AN INVERSE FRESNEL PROBLEM
IN GLOBAL NAVIGATION SATELLITE
SYSTEM REFLECTOMETRY.

P. SAVI, Dip. di Elettronica e Telecomunicazione,
Politecnico di Torino;
A. MILANI, Dept. of Mathematics,
Botswana International University of Science and Technology.

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1 Global Navigation Satellite System Reflectometry.

- The main goal of GNSS-R is to derive information on the properties of a portion of soil (e.g., soil moisture, snow depth, wave configurations, ...), by remote sensing; that is, by analyzing signals emitted by GNSS satellites, and the reflected signals captured by an antenna.

- Example: Moisture increases the dielectric constant of the soil medium;

- Dielectric constant can be retrieved from measurements of reflectivity or transmissivity of surface;

- To recover dielectric constant, one has to solve an inverse problem concerning the Fresnel coefficients.
Passive bi-static radar

Detect the surface characteristics by using received GNSS signals

GPS transmitters, 24 satellites, 1.5 (1.2) GHz, PRN modulation, RHCP

Direct signal path

GPS receiver, zenith & nadir antennas

Reflected signal path

Ground surface

GPS BISTATIC RADAR GEOMETRY

Rough surface

Glistening zone

First Fresnel zone

Specular scattering

Advantages
APPLICATIONS:

Sea wind and wave height

Snow depth

Ice thickness and topography

Land soil moisture

Vegetation coverage
• Again: The GPS receiver measures a number of quantities, related to the perpendicular and parallel polarization of the signals.

• These quantities depend principally on the value of the incidence angle \( \theta \), and on the dielectric constant \( \varepsilon \).

• The dielectric constant is intrinsic of the soil, and provides information on its composition and properties.

• For “dispersive” soils, \( \varepsilon \in \mathbb{C} \); for non-dispersive ones, \( \varepsilon \in \mathbb{R}_{>0} \).

• In fact, assume \( \Re(\varepsilon) > 1 \); i.e., the soil is denser than the air (for which, by convention, \( \varepsilon = 1 \)).

• Also, here, neglect scattering due to the ‘roughness’ of the soil.
2 The Fresnel Coefficients.

- For a smooth, perfectly flat, non-magnetic surface, the Fresnel reflection coefficients are a combination of the horizontal and vertical polarization coefficients

\[
\Gamma_n = \frac{\cos \theta - \sqrt{\varepsilon - \sin^2 \theta}}{\cos \theta + \sqrt{\varepsilon - \sin^2 \theta}}, \tag{2.1}
\]

\[
\Gamma_p = \frac{\varepsilon \cos \theta - \sqrt{\varepsilon - \sin^2 \theta}}{\varepsilon \cos \theta + \sqrt{\varepsilon - \sin^2 \theta}}. \tag{2.2}
\]

- Often, only measurements of moduli \(|\Gamma_n|, |\Gamma_p|\), or of combinations such as \(\frac{1}{2} |\Gamma_n - \Gamma_p|\) or \(\frac{1}{2} |\Gamma_n + \Gamma_p|\) (circular polarization).

- In each case, the goal is to recover the value of \(\varepsilon\) from the available measurements on \(\Gamma_n\) and \(\Gamma_p\), via system (2.1)+(2.2).
3 Background.

- Maxwell’s equations (linear).

- Assume: Time-harmonic dependence, yielding “elliptic” equations for the fields; e.g.,

\[ \Delta E + k^2 E = 0. \] (3.1)

- In fact, a family of such equations, parametrized by \( t \):

\[ E(t, x) = e^{j\omega t} E_0(x) \quad (j^2 := -1). \] (3.2)

- Assume: Plane waves:

\[ E_0(x) = E_0 e^{-j k u \cdot x}, \quad E_0 \in \mathbb{R}^3, \quad |u| = 1. \] (3.3)

- Note: \( E \) and \( H \) orthogonal (\( E \cdot H = 0 \)).
(Background, cont.)

- Plane wave incident onto a plane boundary, assumed to be the \((x, y)\)-plane.

