subsets \( V_i \) of \( C \). For each \( i \in I \) choose a point \( a \in N \) such that \( V_i \subseteq V_a \). There is a function \( g_a \in C^{an}(V_a, K) \) such that \( \varphi_a^*(g_a) = f|U_a \). Set \( g_i := g_a|_{V_i} \in C^{an}(V_i, K) \) and \( g_{M \setminus C} := 0 \in C^{an}(M \setminus C, K) \). Then the family \( g := (g_{M \setminus C}, (g_i)_{i \in I}) \in C^{an}(M, K) \) satisfies \( \varphi^*(g) = f \), proving the surjectivity of \( \varphi^* \).

If \( (M_i)_{i \in I} \) is a covering of \( M \) by disjoint compact open subsets, \( N_i := \varphi^{-1}(M_i) \) and \( \varphi_i := \varphi|N_i \) for \( i \in I \) then \( \varphi^* \) is open if and only if all \( \varphi_i^* \) are. Hence we may assume \( M \) and \( N \) to be compact.

In this case both \( C^{an}(M, K) \) and \( C^{an}(N, K) \) are locally convex \( K \)-vector spaces of compact type. In particular, they carry the locally convex final topology with respect to a countable family of BH-spaces. Therefore, the claim follows from [22], Proposition 8.8, and the surjectivity of \( \varphi^* \).

If \( (M_i)_{i \in I} \) and \( (N_i)_{i \in I} \) are as above then \( \varphi_* \) is the direct sum of the maps \( (\varphi_i)_* : D(N_i, K) \to D(M_i, K) \). Since \( \varphi_i^* \) is strict surjective and \( (\varphi_i)_* \) is the corresponding dual map, \( (\varphi_i)_* \) is a topological embedding according to [25], Proposition 1.2 (i). The same is then true for \( \varphi_* \) by [22], Lemma 5.3 (i). \( \square \)

In the situation of Proposition 1.1.2 we will from now on write \( D(N, K) \subseteq D(M, K) \) for the topological embedding \( \varphi_* : D(N, K) \to D(M, K) \) of locally convex \( K \)-vector spaces.

If we assume \( M = G \) to be a finite dimensional, locally \( L \)-analytic group then \( D(G, K) \) carries the structure of a unital, associative \( K \)-algebra with separately continuous multiplication such that the natural inclusion \( K[G] \hookrightarrow D(G, K) \) becomes a homomorphism of rings (cf. [25], section 2). It is explicitly given by

\[
(\delta \cdot \delta')(f) = \delta'(g' \mapsto \delta(g \mapsto f(gg')))
\]

with \( \delta, \delta' \in D(G, K) \) and \( f \in C^{an}(G, K) \). If \( G_0 \) is an open subgroup of \( G \) then according to (1.2)

\[
D(G, K) \simeq \bigoplus_{g \in G/G_0} D(g \cdot G_0, K) \simeq \bigoplus_{g \in G/G_0} \delta_g \cdot D(G_0, K).
\]

If \( H \) is a closed locally \( L \)-analytic subgroup of \( G \) then the topological embedding \( D(H, K) \subseteq D(G, K) \) is a homomorphism of algebras.

### 1.2 The notion of support

**Definition 1.2.1.** The support \( supp(\delta) \) of a distribution \( \delta \in D(M, K) \) is the complement of the largest open subset \( U \) of \( M \) such that \( \delta(f) = 0 \) for all \( f \in C^{an}(M, K) \) with \( supp(f) \subseteq U \). If \( C \) is a subset of \( M \) and \( V \subseteq D(M, K) \) a subspace then we denote by \( V_C \) the subspace of all distributions \( \delta \in V \) whose support is contained in \( C \). Similarly, if \( W \) is a subspace of \( C^{an}(M, K) \) then \( W_C \) denotes the subspace of all locally analytic functions \( f \in W \) with \( supp(f) \subseteq C \).
Remark 1.2.2. The existence of $\text{supp}(\delta)$ for $\delta \in D(M,K)$ follows from the strict paracompactness of $M$: Let $U_1, U_2$ be open subsets of $M$ such that $\delta(f) = 0$ for all $f \in C^\text{an}(M,K)$ with $\text{supp}(f) \subset U_1$ or $\text{supp}(f) \subset U_2$, and let $f \in C^\text{an}(M,K)$ be supported on $U_1 \cup U_2$. There is a closed and open subset $A$ of $M$ with $\text{supp}(f) \subset A \subset U_1 \cup U_2$ (cf. [23], p. 37). Choose a refinement $(V_i)_{i \in I}$ of the covering $(U_1 \cap A, U_2 \cap A)$ of $A$ consisting of disjoint open subsets $V_i$ of $A$. Then $f|_A \in C^\text{an}(A,K) = \prod_{i \in I} C^\text{an}(V_i,K)$, i.e. $f|_A = (f_i)_{i \in I}$ with $f_i \in C^\text{an}(V_i,K)$ for all $i \in I$. Set $f^j := (f_i^j)_{i \in I}$, $j = 1,2$, with $f_i^1 := 0$ if $V_i \not\subset U_1 \cap A$ (i.e. $V_i \cap U_1 = \emptyset$), $f_i^2 := f_i$ if $V_i \subset U_1 \cap A$, $f_i^2 := 0$ if $V_i \subset U_1 \cap A$ and $f_i^2 := f_i$ if $V_i \not\subset U_1 \cap A$. Then $f^1, f^2 \in C^\text{an}(A,K)$ with $f^1 + f^2 = f|_A$. Extending $f^1, f^2$ by zero outside of $A$ we obtain functions $f^1, f^2 \in C^\text{an}(M,K)$ with $f^1 + f^2 = f$ and $\text{supp}(f^j) \subset U_j$, $j = 1,2$. By assumption $\delta(f) = \delta(f^1) + \delta(f^2) = 0$.

Remark 1.2.3. It follows from (1.2) that all locally analytic distributions on $M$ are compactly supported, i.e. $\text{supp}(\delta)$ is a compact subset of $M$ for all $\delta \in D(M,K)$.

If $M = G$ is a locally $L$-analytic group, $g \in G$ and $\delta \in D(G,K)$ then according to (1.3)

$$\text{supp}(\delta_\gamma \cdot \delta) = g \cdot \text{supp}(\delta) \text{ and } \text{supp}(\delta \cdot \delta_\gamma) = \text{supp}(\delta) \cdot g.$$  

More generally we still have:

Lemma 1.2.4. If $\delta_1, \delta_2 \in D(G,K)$ then $\text{supp}(\delta_1 \cdot \delta_2) \subset \text{supp}(\delta_1) \cdot \text{supp}(\delta_2)$.

Proof: Let $g \in \text{supp}(\delta_1 \cdot \delta_2)$. Then for any open subgroup $H \subset G$ there is a function $f \in C^\text{an}(G,K)$ supported on $gH$ with $(\delta_1 \delta_2)(f) = \delta_2(h \mapsto \delta_1(R_h f)) \neq 0$ (here $R_h$ is the right translation operator associated with $h$). Hence there are elements $\gamma_1 \in \text{supp}(\delta_1)$ and $h \in H$ such that $\text{supp}(\delta_1) \cap (\text{supp}(f) \cdot h^{-1}, \gamma_1^{-1}) \neq \emptyset$. Since $\text{supp}(f) \subset gH$ there is $h' \in H$ and $\gamma_1 \in \text{supp}(\delta_1)$ such that $\gamma_1 = g h'^{-1} \gamma_1^{-1}$, i.e. $g = \gamma_1 \gamma_2 (h')^{-1}$. It follows that $g \in \text{supp}(\delta_1) \cdot \text{supp}(\delta_2)$ because $H$ is arbitrary and $\text{supp}(\delta_1) \cdot \text{supp}(\delta_2)$ is closed (even compact). \hfill \Box

For a closed subset $C$ of $G$ the locally convex $K$-vector space $C^\infty_C(G,K)$ of generalized germs in $C$ is the quotient space

$$C^\infty_C(G,K) := C^\text{an}(G,K)/C^\text{an}(G,K)_{G \setminus C}$$

(cf. [14], Definition 2.3.3). If $C$ is compact then there is a topological isomorphism

$$C^\infty_C(G,K) = \lim_{\rightarrow U} C^\text{an}(U,K)$$

with $U$ running through the inductive system of open subsets of $G$ containing $C$ and transition maps defined by restriction of functions. In this case the inductive limit topology on $C^\infty_C(G,K)$ is Hausdorff. If $C = \{g\}$ is a singleton we write $C^\infty_g(G,K)$ instead of $C^\infty_{\{g\}}(G,K)$.

Lemma 1.2.5. $C^\text{an}(G,K)_C$ is a closed subspace of $C^\text{an}(G,K)$ for any subset $C$ of $G$. If $C$ is closed then $D(G,K)_C$ is a closed subspace of $D(G,K)$ and there is a topological isomorphism

$$D(G,K)_C \simeq C^\infty_C(G,K)_b.$$  

If $C$ is compact then this is an isomorphism of nuclear $K$-Fréchet spaces.
Proof: Let $C$ be a subset of $G$. As mentioned in [loc.cit.], section 2.3.1, $C^{an}(G,K)_C$ is equal to the intersection of the kernels of all continuous surjections $C^{an}(G,K) \to C^{an}(G,K)_{g} \; , \; g \in G \setminus C$, hence is closed in $C^{an}(G,K)$. If $C$ is closed in $G$ then $D(G,K)_C$ is the orthogonal space of $C^{an}(G,K)_{g} \setminus C$ with respect to the natural pairing $D(G,K) \times C^{an}(G,K) \to K$ so that $D(G,K)_C$ is closed, as well. Further, the reflexivity of $D(G,K)$ implies by means of [6], IV.2.2 Corollary, that

$$(D(G,K)_C)'_{b} \simeq D(G,K)'_{b}/D(G,K)_{C}$$

where $D(G,K)'_{b}$ denotes the orthogonal subspace of $D(G,K)_C$ with respect to the pairing $D(G,K)'_{b} \times D(G,K) \to K$. Since $C^{an}(G,K)$ is reflexive and $C^{an}(G,K)_{g}$ is closed $D(G,K)'_{b} \simeq C^{an}(G,K)'_{b} = C^{an}(G,K)_{g} \cap C$. It follows that

$$(D(G,K)_C)'_{b} \simeq C^{an}(G,K)/C^{an}(G,K)_{g} \cap C.$$ 

If $G_0$ is a compact open subgroup of $G$ then by (1.2) and [22], Lemma 5.3

$$D(G,K)_C = \oplus_{g \in G/G_0} D(gG_0, K)_{gG_0 \cap C}$$

showing that $D(G,K)_C$ is reflexive $(D(gG_0, K)_{gG_0 \cap C}$ is a closed subspace of the nuclear Fréchet space $D(gG_0, K)$). Thus, (1.6) follows. The last claim follows from $C^{an}_{G}(G,K)$ being of compact type if $C$ is compact (cf. [14], Satz 2.3.2). □

Corollary 1.2.6. If $C$ is a closed subset of $G$ such that $1 \in C$ and $C \cdot C \subseteq C$ then $D(G,K)_C$ is a closed subalgebra of $D(G,K)$. If in addition $C$ is compact then $D(G,K)_C$ is a nuclear $K$-Fréchet algebra. □

Remark 1.2.7. Let $G_0$ be a compact open subgroup of $G$. If $H$ is a locally $L$-analytic subgroup of $G$ and $H_0 := H \cap G_0$ then as seen above

$$D(G,K)_H = \bigoplus_{g \in G/G_0} D(gG_0, K)_{gG_0 \cap H}$$

as locally convex $K$-vector spaces. Noting that $D(gG_0, K)_{gG_0 \cap H} \neq 0$ if and only if $gG_0 \cap H \neq \emptyset$ we get

$$D(G,K)_H = \bigoplus_{b \in H/H_0} \delta_h \cdot D(G_0,K)_{H_0}.$$ 

(1.7)

According to [14], Bemerkung 3.1.2 and Satz 3.3.4, the Lie algebra $g$ of $G$ acts on $C^{an}(G,K)$ via continuous endomorphisms defined by

$$\rho(f)(g) := \frac{d}{dt} f(exp(-tg)|_{t=0}$$ for $\rho \in g$ and $f \in C^{an}(G,K)$. 

This action extends to an action of the universal enveloping algebra $U(g)$ of $g$ on $C^{an}(G,K)$.

According to Lemma 1.2.5 and Corollary 1.2.6 $C^{an}_{G}(G,K)'_{b} \simeq D(G,K)_{(1)}$ is a $K$-Fréchet subalgebra of $D(G,K)$. Fixing an ordered $L$-basis $X = (x_1, \ldots , x_d)$
of \( \mathfrak{g} \) the action of \( U(\mathfrak{g}) \) on \( C^{an}(G, K) \) leads to the following explicit description of \( C^\omega_t(G, K)_b \) (cf. [25], Lemma 2.4):

\[
C^\omega_t(G, K)_b = \left\{ \sum_{\alpha \in \mathbb{N}^d} d_\alpha \mathbf{x}^\alpha | \ d_\alpha \in K, \forall \alpha > 0 : \sup_\alpha |d_\alpha| r^{-|\alpha|} < \infty \right\},
\]

where for \( \alpha = (\alpha_1, \ldots, \alpha_d) \) we set \( |\alpha| := \alpha_1 + \ldots + \alpha_d \) and \( \alpha! := \alpha_1! \cdot \ldots \cdot \alpha_d! \). Further, \( \mathbf{x}^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d} \) is viewed as a distribution via

\[
(1.8) \quad \mathbf{x}^\alpha(f) = (-\mathbf{1})^{\alpha_1} \cdot \ldots \cdot (-\mathbf{1})^{\alpha_d} f(1) \quad \text{for} \quad f \in C^{an}(G, K).
\]

Finally, the Fréchet topology of \( C^\omega_t(G, K)_b \) is defined by the family of norms \( (\nu_r)_{r>0} \) with \( \nu_r(\sum_{\alpha} d_\alpha \mathbf{x}^\alpha) := \sup_\alpha |d_\alpha| r^{-|\alpha|} \).

Letting \( \lambda \mapsto \lambda \) denote the unique anti-automorphism of \( U(\mathfrak{g}) \otimes_L K \) extending multiplication by \(-1\) on \( \mathfrak{g} \), the natural homomorphism \( \lambda \mapsto (f \mapsto \lambda(f(1))) : U(\mathfrak{g}) \otimes_L K \to C^\omega_t(G, K)_b \) of \( K \)-algebras is injective.

**Proposition 1.2.8.** \( U(\mathfrak{g}) \otimes_L K \) is dense in \( C^\omega_t(G, K)_b \). We have

\[
C^\omega_t(G, K)_b = \left\{ \sum_{\alpha} d_\alpha \mathbf{x}^\alpha | \ d_\alpha \in K, \forall \alpha > 0 : \sup_\alpha |d_\alpha| r^{-|\alpha|} < \infty \right\}
\]

and the Fréchet topology of \( C^\omega_t(G, K)_b \) can be defined by the family of norms \( (\nu_r)_{r>0} \) with \( \nu_r(\sum_{\alpha} d_\alpha \mathbf{x}^\alpha) := \sup_\alpha |d_\alpha| r^{-|\alpha|} \).

**Proof:** Since \( |\alpha| \leq 1 \) the right hand side of (1.9) is contained in \( C^\omega_t(G, K)_b \). Conversely, \( |\alpha| \leq 1 \) yields \( |d_\alpha| r^{-|\alpha|} \leq \rho^{-|\alpha|/(p-1)} \), so that if \( \sup_\alpha |d_\alpha| r^{-|\alpha|} < \infty \) for all \( r > 0 \) then also \( \sup_\alpha |d_\alpha| r^{-|\alpha|} < \infty \) for all \( r > 0 \). This proves the reverse inclusion as well as the fact that the two families of norms \( (\nu_r')_{r>0} \) and \( (\nu_r)_{r>0} \) are equivalent.

The density statement is clear. \( \square \)

**Remark 1.2.9.** When working with \( C^\omega_t(G, K)_b \) we will henceforth use the description given by (1.9) and assume its topology to be defined by the family of norms \( (\nu_r)_{r>0} \). To simplify notation we write \( U(\mathfrak{g}, K) := C^\omega_t(G, K)_b \).

**Lemma 1.2.10.** If \( C \) is a closed subset of \( G \) then the \( U(\mathfrak{g}, K) \)-submodule of \( D(G, K)_C \) generated by all Dirac distributions \( \delta_c, c \in C \), is dense.

