Time-Multipatch Discontinuous Galerkin
Space-Time Isogeometric Analysis
of Parabolic Evolution Problems

Ulrich Langer
Institute of Computational Mathematics (NuMa)
Johannes Kepler University Linz

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (ÖAW)
Linz, Austria

AANMPDE 2017, Paleochora, Crete, 2 - 6 October 2017
Joint work with my collaborators

- Christoph Hofer (JKU, DK)
- Martin Neumüller (JKU, NuMa)
- Ioannis Touloupoulos (RICAM, CM4PDE)

Main results have just been published in


Outline

1. Introduction
2. Time Multi-Patch Space-Time IgA
3. Solver
4. Numerical Results
5. Conclusions & Outlooks
Let us consider the IBVP problem: Find \( u : \overline{Q} \rightarrow \mathbb{R} \) such that

\[
\partial_t u - \Delta u = f \quad \text{in} \quad Q := \Omega \times (0, T), \\
u = u_D := 0 \quad \text{on} \quad \Sigma := \partial\Omega \times (0, T), \\
u = u_0 \quad \text{on} \quad \overline{\Sigma}_0 := \overline{\Omega} \times \{0\},
\]

as the typical model problem for a linear parabolic evolution equation posed in the space-time cylinder \( \overline{Q} = \overline{\Omega} \times [0, T] \).

Our space-time technology can be generalized to more general parabolic equations like

\[-\text{div}_x (A(x, t) \nabla u) + b(x, t) \cdot \nabla_x u + c(x, t) \partial_t u + a(x, t) u = f,\]

eddy-current problems, non-linear problems etc.
Let us consider the IBVP problem: Find $u : \overline{Q} \rightarrow \mathbb{R}$ such that

$$
\partial_t u - \Delta u = f \quad \text{in} \quad Q := \Omega \times (0, T),
$$

$$
\begin{align*}
\quad u &= u_D := 0 \quad \text{on} \quad \Sigma := \partial \Omega \times (0, T), \\
\quad u &= u_0 \quad \text{on} \quad \Sigma_0 := \overline{\Omega} \times \{0\},
\end{align*}
$$

as the typical model problem for a linear parabolic evolution equation posed in the space-time cylinder $\overline{Q} = \overline{\Omega} \times [0, T]$.

Our space-time technology can be generalized to more general parabolic equations like

$$
-\text{div}_x (A(x, t) \nabla u) + b(x, t) \cdot \nabla_x u + c(x, t) \partial_t u + a(x, t) u = f,
$$

eddy-current problems, non-linear problems etc.
Standard Weak Space-Time Variational Formulation

Find \( u \in H_{0,0}^{1,0} (Q) \) such that

\[
a(u,v) = \ell (v) \quad \forall v \in H_{0,0}^{1,1} (Q),
\]

with the bilinear form

\[
a(u,v) = - \int_Q u(x,t) \partial_t v(x,t) \, dx \, dt + \int_Q \nabla_x u(x,t) \cdot \nabla_x v(x,t) \, dx \, dt
\]

and the linear form

\[
\ell (v) = \int_Q f(x,t)v(x,t) \, dx \, dt + \int_\Omega u_0(x)v(x,0) \, dx,
\]

see Monograph by Ladyzhenskaya, Solonnikov & Uralceva (1967) or Lecture Notes by Ladyzhenskaya (1973) for solvability and regularity results!
Some References to

**Time-parallel methods:**
Gander (2015): Nice historical overview on 50 years time-parallel method
Parareal introduced by Lions, Maday, Turinici (2001)
Gander & Neumüller (2014): smart time-parallel multigrid:

15 667 822 592 space-time dofs in 30 sec on 262 144 cores at Vulcan - BlueGene/Q, LLNL
perfect week and strong scalings

**Space-time methods for parabolic evolution problems**
Some References to

**Time-parallel methods:**
Gander (2015): Nice historical overview on 50 years time-parallel method
Parareal introduced by Lions, Maday, Turinici (2001)
Gander & Neumüller (2014): smart time-parallel multigrid:
\[15,667,822,592\] space-time dofs in 30 sec on 262,144 cores at Vulcan - BlueGene/Q, LLNL
perfect week and strong scalings

**Space-time methods for parabolic evolution problems**

Space-Time IgA paraphernalia: $Q \subset \mathbb{R}^{d+1}$; $d = 1$ (l) and $d = 2$ (r).