- Parallel (or horizontal) polarization: \(E\) orthogonal to the \((x, z)\)-plane, \(H\) parallel to the \((x, z)\)-plane.

- Perpendicular (or vertical) polarization: viceversa, i.e. \(H\) orthogonal to the \((x, z)\)-plane, \(E\) parallel to the \((x, z)\)-plane.

- Imposing the continuity of the tangential components of the fields across the boundary \(z = 0\), and then implementing SNELL’s laws, deduce the Fresnel system \((2.1) + (2.2)\).
4 Goal.

- Assume have measurements of $\gamma_n := |\Gamma_n|$ and $\gamma_p := |\Gamma_p|$, with $0 < \gamma_p \leq \gamma_n < 1$ (see figure 4).
• Continuous curves of figure are of $\gamma_n$ and $\gamma_p$; such curves are all of same shape, for each $\varepsilon \in IR_{>1}$.

• Given incidence angle $\theta \in ]0, \frac{\pi}{2}[$, and such measured values $\gamma_n$, $\gamma_p \in ]0, 1[$, find $\varepsilon \in \mathbb{C}$, with $\Re(\varepsilon) > 1$, solution of the system

\[
\left| \frac{\cos \theta - \sqrt{\varepsilon - \sin^2 \theta}}{\cos \theta + \sqrt{\varepsilon - \sin^2 \theta}} \right| = \gamma_n, \quad (4.1)
\]
\[
\left| \frac{\varepsilon \cos \theta - \sqrt{\varepsilon - \sin^2 \theta}}{\varepsilon \cos \theta + \sqrt{\varepsilon - \sin^2 \theta}} \right| = \gamma_p. \quad (4.2)
\]
5 Immediate Remarks.

• In the literature, system (4.1)+(4.2) seems to be solved numerically, even though exact, algebraic solution is (almost) elementary.

• If soil is known to be essentially non-dispersive, i.e. $|\Im(\varepsilon)| \ll 1$, then one solves (4.1)+(4.2) for $\varepsilon \in \mathbb{R}_{>1}$.

• However, in this case system is over-determined, and can be solved only under suitable compatibility conditions.

• More specifically, looking for real solutions $\varepsilon \in \mathbb{R}_{>1}$:

  − (4.1) yields $\varepsilon_n = \varphi(\gamma_n, \theta)$; (4.2) yields $\varepsilon_p = \psi(\gamma_p, \theta)$;

  − So, need to make sure that:

    1) $\varepsilon_n = \varepsilon_p =: \varepsilon$;

    2) $\varepsilon$ is independent of $\theta$.

• In all cases, find “optimal” strategy to find $\varepsilon$. 
6 Real Solutions of (4.1).

Theorem 6.1 1) For all \( \theta \in [0, \frac{\pi}{2}] \) and all corresponding \( \gamma_n \in ]0, 1[ \) (as measured), there exists a unique solution \( \varepsilon = \varepsilon_n > 1 \) of equation (4.1), given by

\[
\varepsilon_n = 1 + \frac{4 \gamma_n \cos^2 \theta}{(1 - \gamma_n)^2}.
\] (6.1)

2) This solution is independent of \( \theta \) if and only if the measured values of \( \gamma_n \) satisfy the following condition: There is \( \alpha > 1 \) such that

\[
\gamma_n(\theta) = \frac{\sqrt{\alpha - \sin^2 \theta} - \cos \theta}{\sqrt{\alpha - \sin^2 \theta} + \cos \theta}
\] (6.2)

for all \( \theta \in [0, \frac{\pi}{2}] \). In this case, \( \varepsilon_n(\theta) \equiv \alpha \).

- Solution (6.1) is immediate.

- Condition (6.2) certainly not surprising, as it essentially is the definition of \( \gamma_n \) itself.
7 The Brewster Angle.

- When $\varepsilon \in \mathbb{R}_{>0}$, the numerator of (2.2) can change sign: at angle
  \[ \theta_B = \theta_B(\varepsilon) = \arctan(\sqrt{\varepsilon}). \]  
  (7.1)

- $\theta_B$ called Brewster angle.