**Proof:** Let \( \Delta := \text{closure of } \sum_{c \in C} \delta_c \cdot U(\mathfrak{g}, K) \text{ in } D(G, K) \). It follows from Lemma 1.2.4 and Lemma 1.2.5 that \( \Delta \subseteq D(G, K)_C \). We know that \( C^{an}(G, K)/C^{an}(G, K)_C \) is reflexive. Let \( \ell \) be a continuous functional on \( D(G, K)_C \) vanishing on \( \Delta \). By (1.5) and (1.6), \( \ell \) corresponds to an element \( \overline{\ell} \) of \( C^{an}(G, K)/C^{an}(G, K)_C \). To say \( \ell \) vanishes on \( \Delta \) is to say that any representative \( f \) of \( \overline{\ell} \) in \( C^{an}(G, K) \) vanishes in an open neighborhood of \( C \). Hence \( f \in C^{an}(G, K)_C \), i.e. \( \overline{\ell} = 0 \), and \( \Delta = D(G, K)_C \) by the Hahn-Banach theorem. \( \square \)

**Remark 1.2.11.** Let \( B \) and \( C \) be locally convex \( K \)-vector spaces carrying separately continuous \( K \)-algebra structures with a common \( K \)-subalgebra \( A \). If \( B \otimes K, C \) denotes the Hausdorff completion of the algebraic tensor product \( B \otimes K, C \) endowed with its inductive tensor product topology then we let \( B \otimes A, C \)
be the quotient of $B \hat{\otimes}_A C$ by the closure of the subspace generated by all elements of the form

$$ba \otimes c - b \otimes ac, \quad a \in A, b \in B \text{ and } c \in C.$$  

We endow $B \hat{\otimes}_A C$ with the corresponding quotient topology. If $B$ and $C$ are $K$-Fréchet spaces then the inductive and the projective tensor product topologies on $B \otimes_K C$ coincide. Therefore, we omit the $\iota$ from the notation and simply write $B \hat{\otimes}_K C$ and $B \hat{\otimes}_A C$.

Note that $B \hat{\otimes}_A C$ is naturally a $B$-$C^{\text{op}}$-bimodule ($C^{\text{op}}$ being the $K$-algebra opposite to $C$). If $A$ is contained in the centers of $B$ and $C$ then $B \hat{\otimes}_A C$ is naturally a module over $B \otimes_K C$ and even over $B \otimes_A C$.

Let $H$ be a closed, locally $L$-analytic subgroup of $G$ and $\mathfrak{h}$ its Lie algebra. The multiplication map

$$(1.10) \quad D(H, K) \times U(\mathfrak{g}, K) \rightarrow D(G, K)_H$$

induces a continuous $K$-linear map

$$\mu : D(H, K) \hat{\otimes}_{U(\mathfrak{h}, K)} U(\mathfrak{g}, K) \rightarrow D(G, K)_H.$$  

**Proposition 1.2.12.** If $K$ is discretely valued then $\mu$ is a topological isomorphism of $D(H, K) - U(\mathfrak{g}, K)^{\text{op}}$-bimodules.

Proof: In Corollary 1.3.6 and Corollary 1.4.3 we will prove that there is a compact open subgroup $G_0$ of $G$ with the following properties: $D(G_0, K)$ is a $K$-Fréchet-Stein algebra with respect to a family of norms $\| \cdot \|_{\tau}$, $r \in p^\mathbb{Q}$, $1/p < r < 1$, such that the completion $D_r(G_0, K)$ of $D(G_0, K)$ with respect to the norm $\| \cdot \|_{\tau}$ is finitely generated and free as a module over the closure $U_r(\mathfrak{g}, K)$ of $U(\mathfrak{g}, K)$ in $D_r(G_0, K)$; if $H_0 := H \cap G_0$ then $D(H_0, K)$ is a $K$-Fréchet-Stein algebra with respect to the family of norms $\| \cdot \|_{\tau}$ restricted to $D(H_0, K)$; for each $r$ the closure $D_r(H_0, K)$ of $D(H_0, K)$ in $D_r(G_0, K)$ is finitely generated and free as a module over the closure $U_r(\mathfrak{h}, K)$ of $U(\mathfrak{h}, K)$ in $D_r(H_0, K)$; $U_r(\mathfrak{g}, K)$ and $U_r(\mathfrak{h}, K)$ are noetherian $K$-Banach algebras.

**Lemma 1.2.13.** If $(V_i)_{i \in I}$ and $W$ are Hausdorff locally convex $K$-vector spaces then there is a topological isomorphism

$$(\bigoplus_{i \in I} V_i) \hat{\otimes}_K W \simeq \bigoplus_{i \in I} (V_i \hat{\otimes}_K W).$$

Proof: This is a straightforward generalization of [18], I.3.1 Proposition 14.I, to the non-archimedean setting. \qed

By (1.7), Lemma 1.2.13 and [22], Lemma 5.3, it suffices to show that the map

$$D(H_0, K) \hat{\otimes}_{U(\mathfrak{h}, K)} U(\mathfrak{g}, K) \rightarrow D(G_0, K)_{H_0}$$

is a topological isomorphism. We again denote it by $\mu$. Let $r \in p^\mathbb{Q}$ with $1/p < r < 1$. The multiplication in $D_r(G_0, K)$ induces a continuous $K$-linear map

$$\mu_r : D_r(H_0, K) \otimes_K U_r(\mathfrak{g}, K) \rightarrow D_r(G_0, K)_{H_0}.$$
here $D_r(G_0, K)_{H_0}$ denotes the closure of $D(G_0, K)_{H_0}$ in $D_r(G_0, K)$. In the proof of Corollary 1.4.3 we will show that $D_r(G_0, K)_{H_0}$ is free and finitely generated as a module over $U_r(g, K)$ and has a basis $(b^a)_{a \in A'}$ in $K[H_0]$ which is simultaneously a basis of the free $U_r(h, K)$-module $D_r(H_0, K)$. Hence $\mu_r$ induces a continuous $K$-linear bijection

\[(1.11) \quad D_r(H_0, K) \otimes_{U_r(h, K)} U_r(g, K) \longrightarrow D_r(G_0, K)_{H_0}, \]

$D_r(H_0, K)$ and $U_r(g, K)$ are complete normed modules over the noetherian $K$-Banach algebra $U_r(h, K)$. Further, $D_r(H_0, K)$ is a finitely generated, free $U_r(h, K)$-module and therefore topologically isomorphic to a direct sum of copies of $U_r(h, K)$ (cf. [26], Proposition 2.1 (iii)). A straightforward generalization to the non-commutative setting of [2], 2.1.7 Proposition 6, shows that $D_r(H_0, K) \otimes_{U_r(h, K)} U_r(g, K)$ is a complete normed space with respect to the tensor product norm. By the open mapping theorem (1.11) is a topological isomorphism. In addition,

\[\overline{\ker \mu_r} \quad \text{is the closure of } \ker \mu_r \quad \text{in } D_r(H_0, K) \otimes_K U_r(g, K). \]

Thus, we obtain a short exact sequence of strict continuous $K$-linear maps between Banach spaces

\[0 \longrightarrow \overline{\ker \mu_r} \longrightarrow D_r(H_0, K) \otimes_K U_r(g, K) \longrightarrow D_r(G_0, K)_{H_0} \longrightarrow 0.\]

Recall that $U := \ker(D(0, K) \otimes_K U(g, K) \longrightarrow D(H_0, K) \otimes_{U(h, K)} U(g, K))$ is the closure of the subspace of $D(H_0, K) \otimes_K U(g, K)$ generated by all elements of the form

\[\lambda \eta \otimes \varphi - \lambda \otimes \eta \varphi \quad \text{with } \lambda \in D(0, K), \eta \in U(h, K) \quad \text{and } \varphi \in U(g, K).\]

Since by (1.11) the kernel of $\mu_r$ is the vector space generated by all elements of the form

\[\lambda \eta \otimes \varphi - \lambda \otimes \eta \varphi \quad \text{with } \lambda \in D_r(H_0, K), \eta \in U_r(h, K) \quad \text{and } \varphi \in U_r(g, K)\]

$U \subseteq \overline{\ker \mu_r}$ is dense for all $r$. Therefore, the system $(\overline{\ker \mu_r})$ with $r \in \mathbb{Z}^2$ and $1/p < r < 1$ satisfies the Mittag-Leffler property as formulated in [17], 13.2.4. By [loc.cit.], 13.2.2, we obtain an exact sequence

\[0 \longrightarrow U = \overline{\lim_r \ker \mu_r} \longrightarrow D(H_0, K) \otimes_K U(g, K) \longrightarrow D(G_0, K)_{H_0} \longrightarrow 0,\]

because

\[\overline{\lim_r (D_r(H_0, K) \otimes_K U_r(g, K))} \simeq (\overline{\lim_r D_r(H_0, K)}) \otimes_K (\overline{\lim_r U_r(g, K)})\]

(cf. [12], Proposition 1.1.29). It induces a continuous $K$-linear bijection

\[D(H_0, K) \otimes_{U(h, K)} U(g, K) \longrightarrow D(G_0, K)_{H_0}\]

which is a topological isomorphism by the open mapping theorem. That it coincides with $\mu$ is clear from the fact that for each $r$ the restriction of $\mu_r$ to $D(H_0, K) \otimes_K U(g, K)$ is induced by the multiplication in $D(G_0, K)$. \qed
Remark 1.2.14. Assume there is a compact open subgroup $G_0$ of $G$ and a closed locally $L$-analytic subgroup $C_0$ of $G_0$ such that $G_0 = H_0 \times C_0$ as locally $L$-analytic groups with $H_0 := H \cap G_0$. Then the above proposition can be proved without any allusion to Fréchet-Stein structures and simplifies in the following manner: According to Proposition A.3 and Remark A.4 of [28] there is a topological isomorphism

$$D(H_0, K) \hat{\otimes}_K D(C_0, K) \rightarrow D(G_0, K)$$

induced by multiplication. It follows from Lemma 1.2.10 and [22], Corollary 17.5 (ii) and Proposition 19.10 (i), that the preimage of $D(G_0, K)_{H_0}$ under this map is $D(H_0, K) \hat{\otimes}_K U(\epsilon, K)$ where $\epsilon$ is the Lie algebra of $C_0$. Hence we obtain from Lemma 1.2.13 that

$$D(G, K)_H \simeq D(H, K) \hat{\otimes}_K, U(\epsilon, K).$$

1.3 Restriction of the base field

Let $L_0|\mathbb{Q}_p$ be an extension of fields with $L_0 \subseteq L$ and let $R^{L|L_0}$ be the functor "restriction of the base field from $L$ to $L_0$" from the category of paracompact locally $L$-analytic manifolds to the category of locally analytic manifolds of the same type over $L_0$ (cf. [7], 5.14).

There is a natural embedding

$$\tau : C^{an}(G, K) \rightarrow C^{an}(R^{L|L_0} G, K)$$

mapping $C^{an}(G, K)$ homeomorphically onto its closed image (cf. [24], Lemma 1.2).

Lemma 1.3.1. The dual map $\tau' : D(R^{L|L_0} G, K) \rightarrow D(G, K)$ is a strict surjection and a homomorphism of $K$-algebras.

Proof: Since $\tau'$ restricts distributions on $R^{L|L_0} G$ to the subspace $C^{an}(G, K)$ of $C^{an}(R^{L|L_0} G, K)$ it is clear that $\tau'$ is a homomorphism of $K$-algebras. To show the surjectivity we may assume $G$ to be compact. But then $\tau$ is a topological embedding of spaces of compact type so that the claim follows from [25], Proposition 1.2 (i).

Consider the ideal $I := \ker(\tau')$ of $D(R^{L|L_0} G, K)$. It is the orthogonal subspace of $C^{an}(G, K)$ with respect to the natural pairing

$$D(R^{L|L_0} G, K) \times C^{an}(R^{L|L_0} G, K) \rightarrow K.$$ 

Since $D(R^{L|L_0} G, K)$ is reflexive we obtain by means of [6], IV.2.2 Corollary, that $I_b'$ is topologically isomorphic to $C^{an}(R^{L|L_0} G, K)/C^{an}(G, K)$. The topological isomorphism $I \simeq \oplus_{g \in G/G_0} \ker((\tau |C^{an}(gG_0, K))'$ for a compact open subgroup $G_0$ of $G$ shows that $I$ itself is reflexive. Thus, there is a topological isomorphism

$$I \simeq (C^{an}(R^{L|L_0} G, K)/C^{an}(G, K))'_b.$$

In order to give an explicit description of the locally $L$-analytic functions inside $C^{an}(R^{L|L_0} G, K)$ we follow the arguments given in section 1 of [24]. If we write $g_{L_0}$ for $g$ viewed as a Lie algebra over $L_0$ then $g_{L_0}$ can be identified with the Lie algebra of $R^{L|L_0} G$. 

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Lemma 1.3.2. $C^\alpha(G, K)$ is the closed subspace of all those functions $f \in C^\alpha(R^{L_0} G, K)$ for which $(t\tau)(f) = t \cdot \tau(f)$ for all $t \in L$ and all $\tau \in \mathfrak{g}_{L_0}$.

Proof: If we let $W$ be the subspace of $C^\alpha(R^{L_0} G, K)$ consisting of all functions with the above property then $C^\alpha(G, K) \subseteq W$. Let $f \in W$. If $\tau, \eta \in \mathfrak{g}$ and $t \in L$ then

$$
(t\tau)(\eta(f)) = \eta((t\tau)(f)) + [t\tau, \eta](f) = \eta(t \cdot \tau(f)) + (t \cdot [\tau, \eta])(f)
$$

shows that $W$ is $\mathfrak{g}_{L_0}$-invariant. Therefore, the proof of [loc.cit.], Lemma 1.1, generalizes to the non-commutative setting in the following manner: Fix an $L$-basis $\mathfrak{X} = (x_1, \ldots, x_d)$ of $\mathfrak{g}$. Choose an orthonormal basis $(v_1, \ldots, v_n)$ of $L$ as a vector space over $L_0$ and put $\mathfrak{Y} := (v_1 \otimes v_1, \ldots, v_n \otimes v_n)$. The corresponding system $\theta_{L_0}$ of canonical coordinates of the second kind is defined by

$$
\theta_{L_0}(\sum_{i,j} t_{ij} v_i \otimes v_j) := \exp(t_{11} v_1 \otimes v_1) \exp(t_{21} v_2 \otimes v_1) \cdots \exp(t_{nd} v_n \otimes v_d)
$$

for $t_{ij}$ sufficiently close to zero in $L_0$ (cf. [4], III.4.3 Proposition 3). Given $g \in R^{L_0}G$ we have the expansion

$$(R_g f \circ \theta_{L_0})(\sum_{i,j} t_{ij} v_i \otimes v_j) = \sum_{\beta \in \mathbb{N}^n \times \mathbb{N}^d} c_\beta t^\beta$$

converging for all $t_{ij}$ near zero in $L_0$; here $c_\beta \in K$, $t^\beta := \prod_{i,j} t_{ij}^{\beta_{ij}}$ and $R_g$ is the right translation operator associated with $g$. Letting $\mathfrak{Y}^\beta(R_g f) := (v_1 \otimes v_1)^{\beta_{11}} \circ (v_2 \otimes v_1)^{\beta_{21}} \cdots \circ (v_n \otimes v_1)^{\beta_{nd}}(R_g f)$ it follows from the remarks after Lemma 4.7.2 of [14] that

$$
c_\beta = (-1)^{|\beta|} \frac{|\beta|!}{\beta!} \mathfrak{Y}^\beta(R_g f)(1) = (-1)^{|\beta|} \frac{|\beta|!}{\beta!} \mathfrak{Y}^\beta(f)(g)
$$

for all $\beta \in \mathbb{N}^n \times \mathbb{N}^d$ where $|\beta|$ and $|\beta|$ are as in subsection 1.2. Letting $\varphi(\beta) := (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j := \beta_{11} + \cdots + \beta_{nj}$, $b_\varphi(\beta) := c_{(\alpha_1, 0, \ldots, 0, \alpha_d, 0, \ldots)}$ and $\mathfrak{Y}^\varphi(\beta)(R_g f) := x_1^{\alpha_1} \cdots x_d^{\alpha_d}(R_g f)$ we deduce

$$
\mathfrak{Y}^\beta(f)(g) = \prod_{i=1}^n v_i^{\beta_{i1} + \cdots + \beta_{id}} \cdot \mathfrak{X}^\varphi(\beta)(f)(g)
$$

from the assumption on $f$ and the $\mathfrak{g}_{L_0}$-invariance of $W$. Thus

$$
c_\beta = b_\varphi(\beta) \frac{|\beta|!}{\beta!} \prod_{i=1}^n v_i^{\beta_{i1} + \cdots + \beta_{id}}
$$

for all $\beta$. Since this is precisely the relation given in the proof of [24], Lemma 1.1, we may conclude that $f$ is locally $L$-analytic at $g$. □

The proof of the following lemma uses the same Hahn-Banach argument as the proof of Lemma 1.2.10.

Lemma 1.3.3. If $J := I \cap (U(\mathfrak{g}_{L_0}) \otimes L_0) K$ then the vector space $\sum_{g \in G} \delta_g \cdot J$ is dense in $I$. □
Lemma 1.3.4. Let $C \subseteq G$ be a closed subset, considered also as a subset of $R^{|L_{\mathsf{loc}}|G}$. Then the image of $D(R^{|L_{\mathsf{loc}}|G}, K)_C$ under $\tau'$ is dense in $D(G, K)_C$.