**In this talk:** Generalization to the multipatch case!
Outline

1 Introduction

2 Time Multi-Patch Space-Time IgA

3 Solver

4 Numerical Results

5 Conclusions & Outlooks
Time Multi-Patch Decomposition of $Q$

We decompose the space-time cylinder $Q = \Omega \times (0, T)$ into $N$ non-overlapping space-time subcylinder $Q_n = \Omega \times (t_{n-1}, t_n) = \Phi_n(\hat{Q})$, $n = 1, 2, \ldots, N$, such that

$$\overline{Q} = \bigcup_{n=1}^{N} \overline{Q}_n$$

with the time faces $\Sigma_n = \overline{Q}_{n+1} \cap \overline{Q}_n = \Omega \times \{t_n\}$, $\Sigma_N = \Sigma_T$.
We look for an approximate solution $u_h$ to the IBVP (2) in the globally discontinuous, but patch-wise smooth IgA (B-spline, NURBS) spaces

$$V_{0h} = \{ v_h \in H^{1,0}_0(Q) : v^n_h := v_h|_{Q_n} \in \mathbb{B}_{\Xi_{d+1}^n}(Q_n), n = 1, \ldots, N \}$$

$$= \{ v_h \in L_2(Q) : v^n_h \in V^n_{0h}, n = 1, \ldots, N \} = \text{span}\{ \varphi_i \}_{i \in I},$$

$$V^n_{0h} = \{ v^n_h \in \mathbb{B}_{\Xi_{d+1}^n}(Q_n) : v^n_h = 0 \text{ on } \Sigma \} = \text{span}\{ \varphi_{n,i} \}_{i \in I_n}$$

where $\mathbb{B}_{\Xi_{d+1}^n}(Q_n)$ is the smooth (depending of the polynomial degrees and multiplicity of the knots) IgA space corresponding to the knot vector

$$\Xi_{d+1}^n = \Xi_{d+1}^n(n_1^n, \ldots, n_{d+1}^n; p_1^n, \ldots, p_{d+1}^n) = \ldots$$
Stable Time Multipatch dG IgA Scheme

Multiplying the PDE by $v_h + \theta_n h_n \partial_t v_h$, integrating over $Q_n$, integrating by parts, suming over $n$, and using that the jumps $[|u|]$ across $\Sigma_n$ are 0 at the solution $u \in H^{2,1}(Q)$, we get the multi-patch space-time scheme: Find $u_h \in V_{0h}$ such that

$$a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{0h},$$

where

$$a_h(u_h, v_h) = \sum_{n=1}^{N} \int_{Q_n} \left( \partial_t u_h v_h + \theta_n h_n \partial_t u_h \partial_t v_h + \nabla_x u_h \nabla_x v_h + \theta_n h_n \nabla_x u_h \cdot \nabla_x \partial_t v_h \right) \, dx \, dt$$

$$+ \sum_{n=1}^{N} \int_{\Sigma_n} [u_{h,n-1}^n] v_{h,n-1}^n \, dx,$$

$$\ell_h(v_h) = \sum_{n=1}^{N} \int_{Q_n} f [v_h + \theta_n h_n \partial_t v_h] \, dx \, dt + \int_{\Sigma_0} u_0 v_{h,0}^0 \, dx.$$
Hence, we look for the solution \( u_h \in V_{0h} \) of the IgA scheme

\[
a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{0h}
\]

in the form of

\[
u_h(x, t) = u_h(x_1, \ldots, x_d, x_{d+1}) = \sum_{i \in I} u_i \varphi_i(x, t)
\]

where \( u_h := [u_i]_{i \in I} \in \mathbb{R}^{N_h=|I|} \) is the unknown solution vector of control points defined by the solution of the linear system

\[
L_h u_h = f_h
\]

with huge, non-symmetric, but \textbf{positive definite} system matrix \( L_h \).
**V₀h—Coercivity of the bilinear form aₜ(·, ·)**

We now introduce the mesh-dependent dG norm

\[ \|v\|_{dG}^2 = \sum_{n=1}^{N} \left( \|\nabla_x v\|_{L^2(Q_n)}^2 + \theta_n h_n \|\partial_t v\|_{L^2(Q_n)}^2 + \frac{1}{2} \|[v]_{n-1}\|_{L^2(S_{n-1})}^2 \right) + \frac{1}{2} \|v\|_{L^2(S_N)}^2. \]