Proof: That $\tau'(D(R^{|L_{\mathsf{loc}}|G}, K)_C)$ is contained in $D(G, K)_C$ follows from

$C^{an}(G, K)_C \subseteq C^{an}(R^{|L_{\mathsf{loc}}|G}, K)_C \cap C^{an}(G, K)_C$.

The same equation shows that $\tau$ induces a continuous injection

$C^{an}(G, K)/C^{an}(G, K)_C \leftarrow C^{an}(R^{|L_{\mathsf{loc}}|G}, K)/C^{an}(R^{|L_{\mathsf{loc}}|G}, K)_C$.

We know from the proof of Lemma 1.2.10 that the locally convex $K$-vector spaces on both sides are reflexive so that as a consequence of the Hahn-Banach Theorem the dual map $\tau' : D(R^{|L_{\mathsf{loc}}|G}, K)_C \rightarrow D(G, K)_C$ has to have dense image. \qed

We recall the following basic definitions (cf. [9], Part I): If $G$ is a pro-$p$ group set $P_i(G) := G$ and $P_{i+1}(G) := P_i^p[P_i(G), G]$ for $i \geq 1$. Here $P_i(G)^p[P_i(G), G]$ denotes the subgroup of $G$ generated by the $p$th powers of elements of $P_i(G)$ and by all commutators $[a, b]$ with $a \in P_i(G)$ and $b \in G$; $\tilde{X}$ denotes the topological closure of a subset $X$ of $G$. A pro-$p$ group $G$ is called powerful if $p$ is odd and $G/G^p$ is abelian or if $p = 2$ and $G/G^4$ is abelian. A pro-$p$ group $G$ is called uniform if it is topologically finitely generated, powerful and if $(P_i(G) : P_1(G)) = (G : P_2(G))$ for all $i \geq 1$.

One of the most fundamental properties of a uniform pro-$p$ group $G$ is given by the following theorem ([loc.cit.], Theorem 4.9): If $(a_1, \ldots, a_d)$ is a system of topological generators of $G$ with $d = \dim G$ then every element has a unique expression of the form $a_1^{\lambda_1} \cdots a_d^{\lambda_d}$ with $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}_p$. The resulting bijection $\mathbb{Z}_p^d \simeq G$ is a homeomorphism. In this way, uniform pro-$p$ groups turn out to be the fundamental examples of locally $\mathbb{Q}_p$-analytic groups ([loc.cit.], Theorem 8.32).

Assume $L_0 = \mathbb{Q}_p$. For further applications we need the following technical results:

Proposition 1.3.5. Let $G$ be a locally $L$-analytic group. Then there is an open subgroup $G_0$ of $G$ and a $\mathbb{Z}_p$-lattice $\Lambda \subset \mathfrak{g}_{\mathbb{Q}_p}$ with the following properties:

1. there is an $L$-basis $(\xi_1, \ldots, \xi_d)$ of $\mathfrak{g}$ and a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{g}_L$ such that $(v_1\xi_1, \ldots, v_m\xi_d)$ is a $\mathbb{Z}_p$-basis of $\Lambda$;

2. the corresponding canonical coordinates of the second kind give a well defined isomorphism $\theta_{G_0} : \Lambda \rightarrow R^{|L|G_0}$ of locally $\mathbb{Q}_p$-analytic manifolds;

3. $R^{|L|G_0}$ is a uniform pro-$p$ group.

Proof: Let $(\xi_1, \ldots, \xi_d)$ be an $L$-basis of $\mathfrak{g}$ and $\theta_L$ the corresponding system of canonical coordinates of the second kind. Since $\theta_L$ is étale at $0 \in \mathfrak{g}$ we may choose an open subgroup $G'$ of $G$ and an open neighborhood $U$ of zero in $\mathfrak{g}$ such that $\theta_L : U \rightarrow G'$ is an isomorphism of locally $L$-analytic manifolds. Let $\Phi_L$ be its inverse. According to [4], III.7.3 Théorème 4 and its proof there is $\lambda \in L^*$ such that $\mathfrak{g}_L \cdot \mathfrak{m}_L \subseteq \lambda \cdot \Phi_L(G') = \lambda \cdot U$ and the group structure on $\oplus_i \lambda^{-1} \mathfrak{m}_L \xi_i$ obtained by transport of structure from $G'$ is given by formal power series with
If $p$ is odd set \( \Lambda := \oplus_i \lambda^{-1} m^2_{L_i} \) and \( \Lambda := \oplus_i \lambda^{-1} m^{2e_{L_i}} \) otherwise. By [loc.cit.], III.7.4 Proposition 5, \( G_0 := \theta_L(\Lambda) \) is an open subgroup of \( G \). Choosing a \( \mathbb{Z}_p \)-basis \((v_1, \ldots, v_m)\) of \( \mathfrak{o}_L \) the canonical coordinates of the second kind

\[
\theta_{Q_p} : \mathfrak{g}_{Q_p} \longrightarrow R^{L[Q_p]}G
\]

corresponding to the decomposition \( \mathfrak{g}_{Q_p} = \oplus_{i,j} Q_p \lambda^{-1} v_{ij} \) coincide with \( \theta_L \) as \([v_{ij}, v_{kj}] = 0\) in \( \mathfrak{g}_L \) and because of the properties of the exponential map.

Since \( m^2_{L_i} = p^d \mathfrak{o}_L \) (resp. \( 4 \mathfrak{o}_L \) if \( p = 2 \)) (i) and (ii) are proved if for (i) we choose \((\lambda^{-1} p_{L_i})\) as an \( L \)-basis of \( \mathfrak{g} \) (resp. \((\lambda^{-1} 4_{L_i})\) if \( p = 2 \)).

It remains to show that \( \theta_{Q_p}(\Lambda) = R^{L[Q_p]}G_0 \) is a uniform pro-\( p \) group. According to [9], Theorem 8.31, we only need to show that \( R^{L[Q_p]}G_0 \) is a standard group in the sense of [loc.cit.], Definition 8.22. This follows directly from the construction. \( \square \)

If \( H \) is a closed, uniform subgroup of a uniform pro-\( p \) group \( G \) then we say that \( H \) is compatible with \( G \) if there is a basis of topological generators of \( H \) that can be extended to a basis of topological generators of \( G \).

**Corollary 1.3.6.** Let \( G \) be a locally \( L \)-analytic group and \( H \) a closed locally \( L \)-analytic subgroup. Then there is an open subgroup \( G_0 \) of \( G \) as in Proposition 1.3.5 such that \( H_0 := H \cap G_0 \) as an open subgroup of \( H \), satisfies conditions (i) – (iii) of Proposition 1.3.5 and \( R^{L[Q_p]}H_0 \) is compatible with \( R^{L[Q_p]}G_0 \).

Proof: Extend an \( L \)-basis \((f_1, \ldots, f_d)\) of the Lie algebra \( \mathfrak{h} \) of \( H \) to an \( L \)-basis \((f_1, \ldots, f_d)\) of \( \mathfrak{g} \), \( j \leq d \). We may assume \( U \) and \( G' \) from the proof of Proposition 1.3.5 to satisfy \( \Phi_L(H \cap G') \subseteq \mathfrak{h} \). Starting with \( G' \) define \( \Lambda \subseteq U \) and \( G_0 \subseteq G' \) as before. Then \( \Lambda' := \Lambda \cap \mathfrak{h} \) is an open neighborhood of \( 0 \) in \( \mathfrak{h} \) and a direct summand of \( \Lambda \). Therefore, the restriction of \( \theta_L \) from \( \Lambda \) to \( \Lambda' \) is an isomorphism \( \Lambda' \rightarrow H_0 := G_0 \cap H \) of locally \( L \)-analytic manifolds. It follows as above that \( H_0 \) satisfies conditions (i) – (iii) of Proposition 1.3.5 with respect to \( \Lambda' \). By definition, \( \Lambda \) (resp. \( \Lambda' \)) gives rise to the basis of topological generators \((\exp(v_{ij} f_k))\), \( 1 \leq k \leq m \), \( 1 \leq i \leq d \), (resp. \( 1 \leq k \leq m \), \( 1 \leq i \leq j \)) of \( R^{L[Q_p]}G_0 \) (resp. \( R^{L[Q_p]}H_0 \)). Thus, \( R^{L[Q_p]}G_0 \) and \( R^{L[Q_p]}H_0 \) are compatible. \( \square \)

### 1.4 Explicit Fréchet-Stein structures

The notion of a \( K \)-Fréchet-Stein algebra was first introduced by P. Schneider and J. Teitelbaum (cf. [26], section 3): A \( K \)-Fréchet algebra \( A \) is called a \( K \)-Fréchet-Stein algebra if there is a sequence \( q_1 \leq q_2 \leq \ldots \) of continuous algebra seminorms on \( A \) defining its Fréchet topology such that for all \( n \in \mathbb{N} \) the Hausdorff completion \( A_{q_n} \) of \( A \) with respect to \( q_n \) is a (left) noetherian \( K \)-Banach algebra and a flat \( A_{q_{n+1}} \)-module via the natural map \( A_{q_{n+1}} \rightarrow A_{q_n} \). In this subsection we will assume \( K \) to be discretely valued.

Let \( G_0 \) be a uniform pro-\( p \) group with a basis \((a_1, \ldots, a_d)\) of topological generators. Putting \( b_i := a_i - 1 \) and \( b^{a} := b_1^{a_1} \ldots b_d^{a_d} \) in \( K[G_0] \) for a multi-index
\( \alpha \in \mathbb{N}^d \) \( D(G_0, K) \) admits the explicit description
\[
D(G_0, K) = \left\{ \sum_{\alpha} d_\alpha b^\alpha \mid d_\alpha \in K, \forall 0 < r < 1 : \sup_{\alpha} |d_\alpha|r^{\tau_\alpha} < \infty \right\}
\]
(loc.cit. section 4). Here \( \tau_\alpha = \sum \tau_\alpha \alpha_i \) with rational numbers \( \tau_i \) depending on the structure of \( G_0 \) as a \( p \)-valued group. The Fréchet topology of \( D(G_0, K) \) can be defined by the family of norms \( (\| \cdot \|_r)_{0 < r < 1} \) given by
\[
\left\| \sum_{\alpha} d_\alpha b^\alpha \right\|_r := \sup_{\alpha} |d_\alpha|r^{\tau_\alpha}.
\]
The norms \( \| \cdot \|_r \) are independent of the choice of a basis \( (a_1, \ldots, a_d) \) of topological generators. If we let \( D_r(G_0, K) = \left\{ \sum_{\alpha} d_\alpha b^\alpha \mid \lim_{|\alpha| \to -\infty} |d_\alpha|r^{\tau_\alpha} = 0 \right\} \) be the completion of \( D(G_0, K) \) with respect to the norm \( \| \cdot \|_r \) then
\[
D(G_0, K) = \lim_{r \to 0} D_r(G_0, K)
\]
as \( K \)-Fréchet spaces. We summarize some of the main results of [26] in the following theorem (loc.cit. Theorem 4.5 and Theorem 4.9):

**Theorem** (Schneider-Teitelbaum). If \( K \) is discretely valued, \( r \in p\mathbb{Q} \) and \( 1/p < r < 1 \) then the algebra structure of \( D(G_0, K) \) extends to \( D_r(G_0, K) \) making it a \( K \)-Banach algebra with multiplicative norm \( \| \cdot \|_r \). Moreover, for any two real numbers \( r, r' \in p\mathbb{Q} \) with \( 1/p < r < r' < 1 \) the natural inclusion \( D_r(G_0, K) \hookrightarrow D_{r'}(G_0, K) \) is a flat map of noetherian rings. In other words: \( D(G_0, K) \) is a \( K \)-Fréchet-Stein algebra with respect to the family of norms \( \| \cdot \|_r \), \( r \in p\mathbb{Q}, 1/p < r < 1 \).

For \( 0 < r < 1 \) we let \( U_r(\mathfrak{g}, K) \) be the closure of \( U(\mathfrak{g}, K) \) in \( D_r(G_0, K) \) with respect to the norm \( \| \cdot \|_r \). A careful analysis of orthogonal bases (cf. [16], section 1) leads to the following result (loc.cit. 1.4 Lemma 3, Corollaries 1, 2 and 3):

**Theorem** (Frommer). If \( r \in p\mathbb{Q} \) and \( 1/p < r < 1 \) then \( U_r(\mathfrak{g}, K) \) is a noetherian subalgebra of \( D_r(G_0, K) \). In fact, there are integers \( \ell_i > 0 \) depending on \( r \) such that \( D_r(G_0, K) \) is free as a (right) module over \( U_r(\mathfrak{g}, K) \) with basis consisting precisely of those \( b^\alpha \in K[G_0] \) for which \( 0 \leq \alpha_i < \ell_i \) for all \( i = 1, \ldots, d \). Further, \( U_r(\mathfrak{g}, K) \) is equal to the algebra
\[
U_r(\mathfrak{g}, K) = \left\{ \sum_{\alpha} d_\alpha X^\alpha \mid d_\alpha \in K, \lim_{|\alpha| \to -\infty} |d_\alpha||X^\alpha|_r = 0 \right\},
\]
where \( X \) is the \( \mathbb{Q}_p \)-basis \( (\tau_i := \log(1 + b_i))_{1 \leq i \leq d} \) of \( \mathfrak{g} \). The norm \( \| \cdot \|_r \) can be computed via \( \| \sum_{\alpha} d_\alpha X^\alpha \|_r = \sup_{\alpha} |d_\alpha||X^\alpha|_r \).

Using compatible uniform pro-\( p \) groups we can slightly extend this result:

**Corollary 1.4.1.** Let \( G_0 \) be a uniform pro-\( p \) group with closed, compatible uniform subgroup \( H_0 \). Then \( D(H_0, K) \) is a \( K \)-Fréchet-Stein algebra with respect to the family of norms \( \| \cdot \|_r \), \( r \in p\mathbb{Q}, 1/p < r < 1 \), restricted to \( D(H_0, K) \). The conclusions of Frommer’s theorem hold for \( D(H_0, K) \). If \( r \in p\mathbb{Q} \) is a real number with \( 1/p < r < 1 \) then the closure \( D_r(G_0, K)_{H_0} \) of \( D(G_0, K)_{H_0} \) in \( D_r(G_0, K) \) is a finitely generated, free \( U_r(\mathfrak{g}, K) \)-module possessing a basis contained in \( K[H_0] \).
Proof: Choose a basis \((a_1, \ldots, a_d)\) of topological generators of \(G_0\) such that \((a_1, \ldots, a_j)\) is a basis of topological generators of \(H_0\), \(j := \dim H_0 \leq d\). Clearly, 
\(D(H_0, K)\) is a \(K\)-Fréchet-Stein algebra with respect to the restricted norms 
\(\| \cdot \|_r\), \(r \in p\mathbb{O}\), \(1/p < r < 1\), if \(H_0\) is viewed as a \(p\)-valued group with respect to the valuation coming from \(G_0\). It is also clear that Frommer’s theorem applies to 
\(D(H_0, K)\). Fix \(r \in p\mathbb{O}\) with \(1/p < r < 1\). Let \(A \subset \mathbb{N}^d\) be the set of all multi-indices satisfying \(0 \leq \alpha_i < \ell_i\) for all \(i\) and \(A' \subseteq A\) be the subset of all \(\alpha\) such that \(\alpha_{j+1} = \cdots = \alpha_d = 0\). If \(\mathfrak{g}\) denotes the Lie algebra of \(H\) then \((\mathfrak{b}^\alpha)_{\alpha \in A'}\) is a basis of the free \(U_r(\mathfrak{g}, K)\)-module \(D_r(H_0, K)\): The proof of [16], 1.4 Lemma 3, shows that writing 
\[ t_i = \log(1 + b_i) = \sum_{n \geq 1} (-1)^{n+1} b_i^n/n \] 
onumber
one can choose 
\[ \ell_i = \max\{m \geq 1 \mid \sup_{n \geq 1} |1/n|^r m^r = |1/m|^r m^r\}. \]
Hence for \(1 \leq i \leq j\) the integers \(\ell_i\) do not depend on whether we consider \(b_i\) as an element of \(K[G_0]\) or \(K[H_0]\).

If \(D\) denotes the free \(U_r(\mathfrak{g}, K)\)-submodule of \(D_r(G_0, K)\) generated by \((\mathfrak{b}^\alpha)_{\alpha \in A'}\), 
then \(D \subseteq D_r(G_0, K)_{H_0}\). Conversely, \(D\) contains \(D_r(H_0, K)\) and \(U_r(\mathfrak{g}, K)\) and thereby a dense subspace of \(D_r(G_0, K)_{H_0}\) (cf. Lemma 1.2.10). According to [26], Proposition 2.1 (ii), \(D\) is closed. Hence \(D = D_r(G_0, K)_{H_0}\). \(\square\)

We are now going to extend Frommer’s theorem and Corollary 1.4.1 to the case of a finite extension \(L|\mathbb{Q}_p\). Recall that if \(A\) is a \(K\)-Fréchet-Stein algebra with respect to a sequence \((q_n)_{n \geq 1}\) of continuous algebra seminorms and if \(I\) is a closed ideal of \(A\) then according to [26], Proposition 3.7, \(A/I\) is a \(K\)-Fréchet-Stein algebra with respect to the sequence \((\overline{q}_n)_{n \geq 1}\) of residue norms \(\overline{q}_n\). It follows that if \(G_0\) is a locally \(L\)-analytic group such that \(R^{L|\mathbb{Q}_p} G_0\) is uniform pro-\(p\) then \(D(G_0, K)\) is a \(K\)-Fréchet-Stein algebra (loc.cit. Theorem 5.1). Namely, \(D(G_0, K)\) is topologically isomorphic to the quotient of \(D(R^{L|\mathbb{Q}_p} G_0, K)\) by \(I := \ker(\tau)\) (cf. Lemma 1.3.1).

For \(1/p < r < 1\) we denote by \(\| \cdot \|_r\) the residue norm on \(D(G_0, K)\) induced by \(\| \cdot \|\). The completion of \(D(G_0, K)\) with respect to \(\| \cdot \|_r\) is denoted by \(D_r(G_0, K)\). Let \(I_r\) be the closure of \(I\) in \(D_r(R^{L|\mathbb{Q}_p} G_0, K)\) and consider the projection 
\[ \tau_r : D_r(R^{L|\mathbb{Q}_p} G_0, K) \longrightarrow D_r(R^{L|\mathbb{Q}_p} G_0, K)/I_r. \]
According to the proof of [26], Proposition 3.7, we have 
\[ D_r(G_0, K) = D_r(R^{L|\mathbb{Q}_p} G_0, K)/I_r. \]
As before we let \(U_r(\mathfrak{g}, K)\) (resp. \(U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)\)) denote the closure of \(U(\mathfrak{g}, K)\) (resp. \(U(\mathfrak{g}_{\mathbb{Q}_p}, K)\)) in \(D_r(G_0, K)\) (resp. \(D_r(R^{L|\mathbb{Q}_p} G_0, K)\)). Set further \(J_r := I_r \cap U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)\).

**Theorem 1.4.2.** Let \(G\) be a locally \(L\)-analytic group and \(G_0\) as in Proposition 1.3.5. If \(r \in p\mathbb{O}\) with \(1/p < r < 1\) then \(D_r(G_0, K)\) is a free, finitely generated module over the noetherian subalgebra \(U_r(\mathfrak{g}, K)\) with the same basis in \(K[G_0]\) as in Frommer’s theorem applied to \(R^{L|\mathbb{Q}_p} G_0\). Further, there is an \(L\)-basis \(X\) of \(\mathfrak{g}\) and a norm \(\nu\) on \(U_r(\mathfrak{g}, K)\) equivalent to \(\| \cdot \|_r\) such that 
\[ U_r(\mathfrak{g}, K) = \left\{ \sum_{\alpha} d_\alpha X^\alpha \mid d_\alpha \in K, \lim_{|\alpha| \to \infty} |d_\alpha|_\nu(x^\alpha) = 0 \right\}. \]
The norm $\nu_r$ can be computed via $\nu_r(\sum_\alpha d_\alpha \mathbf{x}^\alpha) = \sup_\alpha |d_\alpha| \nu_r(\mathbf{x}^\alpha)$.

Proof: Let $(b^\alpha)_{\alpha \in A}$ be the $D_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-basis of $D_r(R^{[G_0]}G_0, K)$ considered before and $D$ the (right) $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-submodule $D := \oplus_{\alpha \in A} b^\alpha J_r$. Since $I_r$ is an ideal of $D_r(R^{[G_0]}G_0, K)$ containing $J_r$, we naturally have $D \subseteq I_r$. On the other hand, $D$ contains a dense subspace of $I_r$ according to Lemma 1.3.3 since $J := I \cap (U(\mathfrak{g}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} K) \subseteq J_r$. Since $D$ is closed according to [26], Proposition 2.1 (ii), we also have $I_r \subseteq D$. Hence $D = I_r$. It follows from (1.13) and Frommer’s theorem that there is an isomorphism

$$D_r(G_0, K) \simeq \oplus_{\alpha \in A} b^\alpha(U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r)$$

of (right) $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-modules. It becomes topological if $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r$ carries the (Banach) quotient topology (cf. [26], Proposition 2.1). In particular, the image of $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ under $\tau_r$ is closed. According to Lemma 1.3.4 it contains a dense subspace of $U_r(\mathfrak{g}, K)$ whence there is a topological isomorphism

$$(1.14) U_r(\mathfrak{g}, K) \simeq U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r.$$  

This proves the first statement of the theorem. We claim that the assertions concerning the explicit description of $U_r(\mathfrak{g}, K)$ hold if we equip $U_r(\mathfrak{g}, K)$ with the residue norm $\nu_r$ coming from (1.14).

According to Proposition 1.3.5 there is an $L$-basis $\mathbf{x} = (x_1, \ldots, x_d)$ of $\mathfrak{g}$ and a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{a}_L$ such that the family $\mathcal{R} := (v_i \mathcal{R}^\alpha)$ of $\mathfrak{a}_L$-basis $(v_i \mathcal{R}^\alpha)$ gives rise to the set of topological generators $(\exp(v_i \mathcal{R}^\alpha))_{i,j}$ of $R^{[G_0]}G_0$. By Frommer’s theorem

$$U_r(\mathfrak{g}_{\mathbb{Q}_p}, K) = \left\{ \sum_\beta c_\beta \mathcal{R}^\beta \left| \lim_{|\beta| \to \infty} |c_\beta||\mathcal{R}^\beta||_r = 0 \right\}$$

with multiplicative norm $||\sum_\beta c_\beta \mathcal{R}^\beta||_r = \sup_\beta |c_\beta||\mathcal{R}^\beta||_r$. If $\beta = (\beta_i) \in \mathbb{N}^m \times \mathbb{N}^d$ let $\varphi(\beta) := (\sum_{i=1}^m \beta_i)_{1 \leq j \leq d} \in \mathbb{N}^d$. For any $\beta$ with $\varphi(\beta) = \alpha$ we have $\tau^r(\mathcal{R}^\beta) = \prod_{i,j} v_{ij}^{\alpha_{ij}} \mathbf{x}^\alpha$ and $|\alpha| = |\beta|$. Since $\tau_r$ continuously extends $\tau^r$ we have $\tau_r(\sum_\beta c_\beta \mathcal{R}^\beta) = \sum_\beta \tau_r(c_\beta \mathcal{R}^\beta) = \sum_{\alpha \in \mathbb{N}^d} \left( \sum_{\varphi(\beta) = \alpha} c_\beta \prod_{i,j} v_{ij}^{\alpha_{ij}} \right) \mathbf{x}^\alpha$

and also

$$\left| \sum_{\varphi(\beta) = \alpha} c_\beta \prod_{i,j} v_{ij}^{\beta_{ij}} \right| \nu_r(\mathbf{x}^\alpha) \leq \max_{\varphi(\beta) = \alpha} |c_\beta||\mathcal{R}^\beta||_r \to 0 \text{ as } |\alpha| \to \infty.$$

Therefore, $U_r(\mathfrak{g}, K) \subseteq \left\{ \sum_\alpha d_\alpha \mathbf{x}^\alpha \mid \lim_{|\alpha| \to \infty} |d_\alpha| \nu_r(\mathbf{x}^\alpha) = 0 \right\}$. The converse inclusion is clear.

We claim that $J$ is dense in $J_r$. Note first that $J$ is dense in $I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$: If $\delta = \sum_\beta c_\beta \mathcal{R}^\beta \in U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ then by (1.9) $\lim_{|\beta| \to \infty} |c_\beta| r^{-|\beta|} = 0$ for all $\rho > 0$. Hence $\tau^r(\delta) = \sum_\alpha (\sum_{\varphi(\beta) = \alpha} c_\beta \prod_{i,j} v_{ij}^{\beta_{ij}}) \mathbf{x}^\alpha$ converges in $U(\mathfrak{g}, K)$. If $\delta \in \mathfrak{g} \subseteq U(\mathfrak{g}_{\mathbb{Q}_p}, K)$
then due to uniqueness in $U(g, K)$ we have $\sum_{\varphi(\beta) = \alpha} c_\beta \prod_i \gamma_i^{\beta_i} = 0$ and hence $\sum_{\varphi(\beta) = \alpha} c_\beta \gamma_i^{\beta_i} \in J$ for all $\alpha$. Now $(\sum_{\alpha \leq N} \sum_{\varphi(\beta) = \alpha} c_\beta \gamma_i^{\beta_i})_{N \geq 0}$ converges to $\delta$ as $N \to \infty$, proving the claim.

To see that $I \cap U(g_{G_0}, K)$ is dense in $J_r$ we note that as a direct consequence of Frommer’s theorem $U(g_{G_0}, K)$ is a $K$-Fréchet-Stein algebra with respect to the norms $\| \cdot \|_r$. As a closed ideal $I \cap U(g_{G_0}, K)$ is a coadmissible module over $U(g_{G_0}, K)$. Since $J$ is dense in $I \cap U(g_{G_0}, K)$ we know from Theorem A (cf. [26], section 3) that the corresponding coherent sheaf is given by the $U_r(g_{G_0}, K)$-ideals $J_r'$ where $J_r'$ is the closure of $J$ in $U_r(g_{G_0}, K)$. The same reasoning as above shows that $I_r = \oplus_{\alpha \in \mathbb{A}} b^\alpha J_r'$. Since also $I_r = \oplus_{\alpha \in \mathbb{A}} b^\alpha J_r$ and $J_r' \subseteq J_r$ we obtain $J_r' = J_r$.

Let us now prove the last assertion on $\nu_\tau$. Assume $\delta = \sum_0^\infty d_\alpha \mathcal{X}_\alpha \in U_r(g, K)$, i.e. $\lim_{|\alpha| \to \infty} |d_\alpha| \nu_\tau(\mathcal{X}_\alpha) = 0$. Let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ so large that

$$
\sup_{|\alpha| \leq N} |d_\alpha| \nu_\tau(\mathcal{X}_\alpha) = \sup_\alpha |d_\alpha| \nu_\tau(\mathcal{X}_\alpha) \text{ and } \nu_\tau(\sum_0^N d_\alpha \mathcal{X}_\alpha) \leq \varepsilon.
$$

Note that the preimage of $\sum_0^\infty d_\alpha \mathcal{X}_\alpha$ under $\tau_r$ contains elements in $U(g_{G_0}) \otimes_Q K$. By our above claim there is an element $\sum_\beta c_\beta \mathcal{Y}_\beta \in U(g_{G_0}) \otimes_Q K$ mapping to $\sum_0^\infty d_\alpha \mathcal{X}_\alpha$ under $\tau$, such that

$$
\nu_\tau(\sum_0^\infty d_\alpha \mathcal{X}_\alpha) \geq \| \sum_\beta c_\beta \mathcal{Y}_\beta \|_r - \varepsilon.
$$

Uniqueness in $U(g) \otimes_L K$ implies that $\tau_r(\sum_{\varphi(\beta) = \alpha} c_\beta \gamma_i^{\beta_i}) = d_\alpha \mathcal{X}_\alpha$ for all $\alpha$ with $|\alpha| \leq N$. Therefore,

$$
\| \sum_\beta c_\beta \mathcal{Y}_\beta \|_r = \sup_\beta |c_\beta| \| \mathcal{Y}_\beta \|_r \geq \sup_{|\alpha| \leq N} \left\{ \sup_{\varphi(\beta) = \alpha} |c_\beta| \| \gamma_i^{\beta_i} \|_r \right\} \geq \sup_\alpha |d_\alpha| \nu_\tau(\mathcal{X}_\alpha).
$$

Hence for all $\varepsilon > 0$

$$
\max\{\varepsilon, \nu_\tau(\delta)\} \geq \nu_\tau(\sum_0^\infty d_\alpha \mathcal{X}_\alpha) \geq \sup_\alpha |d_\alpha| \nu_\tau(\mathcal{X}_\alpha) - \varepsilon,
$$

i.e. $\nu_\tau(\delta) \geq \sup_\alpha |d_\alpha| \nu_\tau(\mathcal{X}_\alpha)$. As one always has $\nu_\tau(\delta) \leq \sup_\alpha |d_\alpha| \nu_\tau(\mathcal{X}_\alpha)$, this finishes the proof. \hfill $\square$

**Corollary 1.4.3.** Let $G$ be a locally $L$-analytic group, $H$ a closed locally $L$-analytic subgroup and $G_0$ as in Corollary 1.3.6. If $H_0 := H \cap G_0$ then $D(H_0, K)$ is a $K$-Fréchet-Stein algebra with respect to the family of norms $\| \cdot \|_r$, $r \in p^\infty$, $1/p < r < 1$, restricted from $D(G_0, K)$ to $D(H_0, K)$. The conclusions of Theorem 1.4.2 hold for $D(H_0, K)$. If $r \in p^\infty$ is a real number with $1/p < r < 1$ then the closure $D_r(G_0, K)_{H_0}$ of $D(G_0, K)_{H_0}$ in $D_r(G_0, K)$ is finitely generated, free $U_r(g, K)$-module with the same basis in $K[H_0]$ as in Corollary 1.4.1 applied to the pair $(R^L \otimes_{Q^L} G_0, R^L \otimes_{Q^L} H_0)$.

**Proof:** Since $R^L \otimes_{Q^L} H_0$ is compatible with $R^L \otimes_{Q^L} G_0$ we know from Corollary 1.4.1 that $D(R^L \otimes_{Q^L} H_0, K)$ is a $K$-Fréchet-Stein algebra with respect to the family of
norms $\| \cdot \|_r$, $r \in p^\mathbb{Q}$, $1/p < r < 1$, obtained by restriction from $D(R^{L[Q]}G_0, K)$. The commutativity of the diagram

$$
\begin{array}{ccc}
D(R^{L[Q]}H_0, K) & \xrightarrow{\sim} & D(R^{L[Q]}G_0, K) \\
\downarrow & & \downarrow \\
D(H_0, K) & \xrightarrow{\sim} & D(G_0, K)
\end{array}
$$

shows that the kernel of the left vertical arrow is $I' := I \cap D(R^{L[Q]}H_0, K)$. Applying Theorem 1.4.2 to $H_0$ shows that if we let $I'_r$ be the closure of $I'$ in $D_r(R^{L[Q]}H_0, K)$ then $D(H_0, K)$ is a $K$-Fréchet-Stein algebra with respect to the corresponding quotient norms and

$$D_r(H_0, K) = D_r(R^{L[Q]}H_0, K)/I'_r$$

(cf. (1.13) applied to $H_0$). Recall that we have

$$D_r(R^{L[Q]}G_0, K) = \bigoplus_{\alpha \in A} b^\alpha U_r(\mathfrak{g}_{\mathfrak{q}_p}, K)$$

as $K$-Banach spaces and similarly

$$D_r(R^{L[Q]}H_0, K) = \bigoplus_{\alpha \in A'} b^\alpha U_r(\mathfrak{h}_{\mathfrak{q}_p}, K)$$

with $A' \subseteq A$ (cf. Corollary 1.4.1 and its proof). Moreover, we know from the proof of Theorem 1.4.2 that $I_r = \bigoplus_{\alpha \in A} b^\alpha J_r$ with $J_r := I_r \cap U_r(\mathfrak{g}_{\mathfrak{q}_p}, K)$ and similarly $I'_r = \bigoplus_{\alpha \in A'} b^\alpha (I'_r \cap U_r(\mathfrak{g}_{\mathfrak{q}_p}, K))$. It follows that $I'_r = I_r \cap D_r(R^{L[Q]}H_0, K)$ and hence that

$$D_r(H_0, K) = D_r(R^{L[Q]}H_0, K)/(I_r \cap D_r(R^{L[Q]}H_0, K)).$$

We need to show that the image of $D_r(R^{L[Q]}H_0, K)$ under $\tau_r$ is closed. Making use of the above direct sum decompositions it suffices to show that the image of $U_r(\mathfrak{g}_{\mathfrak{q}_p}, K)$ under $\tau_r$ is closed. We make use of the notation introduced earlier: By construction we may assume $\mathfrak{X}' := (\mathfrak{X}_j, 1 \leq j \leq \dim H_0 \leq d)$, $1 \leq j \leq \dim H_0 \leq d$, to be an $L$-basis of $\mathfrak{h}$. Since $U_r(\mathfrak{g}, K) = \big\{ \sum_{\alpha \in \mathbb{N}^d} d_\alpha \mathfrak{X}^\alpha | \lim_{|\alpha| \to \infty} |d_\alpha| |\nu_r(\mathfrak{X}^\alpha) = 0\big\}$ with $\nu_r(\sum_{\alpha} d_\alpha \mathfrak{X}^\alpha) = \sup_{\alpha} |d_\alpha| \nu_r(\mathfrak{X}^\alpha)$, a straightforward calculation shows that

$$\tau_r(U_r(\mathfrak{g}_{\mathfrak{q}_p}, K)) = \big\{ \sum_{\alpha \in \mathbb{N}^d} d_\alpha (\mathfrak{X}')^\alpha | \lim_{|\alpha| \to \infty} |d_\alpha| |\nu_r((\mathfrak{X}')^\alpha) = 0\big\}$$

which is a closed subspace of $U_r(\mathfrak{g}, K)$.