**Lemma (Coercivity / Ellipticity on V₀h)**

The bilinear form \(aₜ(·, ·) : V₀h \times V₀h \rightarrow \mathbb{R}\) is \(V₀h\)–coercive wrt the norm \(\|·\|_h\), i.e., there exists a constant \(\mu_c = 1/2\) such that

\[ aₜ(vₜ, vₜ) \geq \mu_c \|vₜ\|_h^2, \quad \forall vₜ \in V₀h. \]  (5)

provided that \(\theta_n \leq c_{inv,0}^{-2}\), where \(\|vₜ\|_{L^2(\partial E)} \leq c_{inv,0} h_n^{-1} \|vₜ\|_{L^2(E)}\)

This lemma immediately yields uniqueness and existence of the solution \(uₜ \in V₀h\) and \(uₜ \in \mathbb{R}^{Nₜ}\) of (9) and (4), respectively.
Uniform Boundedness of $a_h(\cdot, \cdot)$ on $V_{0h,*} \times V_{0h}$

Let us introduce the space $V_{0h,*} = H_0^{1,0}(Q) \cap H_2^1(Q) + V_{0h}$ equipped with the norm

$$
\|v\|_{dG,*} = \left(\|v\|^2_{dG} + \sum_{n=1}^N (\theta_n h_n)^{-1}\|v\|^2_{L^2(Q_n)} + \sum_{n=2}^N \|v_{n-1}^{-1}\|^2_{L^2(\Sigma_{n-1})}\right)^{\frac{1}{2}}.
$$

Lemma (Boundedness)

The bilinear form $a_h(\cdot, \cdot)$ is uniformly bounded on $V_{0h,*} \times V_{0h}$:

$$
|a_h(u, v_h)| \leq \mu_b \|u\|_{dG,*} \|v_h\|_{dG}, \quad \forall \ u \in V_{0h,*}, \forall \ v_h \in V_{0h},
$$

with $\mu_b = \max(c_{inv,1} \theta_{max}, 2)$, where $\theta_{max} = \max_n \{\theta_n\} \leq c_{inv,0}^{-2}$.

and $c_{inv,k} = c_{inv,k}(p)$ are constants in the inverse inequalities

$$
\|\partial_t \partial_x \nu_h\|^2_{L^2(E)} \leq c_{inv,1} h_n^{-2} \|\partial_x \nu_h\|^2_{L^2(E)} \quad \text{and} \quad \|\nu_h\|^2_{L^2(\partial E)} \leq c_{inv,0} h_n^{-1} \|\nu_h\|^2_{L^2(E)}.
$$
Consistency and Galerkin orthogonality

Lemma (Consistency)

If the solution $u \in H^{1,0}_0(Q)$ of the variational problem (2) belongs to $H^{2,1}(Q)$, then it satisfies the consistency identity

$$a_h(u, v_h) = \ell_h(v_h) \quad \forall \ v_h \in V_{0h}.$$  (8)

Lemma (Galerkin orthogonality)

Let $u \in H^{1,0}_0(Q)$ be the solution the variational problem (2), that belongs to $H^{2,1}(Q)$, and let $u_h \in V_{0h}$ the solution of the space-time dG IgA scheme (9) then it holds the Galerkin orthogonality

$$a_h(u - u_h, v_h) = 0 \quad \forall \ v_h \in V_{0h}.$$  (9)
The exact solution $u$ of (2) belongs to $H_{0}^{1,0}(Q) \cap H^{2,1}(Q)$, and let $u_h$ be the solution of the space-time IgA scheme (9). Then it holds the a priori discretization error estimate

$$
\| u - u_h \|_{dG} \leq (1 + \frac{\mu b}{\mu c}) \inf_{v_h \in V_0^h} \| u - v_h \|_{dG, \ast},
$$

where $\| v \|_{dG, \ast} = \left( \| v \|_{dG}^2 + \sum_{n=1}^{N} (\theta_n h_n)^{-1} \| v \|_{L^2(Q_n)}^2 + \sum_{n=2}^{N} \| v_n^{-1} \|_{L^2(\Sigma_{n-1})}^2 \right)^{1/2}$ and

$$
\| v \|_{dG} = \left( \sum_{n=1}^{N} \left( \frac{1}{2} \| \nabla_x v \|_{L^2(Q_n)}^2 + \theta_n h_n \| \partial_t v \|_{L^2(Q_n)}^2 + \frac{1}{2} \| v \|_{L^2(\Sigma_n)}^2 + \frac{1}{2} \| v \|_{L^2(\Sigma_{N-1})}^2 \right) \right)^{1/2}.
$$