According to the proof of Corollary 1.4.1 there is a finite basis $(b^\alpha)_{\alpha \in A'}$ of the free $U_r(\mathfrak{g}_{\mathfrak{q}_p}, K)$-module $D_r(R^{L[Q]}G_0, K)$ and a subset $A' \subseteq A$ such that $(b^\alpha)_{\alpha \in A'}$ is a basis of the free, finitely generated $U_r(\mathfrak{g}_{\mathfrak{q}_p}, K)$-module $D_r(R^{L[Q]}G_0, K)_{H_0}$. It follows from the decomposition $I_r = \bigoplus_{\alpha \in A} b^\alpha J_r$ that $I_r \cap D_r(R^{L[Q]}G_0, K)_{H_0} = \bigoplus_{\alpha \in A} b^\alpha J_r$. Thus, by (1.14)

$$D_r(R^{L[Q]}G_0, K)_{H_0}/(I_r \cap D_r(R^{L[Q]}G_0, K)_{H_0}) \simeq \bigoplus_{\alpha \in A'} b^\alpha U_r(\mathfrak{g}, K).$$
In particular, the image of $D_r(R^L|\mathbb{Q}_p G_0, K)_{H_\mathfrak{p}}$ under $\tau_r$ is closed. It follows by means of Lemma 1.3.4 and (1.13) that the left hand side of (1.16) is topologically isomorphic to $D_r(G_0, K)_{H_\mathfrak{p}}$.

Note that by Theorem 1.4.2 $(b^\alpha)_{\alpha \in A'}$ is also a basis of the free $U_r(\mathfrak{h}, K)$-module $D_r(H_0, K)$ and the free $U_r(H_0, K)$-module $D_r(R^L|\mathbb{Q}_p H_0, K)$. □

**Corollary 1.4.4.** If $L_0|\mathbb{Q}_p$ and $L|L_0$ are finite extensions of fields and if $G$ is a locally $L$-analytic group then the natural homomorphism

$$D(R^L|L_0 G, K) \hat{\otimes}_{U(\mathfrak{g}_{L_0}, K)} U(\mathfrak{g}, K) \longrightarrow D(G, K)$$

of $D(R^L|L_0 G, K)-U(\mathfrak{g}, K)^{op}$-bimodules is a topological isomorphism.

Proof: Let $G_0$ be an open subgroup of $G$ as in Proposition 1.3.5. Using $D(G, K) = \bigoplus_{g \in G/G_0} \delta_g \cdot D(\mathfrak{g}_0, K)$ (resp. with $R^L|L_0 G$ and $R^L|L_0 G_0$) it suffices to show that the map

$$D(R^L|L_0 G_0, K) \hat{\otimes}_{U(\mathfrak{g}_{L_0}, K)} U(\mathfrak{g}, K) \longrightarrow D(\mathfrak{g}_0, K)$$

is a topological isomorphism. One easily verifies that also $R^L|L_0 G_0$ satisfies conditions (i) – (iii) of Proposition 1.3.5 (replacing $L$ by $L_0$) so that according to Theorem 1.4.2 the modules $D_r(R^L|L_0 G_0, K)$, resp. $D_r(G_0, K)$, are finitely generated and free over the noetherian Banach algebras $U_r(\mathfrak{g}_{L_0}, K)$, resp. $U_r(\mathfrak{g}, K)$, with a common basis $(b^\alpha)_{\alpha \in A'}$. It follows that the base change map

$$D_r(R^L|L_0 G_0, K) \otimes_{U_r(\mathfrak{g}_{L_0}, K)} U(\mathfrak{g}, K) \longrightarrow D_r(G_0, K)$$

is an isomorphism of $D_r(R^L|L_0 G_0, K)-U_r(\mathfrak{g}, K)^{op}$-bimodules. As in Proposition 1.2.12 one shows that it is bi-continuous and that we may pass to the projective limit in order to obtain that (1.17) is a topological isomorphism. □

The same line of proof gives:

**Corollary 1.4.5.** Let $L_0|\mathbb{Q}_p$ and $L|L_0$ be finite extensions of fields and $G$ be a locally $L$-analytic group. If $H$ is a closed, locally $L$-analytic subgroup of $G$ then the map $\tau^*: D(R^L|L_0 G, K)_H \rightarrow D(G, K)_H$ is surjective. □

**2 Invariant distributions**

$G$ acts on itself via conjugation inducing an action by continuous automorphisms on the space $C^{an}(G, K)$ of locally analytic functions on $G$. The contragredient action on $D(G, K)$ is explicitly given by $(g * \delta)(f) = \delta(h \mapsto f(ghg^{-1})) = (\delta_\delta^\delta\delta_{\delta^{-1}})(f)$ for $g \in G$, $\delta \in D(G, K)$ and $f \in C^{an}(G, K)$, i.e.

$$(2.1) \quad g * \delta = \delta_g \delta \delta_{\delta^{-1}}.$$  

We call a distribution $\delta \in D(G, K)$ invariant if $g * \delta = \delta$ for all $g \in G$. If $U$ is a $G$-invariant subspace of $D(G, K)$ we denote by $U^G$ the subspace of all invariant distributions contained in $U$.

The separate continuity of the multiplication together with the density of $K[G]$ in $D(G, K)$ imply by means of (2.1) that the subspace $D(G, K)^G$ of all invariant
distributions on $G$ coincides with the center of the ring $D(G, K)$.

For later use we introduce the subspace

$$D^p(G, K) := \sum_{g \in G} \delta_g \cdot (U(g) \otimes_L K)$$

of $D(G, K)$. It is the space of all point distributions in the sense of [7], 13.2.1.

2.1 The infinitesimal center

Viewing $\mathfrak{g}$ as an abelian locally $L$-analytic group the space $C^\omega_1(\mathfrak{g}, K)$ is defined as in (1.5). The exponential map $\exp$ induces a topological isomorphism

$$\exp^* : C^\omega_1(G, K) \sim \rightarrow C^\omega_0(\mathfrak{g}, K)$$

which does not depend on the choice of $\exp$ (cf. the remark following III.4.3 Définition 1 of [4]). Dualizing, we obtain a topological isomorphism

$$\exp_* : C^\omega_0(\mathfrak{g}, K)_b \sim \rightarrow U(\mathfrak{g}, K) = C^\omega_1(G, K)_b$$

of locally convex vector spaces which for $\delta \in C^\omega_0(\mathfrak{g}, K)_b$ and $[f] \in C^\omega_1(G, K)$ is explicitly given by

$$(\exp_* \delta)([f]) = \delta(\exp^* [f]) = \delta([x \mapsto f(\exp(x))]).$$

Here $[f]$ denotes the germ in 1 of a locally analytic function $f$ defined in an open neighborhood of 1 in $G$.

Viewing $\mathfrak{g}$ as its own Lie algebra Proposition 1.2.8 shows that

$$C^\omega_0(\mathfrak{g}, K)_b = \left\{ \sum d_\alpha x^n \mid d_\alpha \in K, \forall r > 0 : \sup |d_\alpha| r^{-|\alpha|} < \infty \right\}$$

in terms of power series with commutative multiplication. Since the symmetric algebra $S(\mathfrak{g}) \otimes_L K$ of $\mathfrak{g}$ is dense in $C^\omega_0(\mathfrak{g}, K)_b$ we prefer to change notation and write $S(\mathfrak{g}, K)$ instead of $C^\omega_0(\mathfrak{g}, K)_b$.

The action of $G$ on $C^{an}(G, K)$ by conjugation descends to $C^\omega_1(G, K)$ (cf. (1.5)) which is a locally analytic $G$-representation in the sense of [25], section 3: if $G_0$ is a compact open subgroup of $G$ then the natural projection $C^{an}(G, K) \rightarrow C^\omega_1(G, K)$ factors $G_0$-equivariantly through $C^{an}(G_0, K)$. By [14], Satz 3.3.4, the $G_0$-action on $C^{an}(G_0, K)$ is locally analytic whence so is the $G_0$-action on the barrelled quotient $C^\omega_1(G, K) = C^\omega_1(G_0, K)$ (cf. [12], Lemma 3.6.14). Since $G_0$ is open in $G$ the claim follows.

Similarly, the action of $G$ on $\mathfrak{g}$ via the adjoint representation $Ad$ induces an action on $C^\omega_0(\mathfrak{g}, K)$. Using the formula $g \cdot \exp(\gamma) \cdot g^{-1} = \exp(Ad(g)(\gamma))$ for $g \in G$ and all $\gamma$ in a neighborhood of zero in $\mathfrak{g}$ depending on $g$ (cf. [4], III.4.4 Corollaire 3) one deduces that $\exp^*$ is $G$-equivariant.
Recall that if \( n \in \mathbb{N}, \eta_1, \ldots, \eta_n \in \mathfrak{g} \) and \( \eta_1 \cdots \eta_n \) is their product in \( S(\mathfrak{g}) \) then the symmetrization map \( \text{sym} : S(\mathfrak{g}) \to U(\mathfrak{g}) \) is defined by
\[
\text{sym}(\eta_1 \cdots \eta_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \eta_{\sigma(1)} \cdots \eta_{\sigma(n)}
\]
through \( L \)-linear continuation. Here \( \mathfrak{S}_n \) denotes the symmetric group on \( n \) letters.

**Proposition 2.1.1.** \( \exp^\ast : C^\omega_c(G, K) \to C^\omega_c(\mathfrak{g}, K) \) is an isomorphism of locally analytic \( G \)-representations on locally convex \( K \)-vector spaces of compact type. The corresponding dual map \( \exp_* : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K) \) is an isomorphism of \( C^\omega_c \)-modules. Its restriction to \( S(\mathfrak{g}) \otimes L K \) coincides with \( \text{sym} \otimes \text{id} \) and maps isomorphically onto \( U(\mathfrak{g}) \otimes L K \). Further, if the \( D(G, K) \)-actions on \( S(\mathfrak{g}, K) \) and \( U(\mathfrak{g}, K) \) are denoted by \( * \) then the following formulae hold:

i) \( \mathfrak{r} * \mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \) for all \( \mathfrak{r}, \mathfrak{r} \in \mathfrak{g} \) where \( \mathfrak{r} \) is considered as an element of \( D(G, K) \) and \( \mathfrak{r}, [\mathfrak{r}, \mathfrak{r}] \) as elements of \( S(\mathfrak{g}, K) \) (or \( U(\mathfrak{g}, K) \));

ii) \( \mathfrak{r} * \mathfrak{r} = \mathfrak{r} \cdot \mathfrak{r} \) in \( U(\mathfrak{g}, K) \) for all \( \mathfrak{r} \in \mathfrak{g} \) and \( \mathfrak{r} \in U(\mathfrak{g}, K) \);

iii) \( \mathfrak{r} * (\mathfrak{r}_1 \cdots \mathfrak{r}_n) = (\mathfrak{r} \cdot \mathfrak{r}_1 \cdots \mathfrak{r}_n + \mathfrak{r}_1 \cdots \mathfrak{r}_n \cdot \mathfrak{r} \cdot \mathfrak{r}_n) \) for all \( \mathfrak{r}, \mathfrak{r}_1, \ldots, \mathfrak{r}_n \in \mathfrak{g} \).

Proof: The first statement follows from what was said above. By general principles the dual map \( \exp_* : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K) \) is a topological isomorphism of nuclear Fréchet spaces carrying separately continuous \( D(G, K) \)-module structures for which \( \exp \) is a homomorphism (cf. [25], Corollary 3.3). For the statement about the restriction of \( \exp_* \) to \( S(\mathfrak{g}) \otimes L K \) confer [4], III.4.3 Théorème 4 and II.1.5 Proposition 9.

For \( g \in G, \eta \in \mathfrak{g} \) and \( f \in C^\omega_c(G, K) \) we have
\[
(g * \eta)(f) = \eta(g^{-1} * f) = \frac{d}{dt} f(g \cdot \exp(t \eta) \cdot g^{-1})|_{t=0} = d \frac{d}{dt} f(\exp(t \text{Ad}(g)(\eta)))|_{t=0} = \text{Ad}(g)(\eta)(f)
\]
showing that \( \mathfrak{g} \otimes L K \) carries the structure of a \( D(G, K) \)-submodule of \( U(\mathfrak{g}, K) \) coming from the adjoint representation of \( G \) on \( \mathfrak{g} \). By [4], III.3.12 Proposition 44, we have \( \mathfrak{r} * \eta = d/dt(\text{Ad}(\exp(t \mathfrak{r}))(\eta))|_{t=0} = \text{ad}(\mathfrak{r})(\eta) = [\mathfrak{r}, \eta] \). Note that if \( V \) is a Banach space then the notion of a locally analytic \( G \)-representation as given in [25], section 3, coincides with the notion of an analytic Banach space representation in the sense of Bourbaki (cf. [14], Korollar 3.1.9).

By [4], III.3.11 Proposition 41 and (i) we have
\[
\mathfrak{r} * (\prod_{i=1}^n \eta_i) = [\mathfrak{r}, \eta_1] \eta_2 \cdots \eta_n + \cdots + \eta_1 \cdots \eta_{n-1} [\mathfrak{r}, \eta_n]
\]
for all \( \mathfrak{r} \in \mathfrak{g} \). Since \( [\mathfrak{r}, \eta_i] = \eta_i \mathfrak{r} - \mathfrak{r} \eta_i \in U(\mathfrak{g}) \) we obtain (ii). The statements on \( S(\mathfrak{g}, K) \) are proved analogously. \( \square \)
If \( \delta = \sum_{\alpha} d_{\alpha} X^{\alpha} \in S(\mathfrak{g}, K) \) or \( U(\mathfrak{g}, K) \) and \( n \geq 0 \) then we let \( \delta^{\leq n} := \sum_{|\alpha| \leq n} d_{\alpha} X^{\alpha} \) and \( \delta^{> n} := \sum_{|\alpha| > n} d_{\alpha} X^{\alpha} \). Note that if \( g \in G \) then \( g \ast \delta^{\leq n} \) is of degree \( \leq n \) for every \( n \in \mathbb{N} \). This follows from writing \( g \ast \xi = \sum_{j} a_{j} \xi_{j} \), \( a_{j} \in L \), and noting that by (2.1)

\[
g \ast (\lambda \prod_{i} \xi_{i}^{\alpha_{i}}) = \lambda \prod_{i} (g \ast \xi_{i})^{\alpha_{i}}.
\]

In particular, \( G \) acts on \( S(\mathfrak{g}) \otimes_{L} K \) and \( U(\mathfrak{g}) \otimes_{L} K \).

**Proposition 2.1.2.** \( U(\mathfrak{g})^{\theta} \otimes_{L} K \) and \( U(\mathfrak{g})^{G} \otimes_{L} K \) are dense in \( U(\mathfrak{g}, K)^{\theta} \) and \( U(\mathfrak{g}, K)^{G} \), respectively.

**Proof:** Since \( \exp_{\cdot} \) is equivariant for the actions of \( \mathfrak{g} \) and \( G \) we may equally well show that \( S(\mathfrak{g})^{\theta} \otimes_{L} K \) and \( S(\mathfrak{g})^{G} \otimes_{L} K \) are dense in \( S(\mathfrak{g}, K)^{\theta} \) and \( S(\mathfrak{g}, K)^{G} \), respectively. If \( \delta \in S(\mathfrak{g}, K) \) is homogeneous of degree \( n \) then it follows from Proposition 2.1.1 that for \( \mathfrak{r} \in \mathfrak{g} \) either \( \mathfrak{r} \ast \delta = 0 \) or \( \mathfrak{r} \ast \delta \) is again homogeneous of degree \( n \) (write \( [\mathfrak{r}, \mathfrak{r}'] = \sum j a_{j} \mathfrak{r}_{j} \mathfrak{r} \mathfrak{r}_{j} \) for \( \mathfrak{r} \in \mathfrak{g}, a_{j} \in L \)). We have seen above that similarly \( \mathfrak{r} \ast \delta \) will again be homogeneous of degree \( n \). Thus, if \( \delta \in S(\mathfrak{g}, K)^{\theta} \) (resp. \( S(\mathfrak{g}, K)^{G} \)) then also \( \delta^{\leq n} \) and \( \delta^{> n} \) are \( \mathfrak{g} \)-invariant (resp. \( G \)-invariant). Since \( \delta^{\leq n} \in S(\mathfrak{g}) \otimes_{L} K \) and \( \delta^{> n} \to \delta \) for \( n \to \infty \), the assertion follows. \( \square \)

**Remark 2.1.3.** If \( G \) is an open subgroup of the group of \( L \)-rational points of a connected algebraic group over \( L \) then [25], Proposition 3.7, shows that \( U(\mathfrak{g})^{\theta} \otimes_{L} K = U(\mathfrak{g})^{G} \otimes_{L} K \). According to Proposition 2.1.2 \( U(\mathfrak{g}, K)^{\theta} = U(\mathfrak{g}, K)^{G} \). Similarly, \( S(\mathfrak{g}, K)^{\theta} = S(\mathfrak{g}, K)^{G} \) in this case.

**Remark 2.1.4.** Let \( \nu \) denote a norm on \( S(\mathfrak{g}) \otimes_{L} K \) with respect to which the action of \( G \) (resp. \( \mathfrak{g} \)) is continuous. If the completion \( S_{\nu}(\mathfrak{g}, K) \) of \( S(\mathfrak{g}) \otimes_{L} K \) with respect to \( \nu \) has the explicit description \( \{ \sum_{\alpha} d_{\alpha} X^{\alpha} \mid \lim_{|\alpha| \to \infty} |d_{\alpha}| \nu(X^{\alpha}) = 0 \} \) with

\[
\nu\left(\sum_{\alpha} d_{\alpha} X^{\alpha}\right) = \sup_{\alpha} |d_{\alpha}| \nu(X^{\alpha}),
\]

then the above proof shows that \( S(\mathfrak{g})^{G} \otimes_{L} K \) and \( S(\mathfrak{g})^{\theta} \otimes_{L} K \) are even dense in \( S_{\nu}(\mathfrak{g}, K)^{G} \) and \( S_{\nu}(\mathfrak{g}, K)^{\theta} \), respectively.

In general, the restriction of \( \exp_{\cdot} \) to \( S(\mathfrak{g}, K)^{\theta} \) is not an isomorphism of algebras although both \( S(\mathfrak{g}, K)^{\theta} \) and \( U(\mathfrak{g}, K)^{\theta} \) are commutative. Making use of a construction by M. Duflo we will show, however, that one does obtain an isomorphism

\[
\eta : S(\mathfrak{g}, K)^{\theta} \to U(\mathfrak{g}, K)^{\theta}
\]

of \( K \)-algebras if \( \exp_{\cdot} \) is suitably normalized. This result is similar to the conjecture of Kashiwara and Vergne for real Lie groups (cf. [1]) involving, however, distributions on germs of functions rather than germs of distributions.

Recall the following construction (cf. [10], p. 55): let \( k \) be a field of characteristic zero and \( \mathfrak{h} \) a Lie algebra of finite dimension over \( k \). Choosing dual \( k \)-bases \( (\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{d}) \) and \( (\mathfrak{r}'_{1}, \ldots, \mathfrak{r}'_{d}) \) of \( \mathfrak{h} \) and \( \mathfrak{h}^{*} \), respectively, we identify \( S(\mathfrak{h}) \) with the algebra of polynomial functions on \( \mathfrak{h}^{*} \) and \( S(\mathfrak{h}^{*}) \) with the algebra of differential operators with constant coefficients on \( \mathfrak{h}^{*} \) (denoting by \( D(\mathfrak{g}) \) the operator defined by an element \( q \in S(\mathfrak{h}^{*}) \)). The completion \( \hat{S}(\mathfrak{h}^{*}) \) of \( S(\mathfrak{h}^{*}) \) with respect to the topology defined by the maximal ideal \( (\mathfrak{r}'_{1}, \ldots, \mathfrak{r}'_{d}) \) may be identified with the
algebra of formal power series in the variables $r^*_i$ over $k$. If $f \in S(\mathfrak{h})$ is given and
the first non-zero coefficient of $q \in S(\mathfrak{h}^*)$ appears in sufficiently high order then
$D(q)(f) = 0$. Hence for $q \in \hat{S}(\mathfrak{h}^*)$ one can define $D(q)(f)$ by continuity and set
$\langle q, f \rangle := D(q)(f)(0)$. This identifies $S(\mathfrak{h})$ with the space $\hat{S}(\mathfrak{h}^*)'$ of continuous
functionals on $\hat{S}(\mathfrak{h}^*)$.

If $S(\mathfrak{h})$ is identified with the algebra of constant coefficient differential operators
on $\mathfrak{h}$ and $f \in S(\mathfrak{h})$ then we let $D^*(f)$ be the corresponding operator. $D^*(f)$
is an endomorphism of $\hat{S}(\mathfrak{h}^*)$. If $q \in \hat{S}(\mathfrak{h}^*)$ is a power series we let $q(0)$ be its
constant term. According to the remarks preceding Lemme II.2 of [loc.cit.] we have
\begin{equation}
D^*(f)(q)(0) = D(q)(f)(0) = \langle q, f \rangle
\end{equation}
for all $q \in \hat{S}(\mathfrak{h}^*)$ and $f \in S(\mathfrak{h})$.

Let $ad(\mathfrak{X}) = M_d(k[\mathfrak{x}]^*_1, \ldots, \mathfrak{x}^*_d])$ be the matrix $ad(\mathfrak{X}) := \sum_i \mathfrak{x}^*_i A_i$, where $A_i \in
M_d(k)$ represents $ad(\mathfrak{x}_i) \in \text{End}_k(\mathfrak{h})$ with respect to the $k$-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{h}$.
If $B_{2n} \in \mathbb{Q}$ denote the Bernoulli numbers of even degree and $exp(t) \in \mathbb{Q}[[t]]$ is
the usual exponential series then the formula
\begin{equation}
q = q(\mathfrak{x}^*_1, \ldots, \mathfrak{x}^*_d) := \text{det} \left( \frac{\exp(ad(\mathfrak{X})/2) - \exp(-ad(\mathfrak{X})/2)}{ad(\mathfrak{X})} \right)^{1/2}
= \exp\left( \sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!} \text{tr}[ad(\mathfrak{X})^{2n}] \right)
\end{equation}
defines a formal power series in the indeterminates $r^*_i$ with coefficients in $k$, i.e.
an element of $\hat{S}(\mathfrak{h}^*)$ (for the second formula cf. [1]). One of the main results of
[11] is the following theorem (loc.cit. Théorème 2):

**Theorem** (Duflo). If $\mathfrak{h}$ is a finite dimensional Lie algebra over a field $k$ of
characteristic zero then the normalized symmetrization map
\[
\eta := \text{sym} \circ D(q) : S(\mathfrak{h})^\mathfrak{h} \rightarrow U(\mathfrak{h})^\mathfrak{h}
\]
is a well-defined isomorphism of $k$-algebras.

It is known that in the case of Lie algebras $\mathfrak{h}$ over the fields $k = \mathbb{R}$ or $\mathbb{C}$, the
formal power series $q$ defines an analytic function around 0 in $\mathfrak{h}$. This is also
true for the Lie algebra $\mathfrak{g}$ over the non-archimedean field $L$:

**Proposition 2.1.5.** The formal power series $q$ defines an analytic function in
a neighborhood of 0 in $\mathfrak{g}$. If we let $[q] \in C^\infty_0(\mathfrak{g}, K)$ denote its germ in 0 then
the normalized exponential map $\eta : S(\mathfrak{g}, K) \rightarrow U(\mathfrak{g}, K)$ defined by
\[
\eta(\delta)([f]) := \delta([q \cdot \exp^*(\langle f \rangle)]) \text{ for } \delta \in S(\mathfrak{g}, K) \text{ and } [f] \in C^\infty_0(G, K),
\]
restricts to a topological isomorphism of $K$-Fréchet algebras
\[
\eta : S(\mathfrak{g}, K)^\mathfrak{g} \xrightarrow{\sim} U(\mathfrak{g}, K)^\mathfrak{g}.
\]
Thus, $W$ and contains a split maximal toral subalgebra $\eta$ of $\mathcal{K}$. If $\delta \in S(\mathfrak{g})$ and $[p] \in C^\infty_0(\mathfrak{g}, K)$ is represented by a formal power series $p \in S(\mathfrak{g}^*)$ then by (2.2) and [10], Lemme II.1,
\[
\delta([q] \cdot [p]) = D^*(\delta)(qp)(0) = D(qp)(\delta)(0) = (qp, \delta) = (p, D(q)(\delta)) = D(q)(\delta)([p]).
\]
Since the restriction of $\exp_\mathfrak{h}$ to $S(\mathfrak{g}) \otimes_L K$ coincides with $\text{sym}$ (cf. Proposition 2.1.1) it follows that $\eta|S(\mathfrak{g}, K)^\text{sym}$ extends Duflo’s isomorphism. Since by Proposition 2.1.2 $S(\mathfrak{g})^\text{sym} \otimes_L K$ (resp. $U(\mathfrak{g})^\text{sym} \otimes_L K$) is dense in $S(\mathfrak{g}, K)^\#$ (resp. $U(\mathfrak{g}, K)^\#$) it follows that $\eta$ is an isomorphism of algebras onto $U(\mathfrak{g}, K)^\#$. \hfill $\square$

We are now going to explicitly compute $U(\mathfrak{g}, K)^\#$ in the case that $\mathfrak{g}$ is semisimple and contains a split maximal toral subalgebra $t$ (cf. [8], 1.9.10). The Weyl group $\mathfrak{W} = \mathfrak{W}(\mathfrak{g}, t)$ acts on $t^*$ by $L$-linear endomorphisms and dually on $t$. Thus, $\mathfrak{W}$ acts continuously on $C^\text{an}(t, K)$. Since the subspace $C^\text{an}(t, K)_{\text{tr}}$ is $\mathfrak{W}$-invariant $\mathfrak{W}$ acts on the quotient $C^\text{an}_\mathfrak{W}(t, K)$ and hence on $S(t, K)$.

\textbf{Theorem 2.1.6.} If $\mathfrak{g}$ is split semisimple with $t$ and $\mathfrak{W}$ as above then there are isomorphisms
\[
U(\mathfrak{g}, K)^\# \simeq S(t, K)^\mathfrak{W} \simeq \mathcal{O}(A^\text{an}_K)
\]
of $K$-Fréchet algebras with $n := \dim_L(t)$. Here $\mathcal{O}(A^\text{an}_K)$ is the $K$-Fréchet algebra of holomorphic functions on the rigid analytic affine space $(A^\text{an}_K)$ of dimension $n$ over $K$.

In order to construct the above isomorphisms we need some preparation. Let $k$ be a field which is complete with respect to a non-trivial, non-archimedean valuation and $(\cdot)^\text{an}$ be the rigid analytification functor on the category of $k$-schemes which are locally of finite type.

\textbf{Proposition 2.1.7.} Let $X$ be an affine scheme of finite type over $k$, $\Gamma$ a finite group of $k$-automorphisms of $X$ and $\pi : X \to X/\Gamma$ be the canonical quotient map. The presheaf $\mathcal{F}$ on $(X/\Gamma)^\text{an}$ defined by $\mathcal{F}(U) := \mathcal{O}_X((\pi^\text{an})^{-1}(U))^\Gamma$ is an $\mathcal{O}_{(X/\Gamma)^\text{an}}$-submodule of $\pi^\text{an}_*\mathcal{O}_X$ via the natural map $\pi^\text{an}_* : \mathcal{O}_{(X/\Gamma)^\text{an}} \to \pi^\text{an}_*\mathcal{O}_X$. In fact, $(\pi^\text{an})^\#$ is an isomorphism onto $\mathcal{F}$.

\textbf{Proof:} With $\pi$ also $\pi^\text{an}$ is surjective and we have the following commutative diagram of locally $G$-ringed spaces:
\[
\begin{array}{ccc}
X^\text{an} & \xrightarrow{\text{an}} & (X/\Gamma)^\text{an} \\
\downarrow \pi^\text{an} & & \downarrow \\
X & \xrightarrow{\pi} & X/\Gamma.
\end{array}
\]

We see that if $U \subseteq (X/\Gamma)^\text{an}$ is admissible open then $V := (\pi^\text{an})^{-1}(U) \subseteq X^\text{an}$ is admissible open and $\Gamma$-invariant. Thus, $\Gamma$ acts on $\mathcal{O}_X(V)$ so that the presheaf
\( F \) is well-defined. It is straightforward to check that it is in fact a sheaf of \( \mathcal{O}_{X^{an}} \)-modules. By [2], 9.4.1 Proposition 2, it suffices to prove the claim for an admissible open covering \((U_i)_{i \in I}\) of \((X/\Gamma)^{an}\). By construction, \((X/\Gamma)^{an}\) admits a countable, ascending covering by open affinoid subdomains \( U_i := \text{Sp}(B_i) \), \( i \in \mathbb{N} \). Setting \( A := \mathcal{O}_X(X) \) and \( B := \mathcal{A}^\Gamma \) the algebra \( A_i := B_i \otimes_B A \) is finite over \( B_i \) and hence \( k\)-affinoid. In fact, the maps \( A_{i+1} \rightarrow A_i \) induced by \( B_{i+1} \rightarrow B_i \) define \((\text{Sp}(A_i))_{i \in I}\) as an admissible covering of \( X^{an} \) with \((\pi^{an})^{-1}(U_i) = \text{Sp}(A_i)\) (this is the way one shows that with \( \pi \) also its analytification is finite). Since \( B_i \) is flat over \( B \) (cf. [20], Satz 2.1) we have

\[
A^\Gamma_i = (B_i \otimes_B A)^\Gamma = B_i \otimes_B A^\Gamma = B_i
\]

(cf. [3], I.2.3 Remark 2 and I.2.6 Remark 1) \( \square \)

**Remark 2.1.8.** It follows from (2.4) that the underlying point space of \((X/\Gamma)^{an}\) is the set theoretical quotient of \( X^{an} \) modulo \( \Gamma \). According to Proposition 2.1.7 the structure sheaf of \((X/\Gamma)^{an}\) is given by \( \mathcal{O}_{X/\Gamma}^{an}(U) = \mathcal{O}_{X^{an}}((\pi^{an})^{-1}(U))^\Gamma \) so that \((X/\Gamma)^{an}\) can be identified with the rigid analytic quotient \( X^{an}/\Gamma \) whose existence is claimed (but not proved) in [15], 6.4.

Proof of Theorem 2.1.6: Let \( t = (t_1, \ldots, t_n) \) be an \( L \)-basis of \( t \otimes_L K \). Proposition 1.2.8 shows that there is a topological isomorphism \( S(t, K) \rightarrow O((\mathbb{A}^n_K)^{an}) \) of \( K \)-Fréchet algebras identifying the subalgebra \( S(t) \otimes_L K \) with the polynomial algebra \( K[t_1, \ldots, t_n] \) in the variables \( t_i \), i.e. with the algebra of regular functions on the affine space \( \mathbb{A}^n_K \) of dimension \( n \) over \( K \). There is a family \( s = (s_1, \ldots, s_n) \) of \( n \) algebraically independent, homogeneous elements in \( S(t) \otimes_L K \) such that the inclusion homomorphism

\[
\varphi : K[s_1, \ldots, s_n] \longrightarrow (S(t) \otimes_L K)^{an}
\]

is an isomorphism (cf. [8], 11.1.14). According to Proposition 2.1.7 it extends to an isomorphism

\[
\varphi : O((\mathbb{A}^n_K)^{an}) \longrightarrow S(t, K)^{an}
\]

of \( K \)-algebras. If \( c \in K^* \) with \( |c| > 1 \) and \( i \in \mathbb{N} \) we denote by \( |\cdot|_i \) the norm on the left hand side for which the family \((s^\alpha)_{\alpha \in \mathbb{N}^n}\) is orthogonal with \( |s^\alpha|_1 = |c|^{|\alpha|} \).

Similarly, \( \nu_i := \nu_{|\cdot|_i} \) is the multiplicative norm on \( S(t, K) \) for which \((t^\alpha)_{\alpha \in \mathbb{N}^n}\) is orthogonal with \( \nu_j(t^\alpha_j) = |c|^{-|\alpha|} \). We view \( S(t, K)^{an} \) as a (closed) subspace of \( S(t, K) \). Given \( i \in \mathbb{N} \) choose \( i_0 \in \mathbb{N} \) such that \( \max_j \{\nu_j(\varphi(s_j))\} \leq |c^{i_0}| \). Then

\[
\nu_i(\varphi(\sum_{\alpha} d_\alpha s^\alpha)) \leq |\sum_{\alpha} d_\alpha s^\alpha|_{i_0},
\]

so that \( \varphi \) is continuous and in fact a topological isomorphism due to the open mapping theorem.