Proof: $\| u - u_h \|_{dG} \leq \| u - v_h \|_{dG} + \| v_h - u_h \|_{dG}$

$$
\mu_c \| v_h - u_h \|_{dG}^2 \leq a_h(v_h - u_h, v_h - u_h) = a_h(v_h - u, v_h - u_h)
$$

$$
\leq \mu_b \| u - v_h \|_{dG, \ast} \| v_h - u_h \|_{dG} \square
Theorem (Approximation Theorem)

Let \( p_n + 1 \geq \ell_n \geq 2 \) and \( p_n + 1 \geq m_n \geq 1 \) be integers, and let \( u \in L_2(Q) \) such that the restriction \( u^n := u|_{Q_n} \) belongs to \( H^{\ell_n,m_n}(Q_n) \) for \( n = 1, \ldots, N \). Then there exists a quasi-interpolant \( \Pi_h u \in V_{0h} \) such that

\[
\|u - \Pi_h u\|_{dG,*}^2 = \left( \sum_{n=1}^{N} \left( \| \nabla_x (u - \Pi^n_h u) \|^2_{L_2(Q_n)} + \theta_n h_n \| \partial_t (u - \Pi^n_h u) \|^2_{L_2(Q_n)} \right) \\
+ \frac{1}{2} \| (u - \Pi_h u)^{n-1} \|^2_{L_2(\Sigma_{n-1})} \right) + \frac{1}{2} \| u - \Pi^N_h u \|^2_{L_2(\Sigma_N)} \\
+ \sum_{n=1}^{N} \frac{1}{\theta_n h_n} \| u - \Pi^n_h u \|^2_{L_2(Q_n)} + \sum_{n=2}^{N} \| (u - \Pi^{n-1}_h u)_n^{-1} \|^2_{L_2(\Sigma_{n-1})} \\
\leq \sum_{n=1}^{N} \left( C_n \left( h_n^{2(\ell_n-1)} + \theta_n h_n^{2\ell_n-1} + h_n^{2m_n-1} + \theta_n h_n^{2m_n-1} \right) \right) \| u \|^2_{H^{\ell_n,m_n}(Q_n)} \\
\leq \sum_{n=1}^{N} \left( \tilde{C}_n \left( h_n^{2(\ell_n-1)} + h_n^{2(m_n-1/2)} \right) \right) \| u \|^2_{H^{\ell_n,m_n}(Q_n)}
\]
A priori Discretization Error Estimate

**Theorem (A priori Discretization Error Estimate)**

\[ \| u - u_h \|_{dG} \leq (1 + \frac{\mu_b}{\mu_c}) \sum_{n=1}^{N} \left( \tilde{C}_n \left( h_n^{2(\ell_n - 1)} + h_n^{2(m_n - \frac{1}{2})} \right) \right) \| u \|_{2H^{\ell_n,m_n}(Q_n)} \]

We remark that for the case of highly smooth solutions, i.e., \( p + 1 \leq \min(\ell_n, m_n) \), the above estimate takes the form

\[ \| u - u_h \|_{dG} \leq C \sum_{n=1}^{N} h_n^p \| u \|_{H^{p+1,p+1}(Q_n)} \leq C h^p \| u \|_{H^{p+1,p+1}(Q)} \]

where the last estimate holds if \( u \in H^{p+1,p+1}(Q) \) and \( h = \max \{ h_n \} \) is assumed.
One huge system of IgA equations

Once the basis is chosen, the IgA scheme (9) can be rewritten as a huge system of algebraic equations of the form

$$L_h u_h = f_h$$ (11)

for determining the vector $u_h = ((u_1,i)_{i \in I_1}, \ldots, (u_N,i)_{i \in I_N}) \in \mathbb{R}^{N_h}$ of the control points of the IgA solution

$$u_h(x, t) = \sum_{i \in I_n} u_{n,i} \varphi_{n,i}(x, t), \ (x, t) \in \overline{Q}_n, \ n = 1, \ldots, N,$$

solving the IgA scheme (9). The system matrix $L_h$ is the usual Galerkin (stiffness) matrix, and $f_h$ is the rhs (load) vector.
The Galerkin matrix $L_h$ can be rewritten in the block form

$$L_h = \begin{pmatrix} A_1 & -B_2 & & \cdots & \cdots & -B_N \\ -B_2 & A_2 & & & & \\ & -B_3 & A_3 & & & \\ & & \ddots & \ddots & & \\ & & & & -B_N & A_N \end{pmatrix},$$

with the matrices

$A_n := M_{n,x} \otimes K_{n,t} + K_{n,x} \otimes M_{n,t}$ for $n = 1, \ldots, N$,