Let \( \Phi = \Phi(g, t) \) be the root system of \( g \) with respect to \( t \) and choose an eigenvector \( X_\alpha \) of \( g \) for any \( \alpha \in \Phi \). Extend \( t \) to the \( L \)-basis \( \mathbf{X} = (t_1, \ldots, t_n, (X_\alpha)_{\alpha \in \Phi}) \) of \( g \) and let \( \mathcal{J} \) be the closed ideal of \( S(g, K) \) generated by \( \{X_\alpha\}_{\alpha \in \Phi} \). The explicit descriptions of \( S(g, K) \) and \( S(t, K) \) show that

\[
S(g, K) = S(t, K) \oplus \mathcal{J}
\]
first as abstract vector spaces but then also topologically due to the open mapping theorem. We claim that the induced continuous, surjective homomorphism $S(g, K) \to S(t, K)$ of $K$-algebras restricts to a topological isomorphism $\theta : S(g, K)^G \xrightarrow{\sim} S(t, K)^G$. By the open mapping theorem we only need to show that $\theta$ is bijective. If $J := S(g) \cap J$ then $S(g) = S(t) \oplus J$ and the corresponding projection $S(g) \to S(t)$ restricts to an isomorphism

\[(2.5) \quad S(g)^G \simeq S(t)^G \]

of algebras (cf. [8], Théorème 7.3.7).

As in the proof of Proposition 2.1.2 one sees that if $\delta$ is an element of $S(g, K)^G$ (resp. $J$) then both $\delta^{\leq n}$ and $\delta^{> n}$ are elements of $S(g, K)^G$ (resp. $J$). Since $(S(g)^G \otimes_L K) \cap (J \otimes_L K) = 0$ it follows that $S(g, K)^G \cap J = 0$.

Let $\tau \in S(t, K)^G$. It follows from (1.3) that for $s_1, s_2 \in S(t) \otimes_L K$ and $w \in \mathfrak{W}$

$$w \cdot (s_1 \cdot s_2) = (w \cdot s_1) \cdot (w \cdot s_2).$$

Thus, the homogeneous components $\tau_k$ of $\tau$ of degree $k$ with respect to the variables $t$ are $\mathfrak{W}$-invariant for all $k \geq 0$. Write $\tau_k = \sum_\alpha d_\alpha(k)s^\alpha$ and let $\xi_1, \ldots, \xi_n \in S(g)^G$ be preimages of $s_1, \ldots, s_n$ under the map (2.5). Then $\gamma_k := \sum_k \gamma_k \in S(g)^G \otimes_L K$ maps to $\tau_k$ and we need to show that the series $\sum_k \gamma_k$ converges in $S(g, K)$. Note that the Fréchet topology on $S(g, K)$ can be defined by a family of multiplicative norms $(\nu_k)_{k \in \mathbb{N}}$ extending the norms $\nu_k$ on $S(t, K)$ because $\mathfrak{X}$ extends the $L$-basis $t$ of $t$ (cf. Proposition 1.2.8). Since $\varphi^{-1}$ is continuous we have $\lim_{k \to \infty} |\varphi^{-1}(\gamma_k)|_i = 0$ for all $i \in \mathbb{N}$. Given $i \in \mathbb{N}$, choose $i_0 \in \mathbb{N}$ such that $\max_j \{|\nu_k(\xi_j)|\} \leq |e^{|t|}}. Then

$$\nu_k(\gamma_k) \leq \sup_\alpha |d_\alpha(k)|\nu_k(\xi^\alpha) \leq \sup_\alpha |d_\alpha(k)||e^{i_0}|^{\alpha}$$

and

$$|\sum_\alpha d_\alpha(k)s^\alpha|_{i_0} = |\varphi^{-1}(\gamma_k)|_{i_0} \to 0$$

as $k \to \infty$. Composing $\theta$ with the inverse of Duflo’s isomorphism we obtain the isomorphism $\xi := \theta \circ \eta^{-1} : U(g, K)^G \to S(t, K)^G$ of $K$-Fréchet algebras.

2.2 Centrally supported invariant distributions

The conjugation action of $G$ on $D(G, K)$ restricts to $U(g, K)$, $D(Z, K)$ and $D(G, K)_Z$, inducing an action on $D(Z, K) \otimes_{U(g, K)} U(g, K)$.

**Theorem 2.2.1.** If $K$ is discretely valued and $G$ is an open subgroup of the group of $L$-rational points of a connected, algebraic group defined over $L$ then there are $K$-linear topological isomorphisms

$$D(Z, K) \otimes_{U(g, K)} U(g, K)^G \simeq (D(Z, K) \otimes_{U(g, K)} U(g, K))^G \simeq D(G, K)^G_Z$$

of separately continuous $K$-algebras induced by multiplication in $D(G, K)_Z$. In particular, the subspace $D^{pl}(G, K)_Z^G$ of centrally supported invariant point distributions is dense in $D(G, K)_Z^G$.  

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Proof: We endow $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G$ and $(D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K))^G$ with the $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G$-module actions of Remark 1.2.11. Since $D(Z, K)$ and $U(\mathfrak{g}, K)^G$ are contained in the center of $D(G, K)$ it is clear that the maps

\[
D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G \to D(G, K)^G \\
(D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K))^G \to D(G, K)^G
\]

induced by multiplication are homomorphisms of $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G$-modules. If we can show them to be topological isomorphisms then, by the density of $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G$ in the space $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G$, both $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G$ and $(D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K))^G$ carry unique $K$-algebra structures extending the action of $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G$ and for which the above maps are homomorphisms.

The $D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^{op}$-bimodule isomorphism

\[\mu : D(Z, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K) \to D(G, K)_{Z}\]

of Proposition 1.2.12 is $G$-equivariant by definition of the respective $G$-actions. This gives the second isomorphism of the theorem.

If $G_0$ is a compact open subgroup of $G$ and $Z_0 := G_0 \cap Z$ then, using Lemma 1.2.13, it suffices to show that the map

\[D(Z_0, K)\otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G \to D(G_0, K)_{Z_0}\]

induced by multiplication is a topological isomorphism.

According to [4], III.7.2 Proposition 3, there are compact open subgroups $\Lambda_0$ and $G_0$ of $\mathfrak{g}$ and $G$, respectively, such that $\Lambda_0$ lies in the domain of the exponential map and $exp : \Lambda_0 \to G_0$ is an isomorphism of locally $L$-analytic manifolds. In fact, $\Lambda_0$ may be chosen to be contained in any open neighborhood of zero in $\mathfrak{g}$. If therefore $\Lambda_1 := \Lambda_0 \cap \mathfrak{z}$ and $Z_0 := G_0 \cap Z$ then we may assume $exp$ to restrict to an isomorphism $\Lambda_1 \to Z_0$ (note that $exp$ is also an exponential map for $Z_0$). The $K$-linear topological isomorphism $exp_* : D(\Lambda_0, K) \to D(G_0, K)$ therefore restricts to isomorphisms

\[exp_* : D(\Lambda_1, K) \to D(Z_0, K) \]
\[id : S(\mathfrak{z}, K) \to U(\mathfrak{z}, K) \]
\[exp_* : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K).\]

**Lemma 2.2.2.** If $\lambda \in D(\Lambda_1, K)$ and $\delta \in D(\Lambda_0, K)$ then $exp_* (\lambda \cdot \delta) = exp_* (\lambda) \cdot exp_* (\delta)$.

Proof: Let $\eta \in \Lambda_1$ and $f \in C^{an}(G_0, K)$. Then

\[exp_* (\delta_0 \cdot \delta)(f) = (\delta_0 \cdot \delta)(exp^* f)\]
\[= \delta(\tau \mapsto f(exp(\eta + \tau)))\]
\[= \delta(\tau \mapsto f(exp(\eta) \cdot exp(\tau)))\]
\[= (exp_* (\delta_0) \cdot exp_* (\delta))(f),\]
since \( \eta \) commutes with all \( r \in \mathfrak{g} \). Since \( K[\Lambda_3] \) is dense in \( D(\Lambda_3, K) \), the assertion follows from the linearity and continuity of \( \exp_r \). \( \square \)

Together with Lemma 1.2.10 we obtain that \( \exp_r \) restricts to an isomorphism \( D(\Lambda_\# K)_{\Lambda_3} \to D(G_0, K)_{Z_0} \) and that the diagram

\[
\begin{array}{ccc}
D(\Lambda_3, K) \hat{\otimes} S(\mathfrak{g}, K) & \xrightarrow{\mu} & D(\Lambda_\# K)_{\Lambda_3} \\
\exp_\otimes \exp_\otimes i & & \exp_\otimes i \\
D(Z_0, K) \hat{\otimes} U(\mathfrak{g}, K) & \xrightarrow{\mu} & D(G_0, K)_{Z_0}
\end{array}
\]

is commutative. \( G \) acts trivially on \( D(\Lambda_3, K) \) and \( D(Z_0, K) \). Moreover, \( G \) acts on \( S(\mathfrak{g}, K) \) in such a way that \( \exp_r : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K) \) is \( G \)-equivariant. Thus, there is an action of \( G \) on \( D(\Lambda_\# K)_{\Lambda_3} \) such that \( \exp_r : D(\Lambda_\# K)_{\Lambda_3} \to D(\Lambda_\# K)_{Z_0} \) is \( G \)-equivariant and we may equally well show the above statements in the setting of \( \Lambda_\# \) and \( \Lambda_3 \).

Passing to an open subgroup of \( \Lambda_\# \), we may assume that \( \Lambda_\# \) and \( \Lambda_3 \) satisfy the compatibility conditions of Corollary 1.3.6. Hence for \( r \in p^\mathbb{Z} \) with \( 1/p < r < 1 \) the \( K \)-Banach algebra \( D_r(\Lambda_\# K)_{\Lambda_3} \) admits a finite direct sum decomposition

\[
D_r(\Lambda_\# K)_{\Lambda_3} = \bigoplus_{\alpha \in A'} b^\alpha S_r(\mathfrak{g}, K)
\]

with \( b^\alpha \in K[\Lambda_3] \) for all \( \alpha \in A' \) (cf. Corollary 1.4.3).

**Lemma 2.2.3.** The action of \( \mathfrak{g} \) on \( D(\Lambda_\# K)_{\Lambda_3} \) induced by that of \( G \) extends to a \( \mathfrak{g} \)-action on \( D_r(\Lambda_\# K)_{\Lambda_3} \).

**Proof:** It suffices to show that the action of \( \mathfrak{g} \) on \( S(\mathfrak{g}, K) \) is continuous with respect to the norm \( || \cdot ||_r \). Note that by Corollary 1.4.5 there is a continuous \( K \)-linear surjection \( \tau' : S(\mathfrak{g}_{\mathbb{Q}_p}, K) \to S(\mathfrak{g}, K) \) which is seen to be \( \mathfrak{g} \)-equivariant (use Proposition 2.1.1). As a direct consequence of Frommer’s theorem \( S(\mathfrak{g}_{\mathbb{Q}_p}, K) \) is a \( K \)-Fréchet-Stein algebra. Therefore, \( S(\mathfrak{g}, K) \) and the kernel \( J \) of \( \tau' \) are coadmissible modules over \( S(\mathfrak{g}_{\mathbb{Q}_p}, K) \). According to Theorem B (cf. [26], section 3) the coherent sheaf corresponding to \( J \) is given by the kernels \( J_r \) of the surjections \( S_r(\mathfrak{g}_{\mathbb{Q}_p}, K) \to S_r(\mathfrak{g}, K) \) (cf. (1.14)). Since \( J \) is \( \mathfrak{g} \)-invariant and dense in \( J_r \) (cf. Theorem A of [26], section 3), we may assume \( L = \mathbb{Q}_p \) and hence \( || \cdot ||_r = || \cdot ||, \) to be multiplicative.

Recall from Frommer’s theorem that there is a \( \mathbb{Q}_p \)-basis \( \mathfrak{X} = (\mathfrak{X}_1, \ldots, \mathfrak{X}_d) \) of \( \mathfrak{g} \) such that

\[
S_r(\mathfrak{g}, K) = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K, \lim_{|\alpha| \to \infty} |d_\alpha||\mathfrak{X}^\alpha|_r = 0 \right\}
\]

with \( || \sum_\alpha d_\alpha \mathfrak{X}^\alpha ||_r = \sup_\alpha \{|d_\alpha| \prod_{i=1}^d |\mathfrak{X}_i|^{\alpha_i} \} \). For \( r \in \mathfrak{g} \) choose \( \lambda \in \mathbb{Q}_p^* \) such that \( ||\text{ad}(\lambda)(\mathfrak{X}_i)||_r \leq ||\mathfrak{X}_i||_r \) for all \( i \). It follows that \( ||r * \delta||_r \leq ||\lambda^{-1}|| \cdot ||\delta||_r \) for all \( \delta \in S(\mathfrak{g}, K) \). \( \square \)
We obtain

\[ D_r(\Lambda_\vartheta, K)^\vartheta = \bigoplus_{\alpha \in A'} b^\alpha S_r(\mathfrak{g}, K)^\vartheta. \]

Since, as remarked in the proof of Corollary 1.4.3, \((b^\alpha)_{\alpha \in A'}\) is also a basis for the free \(S_r(\mathfrak{g}, K)\)-module \(D_r(\Lambda_\vartheta, K)\) we obtain a topological isomorphism

\[ D_r(\Lambda_\vartheta, K) \otimes_{S_r(\mathfrak{g}, K)} S_r(\mathfrak{g}, K)^\vartheta \longrightarrow D_r(\Lambda_\vartheta, K)_{\Lambda_\vartheta}. \]

Passing to the projective limit we obtain a topological isomorphism

\[ D(\Lambda_\vartheta, K) \otimes_{S_r(\mathfrak{g}, K)} S(\mathfrak{g}, K)^\vartheta \longrightarrow D(\Lambda_\vartheta, K)_{\Lambda_\vartheta} \]

as in the proof of Proposition 1.2.12: To satisfy the Mittag-Leffler condition we need to know that \(S(\mathfrak{g}, K)^\vartheta\) is dense in \(S_r(\mathfrak{g}, K)^\vartheta\) for all \(r\). This is true according to Remark 2.1.4 and Theorem 1.4.2 and is in fact the reason for our working with \(\Lambda_\vartheta\) and \(\Lambda_\vartheta\) instead of with \(G_0\) and \(Z_0\). By our assumption on \(G\) and Remark 2.1.3 \(D(\Lambda_\vartheta, K)_{\Lambda_\vartheta} = D(\Lambda_\vartheta, K)^G_{\Lambda_\vartheta}\).

Since by Lemma 1.1.1 and Proposition 2.1.2 \(K[Z_0]\) and \(U(\mathfrak{g})^G \otimes_\mathcal{D} K\) are dense in \(D(Z_0, K)\) and \(U(\mathfrak{g}, K)^G\), respectively, it follows from [22], Lemma 19.10 (i), that the space \(K[Z_0] \otimes_K (U(\mathfrak{g})^G \otimes_\mathcal{D} K)\) is dense in \(D(Z_0, K) \otimes_K U(\mathfrak{g}, K)^G\). Therefore, so is its image in the quotient space \(D(Z_0, K) \otimes_{U(\mathfrak{g}, K)} U(\mathfrak{g}, K)^G\). Since the image of \(K[Z_0] \otimes_K (U(\mathfrak{g})^G \otimes_\mathcal{D} K)\) under \(\mu\) is precisely \(D^{\text{pt}}(G_0, K)_{Z_0}\), the proof of the theorem is complete. \(\square\)

Let \(\mathcal{G}\) be a connected, reductive, linear algebraic group defined over \(L\). \(\mathcal{G}\) is the almost direct product of its center and the finitely many minimal, closed, connected, normal \(L\)-subgroups \(\mathcal{G}_i\) of positive dimension of its derived subgroup \(\mathcal{D}\). Let us call \(\mathcal{G}\) sufficiently \(L\)-isotropic if all \(\mathcal{G}_i\) are \(L\)-isotropic. This is the case, for example, if \(\mathcal{G}\) is \(L\)-split. For the following cf. [29], Theorem 2.4.

**Theorem** (Sit). Assume \(G\) to be the group of \(L\)-rational points of a connected, reductive, sufficiently \(L\)-isotropic \(L\)-group \(\mathcal{G}\). If the conjugacy class of an element \(g \in G\) is relatively compact in \(G\) (endowed with the topology induced from \(L\)) then \(g\) is contained in the center of \(G\).

**Corollary 2.2.4.** Assume \(G\) to be the group of \(L\)-rational points of a connected, reductive, sufficiently \(L\)-isotropic \(L\)-group \(\mathcal{G}\). Then \(D(\mathcal{G}, K)^G = D(\mathcal{G}, K)^G_{Z_0}\). Let \(\mathcal{D}\) be the derived group of \(\mathcal{G}\), \(\mathcal{D}\) the group of \(L\)-rational points of \(\mathcal{D}\) and \(\mathfrak{d}\) the Lie algebra of \(\mathcal{D}\). If \(K\) is discretely valued then there is a topological isomorphism

\[ (2.6) \quad D(\mathcal{G}, K)^G \simeq D(Z, K) \otimes_{K, \mathcal{G}} U(\mathfrak{d}, K)^G \]

of separately continuous \(K\)-algebras.