$B_n := \tilde{M}_{n,x} \otimes N_{n,t}$ for $n = 2, \ldots, N$. 
Parallel Space-Time Multigrid Solvers

Solve:

\[ L_h u_h = f_h, \quad \text{with} \quad L_h = \begin{pmatrix} A_1 & -B_2 & \cdots & -B_N \\ -B_2 & A_2 & & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ -B_N & & & A_N \end{pmatrix} \]

**Smother:** inexact damped block Jacobi \((\omega_t = \frac{1}{2})\):

\[ u_h^{(n+1)} = u_h^{(n)} + \omega_t \, D_h^{-1} \left[ f_h - L_h u_h^{(n)} \right] \quad \text{for} \ n = 1, 2, \ldots \]

with \( D_h := \text{diag}\{A_k\}_{k=1}^{N} \)

- Parallel w.r.t. time
- Replace \( D_h^{-1} \) by multigrid w.r.t. space \( \rightarrow \) parallel in space
Parallel Solver Studies for $d = 3$ and $p = 1$

Figure: Computational spatial domain $\Omega$ decomposed into 4096 elements (left) and distributed over 32 processors (right). The IgA solution

$$u_h(x, t) \approx u(x, t) = \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \sin(\pi t)$$

is plotted at $t = 0.5$. 

---

Introduction  Time Multi-Patch Space-Time IgA  Solver  Numerical Results  Conclusions & Outlooks

www.oeaw.ac.at

U. Langer, Time-Multipatch dG Space-Time IgA of Parabolic Evolution Problems
Parallel Solver Studies for \( d = 3 \) and \( p = 1 \)

Convergence results in the dG-norm for the regular solution

\[
u(x, t) = \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \sin(\pi t)\]

as well as iteration numbers and solving times for the parallel space-time multigrid preconditioned GMRES method on Vulcan

<table>
<thead>
<tr>
<th>( N )</th>
<th>overall dof</th>
<th>( | u - u_h |_{dG} )</th>
<th>eoc</th>
<th>( c_x )</th>
<th>( c_t )</th>
<th>cores</th>
<th>iter</th>
<th>time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1125</td>
<td>3.56223E-01</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.03</td>
</tr>
<tr>
<td>2</td>
<td>13 122</td>
<td>1.77477E-01</td>
<td>1.01</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>13</td>
<td>1.87</td>
</tr>
<tr>
<td>4</td>
<td>176 868</td>
<td>8.86255E-02</td>
<td>1.00</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>15</td>
<td>21.47</td>
</tr>
<tr>
<td>8</td>
<td>2 587 464</td>
<td>4.42868E-02</td>
<td>1.00</td>
<td>4</td>
<td>8</td>
<td>32</td>
<td>15</td>
<td>100.48</td>
</tr>
<tr>
<td>16</td>
<td>39 546 000</td>
<td>2.21376E-02</td>
<td>1.00</td>
<td>32</td>
<td>16</td>
<td>512</td>
<td>17</td>
<td>94.32</td>
</tr>
<tr>
<td>32</td>
<td>618 246 432</td>
<td>1.10675E-02</td>
<td>1.00</td>
<td>256</td>
<td>32</td>
<td>8192</td>
<td>17</td>
<td>162.90</td>
</tr>
<tr>
<td>64</td>
<td>9 777 365 568</td>
<td>5.53340E-03</td>
<td>1.00</td>
<td>2048</td>
<td>64</td>
<td>131072</td>
<td>17</td>
<td>211.33</td>
</tr>
</tbody>
</table>
Parallel Solver Studies for $d = 3$ and $p = 1$