**Proof:** According to (2.1), (1.4) and Remark 1.2.3 any invariant distribution on \(G\) is supported on a union of relatively compact conjugacy classes. As a consequence of Sit’s theorem we have \(D(\mathcal{G}, K)^G = D(\mathcal{G}, K)^G_{Z_0}\).

Since \(G = D \cdot Z\) with finite intersection \(D \cap Z\) it follows from Remark 1.2.14 that there is a topological isomorphism

\[ D(G, K)_{Z} \longrightarrow D(Z, K) \otimes_{K, \mathcal{G}} U(\mathfrak{d}, K) \]

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of $D(Z, K)-U(\mathfrak{a}, K)^{op}$-bimodules. The image of $D^{pt}(G, K)^G$ under this isomorphism is $D^{pt}(Z, K) \otimes_K (U(\mathfrak{a})^0 \otimes_L K)$ (cf. Remark 2.1.3). Since $D^{pt}(G, K)^G$, $D^{pt}(Z, K)$ and $U(\mathfrak{a})^0 \otimes_K K$ are dense in $D(G, K)^G$, $D(Z, K)$ and $U(\mathfrak{a}, K)^0$, respectively, (cf. Theorem 2.2.1, Lemma 1.1.1 and Proposition 2.1.2) the above isomorphism restricts to an isomorphism $D(G, K)^G \sim D(Z, K) \otimes_K U(\mathfrak{a}, K)^0$. The arguments given at the beginning of the proof of Theorem 2.2.1 show that it may naturally be viewed as a homomorphism of $K$-algebras.

2.3 The Fourier transform

Let $k$ be a field which is complete with respect to a non-trivial and non-archimedean absolute value. Recall that a rigid analytic $k$-variety $X$ is called quasi-Stein if there is a countable, admissible affinoid covering $(X_i)_{i \in \mathbb{N}}$ of $X$ such that $X_i \subseteq X_{i+1}$ and the image of the map $\mathcal{O}(X_{i+1}) \rightarrow \mathcal{O}(X_i)$ is dense for all $i \in \mathbb{N}$ (cf. [19], Definition 2.3). It is easy to see that if $X$ and $Y$ are quasi-Stein then so is their fibered product $X \times_k Y$. Also, if $X'$ is a rigid analytic $k$-variety admitting a finite morphism to a quasi-Stein $k$-variety $X$ then $X'$ is quasi-Stein itself. If $k'$ is a complete, valued field extension of $k$ then any rigid analytic, quasi-Stein $k$-variety $X$ admits a base extension to $k'$ and the resulting rigid analytic $k'$-variety $X_{k'}$ is quasi-Stein.

**Remark 2.3.1.** If $X$ is quasi-Stein over $k$ and $k'$ is a complete valued field extension of $k$ then the algebra of global sections of $X_{k'}$ is a $k'$-Fréchet-Stein algebra: If $(X_i)_{i \in \mathbb{N}}$ is a covering of $X$ as a quasi-Stein space then

$$\mathcal{O}_{X_k'}(X_{k'}) = \varprojlim \mathcal{O}_{X_{k'}}((X_i)_{k'}).$$

For each $i \in \mathbb{N}$ the algebra $\mathcal{O}_{X_{k'}}((X_i)_{k'})$ is a noetherian $k'$-Banach algebra for which the map $\mathcal{O}_{X_{k'}}((X_{i+1})_{k'}) \rightarrow \mathcal{O}_{X_{k'}}((X_i)_{k'})$ is flat (cf. [2], 7.3.2 Corollary 6). Moreover, the natural map $\mathcal{O}_{X_{k'}}(X_{k'}) \rightarrow \mathcal{O}_{X_{k'}}((X_i)_{k'})$ has dense image because this is true for all transition maps.

Recall that if $Z$ is a commutative locally $L$-analytic group and $X$ is a rigid analytic $L$-variety then the group $\hat{Z}(X)$ of locally analytic characters of $Z$ with values in $X$ consists of the homomorphisms $Z \rightarrow \mathcal{O}_X(X)^*$ of groups such that for any admissible open affinoid subset $X_0 = \text{Sp}(A)$ of $X$ the induced homomorphism $Z \rightarrow A^*$ is an element of $C^\infty(Z, A)$ (cf. [12], Definition 6.4.2). It is shown in [loc.cit.], Corollary 6.4.4, that $\hat{Z}$ is a functor on the category of all rigid analytic $L$-varieties.

**Theorem** (Emerton-Schneider-Teitelbaum). If $Z$ is a commutative, locally $L$-analytic, topologically finitely generated group then the functor $\hat{Z}$ is representable by a strictly $\sigma$-affinoid rigid analytic space over $L$.

Recall that according to [loc.cit.], Definition 2.1.17, a rigid analytic $L$-variety $X$ is called strictly $\sigma$-affinoid if $X$ has an admissible covering $(X_i)_{i \in \mathbb{N}}$ by affinoid subdomains $X_i$ such that for every $i \in \mathbb{N}$ $X_i$ is relatively compact in $X_{i+1}$ in the sense of [2], 9.6.2. As a corollary to the construction of $\hat{Z}$ we obtain:

**Corollary 2.3.2.** $\hat{Z}$ is quasi-Stein.

Proof: By [12], Proposition 6.4.1, there is an isomorphism $Z \rightarrow \Lambda \times Z_0$ of locally $L$-analytic groups where $\Lambda$ is a free abelian group of finite rank, say $r$, and
$Z_0$ is a compact open subgroup of $Z$. Consequently, there is an isomorphism $\hat{Z} \to \hat{\Lambda} \times \hat{Z}_0$. $\Lambda$ is represented by the $r$-fold direct product of the rigid analytification $\mathbb{G}_{m,L}^{an}$ of the multiplicative group $\mathbb{G}_{m,L}$ over $L$ which is quasi-Stein. Further, $\hat{Z}_0$ admits a finite morphism to a finite direct product of copies of $\hat{\sigma}_L$ which is quasi-Stein by [24], p. 456.

The ring of global sections of the structure sheaf of $\hat{Z}_K$ is denoted by $O(\hat{Z}_K)$. Since $\hat{Z}_K$ is quasi-Stein and strictly $\sigma$-affinoid it follows from Remark 2.3.1 and [12], Proposition 2.1.16, that $O(\hat{Z}_K)$ is a nuclear $K$-Fréchet-Stein algebra.

**Theorem (Emerton-Schneider-Teitelbaum).** If $Z$ is a commutative, locally $L$-analytic, topologically finitely generated group then there is a natural continuous injection $D(Z, K) \to O(\hat{Z}_K)$ of $K$-algebras with dense image.

We briefly recall the construction of this map: As above we choose an isomorphism $Z \to \Lambda \times Z_0$. According to [28], Proposition A.3, there is a topological isomorphism $D(Z, K) \cong D(\Lambda, K) \otimes_{K, \iota} D(Z_0, K)$. $\Lambda$ being discrete, $D(\Lambda, K) = K[\Lambda]$ is the topological direct sum of one dimensional $K$-vector spaces. Hence $D(\Lambda, K) \otimes_{K, \iota} D(Z_0, K)$ is complete (cf. Lemma 1.2.13 and [22], Lemma 7.8) so that

$$D(Z, K) \cong K[\Lambda] \otimes_{K, \iota} D(Z_0, K).$$

On the other hand, the Fourier transform of [24], Theorem 2.3, extends to an isomorphism $D(Z_0, K) \cong O((\hat{Z}_0)_K)$ of $K$-Fréchet algebras. Further, $D(\Lambda, K) = K[\Lambda]$ can be interpreted as the algebra of regular functions on the algebraic Cartier dual $D(\Lambda) = G_{m,K}^r$. It admits an embedding into $O((\mathbb{G}_{m,K}^r)^{an}) = O(\hat{\Lambda}_K)$ with dense image. Since

$$O(\hat{Z}_K) \cong O(\hat{\Lambda}_K) \otimes_{K, \iota} O((\hat{Z}_0)_K) \cong O(\hat{\Lambda}_K) \otimes_{K, \iota} O((\hat{Z}_0)_K)$$

the claim follows.

**Corollary 2.3.3.** Let $G$ be a locally $L$-analytic group and assume that either

i) $G$ is commutative and topologically finitely generated or

ii) $G$ is the group of $L$-rational points of a connected, split reductive $L$-group $G$.

If $K$ is discretely valued then there is a quasi-Stein rigid analytic $L$-variety $X$ and an injective, continuous homomorphism $D(G, K)^G \to O(X_K)$ of $K$-algebras with dense image.

Proof: Case (i) is just the previous theorem because $D(G, K)^G = D(G, K)$. In case (ii) let $Z$ be the center of $G$ and $n$ be the dimension of the derived group of $G$. Since $Z$ is topologically finitely generated we may define $X := \hat{Z} \times L (\hat{A}_L^n)^{an}$. Writing $Z = \Lambda \times Z_0$ we have $O(X_K) \cong O(\hat{\Lambda}_K) \otimes_{K, \iota} O((\hat{Z}_0)_K) \otimes_{K, \iota} O((\hat{A}_K^n)^{an})$. Further, Corollary 2.2.4 yields

$$(2.7) \quad D(G, K)^G \cong K[\Lambda] \otimes_{K, \iota} D(Z_0, K) \otimes_{K, \iota} U(\sigma, K)^9,$$
where \( \mathfrak{d} \) denotes the Lie algebra of the derived group of \( \mathbb{G} \). It follows from our assumptions on \( G \) that \( \mathfrak{d} \) is semisimple and \( L \)-split whence by Theorem 2.1.6 there is a topological isomorphism \( U(\mathfrak{d}, K)^o \simeq \mathcal{O}(\mathfrak{a}_K^n)^o \) of \( K \)-Fréchet algebras. Tensoring the embedding \( K[A] \subseteq \mathcal{O}(\hat{\Lambda}_K) \) with

\[
D(Z_0, K) \hat{\otimes}_K U(\mathfrak{d}, K)^o \simeq \mathcal{O}((\hat{Z}_0)_K) \otimes \mathcal{O}(\mathfrak{a}_K^n)^o
\]

gives a continuous \( K \)-linear injection \( D(G, K)^G \to \mathcal{O}(X_K) \). Since \( K[A] \) is dense in \( \mathcal{O}(\hat{\Lambda}_K) \) it has dense image (cf. [22], Lemma 19.10) and, by construction, is a homomorphism of \( K \)-algebras. □

**Remark 2.3.4.** The isomorphism (2.7) makes it possible to explicitly compute the center of \( D(G, K) \) if \( G \) is \( L \)-split. The structure of \( U(\mathfrak{d}, K)^o \) has been determined in Theorem 2.1.6: if \( n \) is the rank of \( \mathfrak{d} \) then \( U(\mathfrak{d}, K)^o \simeq \mathcal{O}(\mathfrak{a}_K^n)^o \) is the \( K \)-algebra of all power series in \( n \) variables with infinite radius of convergence. Moreover, if \( r \) is the dimension of \( Z \) then \( Z \) contains an open subgroup isomorphic to \( \mathfrak{z}_r \). Thus, \( Z \simeq A \times \mathfrak{z}_r \) as locally analytic groups with a discrete, finitely generated abelian group \( A \). Consequently,

\[
D(Z, K) \simeq K[A] \hat{\otimes}_K D(\mathfrak{a}_L, K) \hat{\otimes}_K \cdots \hat{\otimes}_K D(\mathfrak{a}_L, K)
\]

(\( r \)-times)

(cf. [28], Proposition A.3). The structure of \( D(\mathfrak{a}_L, K) \) has been investigated in [24]. It is the \( K \)-algebra of holomorphic functions on a twisted form of the open unit disk.

**Corollary 2.3.5.** Under the assumptions of Corollary 2.3.3 any maximal ideal of \( D(G, K)^G \) which is closed with respect to the topology induced by \( \mathcal{O}(X_K) \) is of finite codimension.

Proof: Let \( \mathfrak{m} \) be a maximal ideal of \( \hat{A} := D(G, K)^G \) which is closed with respect to the metric topology induced by \( \hat{A} := \mathcal{O}(X_K) \) and let \( \hat{\mathfrak{m}} \) be the closure of \( \mathfrak{m} \) in \( \hat{A} \). \( \hat{A}/\hat{\mathfrak{m}} = \hat{A}/\mathfrak{m} \) gives rise to a non-zero, coherent module \( \mathcal{F} \) on \( X_K \) (cf. [26], Lemma 3.6). There is a point \( x \in X_K \) such that \( \mathcal{F}_x \neq 0 \). By Nakayama’s lemma also \( \mathcal{F}_x/\mathfrak{m}_x \neq 0 \), where \( \mathfrak{m}_x \) is the maximal ideal of \( \mathcal{O}_{X_K, x} \). However, \( \dim_{K} \mathcal{F}_x/\mathfrak{m}_x < \infty \), and \( \mathcal{F}_x/\mathfrak{m}_x \) is also a module over \( A/\mathfrak{m} \). □

### 2.4 An extension of Harish-Chandra’s isomorphism

Let \( \mathbb{G} \) be a connected, split reductive, linear algebraic group defined over \( L \) with a maximal \( L \)-split torus \( T \). Let \( \mathfrak{d} \) and \( \mathfrak{z} \) be the center and the derived group of \( \mathbb{G} \), respectively. Then \( \mathbb{D} \) is \( L \)-split and \( \mathbb{D}' := (\mathbb{D} \cap T')^o \) is a maximal \( L \)-split torus of \( \mathbb{D} \). Let \( G, Z, D, T \) and \( T' \) be the group of \( L \)-rational points of \( \mathbb{G}, \mathbb{Z}, \mathbb{D}, \mathbb{T} \) and \( \mathbb{T}' \), respectively, and \( \mathfrak{g}, \mathfrak{z}, \mathfrak{d}, \mathfrak{t} \) and \( \mathfrak{t}' \) be the respective Lie algebras. Note that \( \mathfrak{d} = [\mathfrak{g}, \mathfrak{g}] \) is a semisimple Lie algebra and that \( \mathfrak{t}' \) is an \( L \)-split maximal toral subalgebra of \( \mathfrak{d} \). Let finally \( W = W(G, T) := N_G(T)/T \) be the Weyl group of \( G \) with respect to \( T \), \( W \) acts on \( T \) by conjugation and hence on \( D(T, K) \) such that the subalgebra \( S(t, K) \) of \( D(T, K) \) is stable under \( W \). \( W \) is also the Weyl group of \( D \) with respect to \( T' \), hence acts on \( T' \) and \( D(T', K) \). The corresponding action on \( S(t', K) \) is induced by the adjoint action of \( W \) on \( t' \) (cf. the proof of Proposition 2.1.1). Recall that \( S(t', K) \) is also acted on by the Weyl group \( \mathfrak{W} = \mathfrak{W}(\mathfrak{d}, \mathfrak{t}') \) of the pair \( (\mathfrak{d}, t') \) (cf. subsection 2.1). This action, too, is induced by viewing \( \mathfrak{W} \) as a subgroup of \( \text{Aut}_L(t') \). The following fact is well known.
Lemma 2.4.1. Ad : W → W is an isomorphism of groups. In particular, S(t′, K)^W = S(t′, K)^W. □

Theorem 2.4.2. Let G be the group of L-rational points of a connected, split reductive L-group G with T and W as above. If K is discretely valued then there is a topological isomorphism

\[ D(G, K)^G \simeq D(T, K)^W \]

of separately continuous K-algebras.

Proof: According to Corollary 2.2.4 there is a topological isomorphism

\[ \kappa : D(G, K)^G \rightarrow D(Z, K) \otimes_{K} U(g, K)^p \]

of separately continuous K-algebras.

Since T = Z · T′ with finite intersection Z ∩ T′ one proves in an analogous manner that there is a topological isomorphism of separately continuous K-algebras

\[ \psi : D(Z, K) \otimes_{K} S(t′, K)^W \rightarrow D(T, K)^W \]

According to Theorem 2.1.6 and Lemma 2.4.1 there is a topological isomorphism \( \xi : U(g, K)^p \rightarrow S(t′, K)^W \) of K-Fréchet algebras so that

\[ \psi \circ (id \otimes \xi) \circ \kappa : D(G, K)^G \rightarrow D(T, K)^W \]

is as required. □

Remark 2.4.3. If G is semisimple then Z is finite and \( \kappa \) and \( \psi \) are the obvious isomorphisms

\[ K[Z] \otimes_K U(g, K)^G \rightarrow D(G, K)^G \]

\[ K[Z] \otimes_K S(t, K)^W \rightarrow D(T, K)^W \]

Since the isomorphism \( \xi : U(g, K)^G \rightarrow S(t, K)^W \) was constructed without any restriction on K it follows that we have an isomorphism \( D(G, K)^G \simeq D(T, K)^W \) for any spherically complete coefficient field K.

References


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