Convergence results in the dG-norm for the low regularity solution

$$u(x, t) = \cos(\beta x_1) \cos(\beta x_2) \cos(\beta x_3)(1 - t)^\alpha \in H^{s, \alpha + \frac{1}{2} - \varepsilon}(Q),$$

with $\alpha = 0.75$ and $\beta = 0.3$, for an arbitrary $s \geq 2$ and for an arbitrary small $\varepsilon > 0$, as well as iteration numbers and solving times for the parallel space-time multigrid preconditioned GMRES method on Vulcan BlueGene/Q at LLNL

<table>
<thead>
<tr>
<th>$N$</th>
<th>overall dof</th>
<th>$| u - u_h |_{dG}$</th>
<th>eoc</th>
<th>$c_x$</th>
<th>$c_t$</th>
<th>cores</th>
<th>iter</th>
<th>time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 125</td>
<td>1.58022E-02</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.03</td>
</tr>
<tr>
<td>2</td>
<td>13 122</td>
<td>8.88627E-03</td>
<td>0.83</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>13</td>
<td>2.00</td>
</tr>
<tr>
<td>4</td>
<td>176 868</td>
<td>5.41668E-03</td>
<td>0.71</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>15</td>
<td>21.48</td>
</tr>
<tr>
<td>8</td>
<td>2 587 464</td>
<td>3.33881E-03</td>
<td>0.70</td>
<td>4</td>
<td>8</td>
<td>32</td>
<td>15</td>
<td>100.57</td>
</tr>
<tr>
<td>16</td>
<td>39 546 000</td>
<td>2.05545E-03</td>
<td>0.70</td>
<td>32</td>
<td>16</td>
<td>512</td>
<td>17</td>
<td>94.43</td>
</tr>
<tr>
<td>32</td>
<td>618 246 432</td>
<td>1.25859E-03</td>
<td>0.71</td>
<td>256</td>
<td>32</td>
<td>8192</td>
<td>17</td>
<td>171.83</td>
</tr>
<tr>
<td>64</td>
<td>9 777 365 568</td>
<td>7.65921E-04</td>
<td>0.72</td>
<td>2048</td>
<td>64</td>
<td>131072</td>
<td>17</td>
<td>211.49</td>
</tr>
</tbody>
</table>

**Introduction**

Time Multi-Patch Space-Time IgA

Solver

Numerical Results

Conclusions & Outlooks

www.oeaw.ac.at

U. Langer, Time-Multipatch dG Space-Time IgA of Parabolic Evolution Problems
(sp cG, mp dG) IgA for $d = 1$ and $p = 2, 3, 4$
(sp cG, mp dG) IgA for $d = 1$ and $p = 2, 3, 4$

Error in the dG-norm and convergence rate for the exact solution

$$u(x, t) = \sin(\pi x) \sin\left(\frac{\pi}{2}(t + 1)\right)$$

and for B-Spline degrees 2, 3 and 4

<table>
<thead>
<tr>
<th>refinement</th>
<th>$p = 2$</th>
<th></th>
<th>$p = 3$</th>
<th></th>
<th>$p = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>error</td>
<td>eoc</td>
<td>error</td>
<td>eoc</td>
<td>error</td>
</tr>
<tr>
<td>0</td>
<td>2.85633E-02</td>
<td>-</td>
<td>3.85617E-02</td>
<td>-</td>
<td>9.18731E-03</td>
</tr>
<tr>
<td>1</td>
<td>5.68232E-02</td>
<td>2.33</td>
<td>7.15551E-03</td>
<td>2.43</td>
<td>7.87619E-04</td>
</tr>
<tr>
<td>2</td>
<td>1.34212E-02</td>
<td>2.08</td>
<td>8.11296E-04</td>
<td>3.14</td>
<td>4.62549E-05</td>
</tr>
<tr>
<td>3</td>
<td>3.30721E-03</td>
<td>2.02</td>
<td>9.84754E-05</td>
<td>3.04</td>
<td>2.90675E-06</td>
</tr>
<tr>
<td>4</td>
<td>8.23704E-04</td>
<td>2.01</td>
<td>1.22142E-05</td>
<td>3.01</td>
<td>1.84067E-07</td>
</tr>
<tr>
<td>5</td>
<td>2.05716E-04</td>
<td>2.00</td>
<td>1.52376E-06</td>
<td>3.00</td>
<td>1.16139E-08</td>
</tr>
<tr>
<td>6</td>
<td>5.14138E-05</td>
<td>2.00</td>
<td>1.90375E-07</td>
<td>3.00</td>
<td>7.29917E-10</td>
</tr>
<tr>
<td>7</td>
<td>1.28522E-05</td>
<td>2.00</td>
<td>2.37936E-08</td>
<td>3.00</td>
<td>4.85647E-11</td>
</tr>
</tbody>
</table>
(mp cG, mp dG) IgA for \( d = 1 \) and \( p = 3, 4 \)

<table>
<thead>
<tr>
<th>ref.</th>
<th>( p = 2 )</th>
<th>( p = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>error</td>
<td>eoc</td>
</tr>
<tr>
<td>0</td>
<td>0.0724164</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>0.0155074</td>
<td>2.22</td>
</tr>
<tr>
<td>2</td>
<td>0.00357715</td>
<td>2.12</td>
</tr>
<tr>
<td>3</td>
<td>0.000858059</td>
<td>2.06</td>
</tr>
<tr>
<td>4</td>
<td>0.000210051</td>
<td>2.03</td>
</tr>
<tr>
<td>5</td>
<td>5.19582e-05</td>
<td>2.02</td>
</tr>
<tr>
<td>6</td>
<td>1.29204e-05</td>
<td>2.01</td>
</tr>
</tbody>
</table>
Space mp cG & time mp dG IgA for $d = 2$ and $p = 2$
Numerical experiments:

Parallelization in time - Real Schur

- Degree $p_x, p_t = (3, 3)$, refinement $r_x, r_t = (2, 3)$
- $\theta = 0.01$ and $|t_i - t_{i-1}| = 0.1$;
- $(K_x + (\alpha_i + |\beta_i|)M_x)^{-1} \rightsquigarrow$ Direct solver
- Direct Solver: PARDISO
- Tolerance $\varepsilon = 10^{-8}$
- #slaps = #Processors.

<table>
<thead>
<tr>
<th>#dofs</th>
<th>#slaps</th>
<th>It.</th>
<th>Setup</th>
<th>Solving</th>
</tr>
</thead>
<tbody>
<tr>
<td>48510</td>
<td>2</td>
<td>7</td>
<td>0.088</td>
<td>2.8</td>
</tr>
<tr>
<td>97020</td>
<td>4</td>
<td>7</td>
<td>0.092</td>
<td>3.1</td>
</tr>
<tr>
<td>194040</td>
<td>8</td>
<td>7</td>
<td>0.093</td>
<td>3.2</td>
</tr>
<tr>
<td>388080</td>
<td>16</td>
<td>7</td>
<td>0.093</td>
<td>3.3</td>
</tr>
<tr>
<td>776160</td>
<td>32</td>
<td>7</td>
<td>0.094</td>
<td>3.4</td>
</tr>
<tr>
<td>1552320</td>
<td>64</td>
<td>7</td>
<td>0.096</td>
<td>3.5</td>
</tr>
<tr>
<td>3104640</td>
<td>128</td>
<td>7</td>
<td>0.100</td>
<td>3.5</td>
</tr>
<tr>
<td>6209280</td>
<td>256</td>
<td>7</td>
<td>0.104</td>
<td>3.9</td>
</tr>
</tbody>
</table>
Outline

1 Introduction
2 Time Multi-Patch Space-Time IgA
3 Solver
4 Numerical Results
5 Conclusions & Outlooks
Conclusions & Outlook

- Space-time IgA: space singlepatch cG and time-multipatch dG
- Space-time IgA: space multipatch cG and time-multipatch dG
- Space-time IgA: Fast generation
- Space-time IgA: Fast parallel solvers
- (Functional) a posteriori estimates and THB adaptivity
  ➞ talk by S. Matculevich
- space-time multipatch dG IgA + adaptivity + fast generation + efficient parallel solvers
Conclusions & Outlook

- Space-time IgA: space singlepatch cG and time-multipatch dG
- Space-time IgA: space multipatch cG and time-multipatch dG
- Space-time IgA: Fast generation
- Space-time IgA: Fast parallel solvers
- (Functional) a posteriori estimates and THB adaptivity
  → talk by S. Matculevich
- space-time multipatch dG IgA + adaptivity + fast generation
  + efficient parallel solvers

Introduction  Time Multi-Patch  Space-Time IgA  Solver  Numerical Results  Conclusions & Outlooks

www.oeaw.ac.at  U. Langer, Time-Multipatch dG Space-Time IgA of Parabolic Evolution Problems
THANK YOU VERY MUCH!