- $\varepsilon > 1 \iff \theta_B > \frac{\pi}{4}$.

- It is in fact often observed in measurements that there indeed is an angle $\tilde{\theta}$ such that, correspondingly, $\gamma_p \approx 0$ (recall figure 4). In this case, $\tilde{\theta}$ is taken as an approximation of $\theta_B$. 

8 Real Solutions of \((4.2)\).

Solution of \((4.2)\) in three steps:

1. Algebraic solution [slightly less immediate than for \((4.1)\)]. In contrast with \((4.1)\), find three different solutions to \((4.2)\):
   
   1.1 A solution \(\varepsilon_0^p\) defined for all \(\theta \in \left]0, \frac{\pi}{2}\right[\), and
   
   1.2 Two other solutions, \(\varepsilon_1^p\) and \(\varepsilon_2^p\), defined in a smaller interval \([\theta_0, \frac{\pi}{2}[^{\subset} \left]\frac{\pi}{4}, \frac{\pi}{2}\right[ \]

- In the common interval \([\theta_0, \frac{\pi}{2}[\), \(\varepsilon_0^p \geq \varepsilon_1^p \geq \varepsilon_2^p \geq 1\).

2. Condition for the existence of a solution \(\varepsilon_p\) of \((4.2)\) independent of \(\theta\). Again not surprisingly, this condition is essentially the definition of \(\gamma_p\) itself.

3. Obtain this constant solution \(\varepsilon_p\) by suitably patching together the three above different solutions of \((4.2)\).
(Real Solutions of (4.2), cont.)

**Theorem 8.1** Assume there is \( \theta_B \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right] \) such that, correspondingly, \( \gamma_p = 0 \). For \( \gamma_p \in [0, 1] \), set

\[
\lambda_p := \frac{1 + \gamma_p}{1 - \gamma_p} .
\] (8.1)

- \( \lambda_p \) depends on \( \theta \), via \( \gamma_p \), and \( \lambda_p \geq 1 \). Then:

1) For all \( \theta \in [0, \frac{\pi}{2}] \) and all corresponding \( \gamma_p \in [0, 1] \) (as measured), there exists a solution \( \varepsilon_p^0 > 1 \) of problem (4.2), given by

\[
\varepsilon_p^0 = \frac{\lambda_p^2}{2 \cos^2 \theta} \left( 1 + \sqrt{1 - \frac{\sin^2(2\theta)}{\lambda_p^2}} \right),
\] (8.2)

with

\[
\varepsilon_p^0 \geq \tan^2 \theta \quad \forall \theta \in [0, \frac{\pi}{2}] .
\] (8.3)

Thus, \( \varepsilon_p^0 \) cannot be independent of \( \theta \) in all of \( [0, \frac{\pi}{2}] \).
(Theorem 8.1, cont.)

2) Assume that

\[ \lambda_p(\theta) \sin(2\theta) \leq 1 \]  \hspace{1cm} (8.4)

(see below). Then, for all \( \theta \in [\theta_B, \frac{\pi}{2}] \) and all corresponding \( \gamma_p \in [0, 1] \), there is a solution \( \varepsilon_1^p > 1 \) of problem (4.2), given by

\[
\varepsilon_p^1 = \frac{1}{2 \lambda_p^2 \cos^2 \theta} \left( 1 + \sqrt{1 - \lambda_p^2 \sin^2(2\theta)} \right).
\]  \hspace{1cm} (8.5)

This solution satisfies the conditions

\[
2 \sin^2 \theta \leq \varepsilon_p^1 \leq \tan^2 \theta \quad \forall \theta \in [\theta_B, \frac{\pi}{2}].
\]  \hspace{1cm} (8.6)

Thus, again, \( \varepsilon_p^1 \) cannot be independent of \( \theta \) in all of \( [\theta_B, \frac{\pi}{2}] \) (unless \( \varepsilon_p^1 \geq 2 \) for all \( \theta \in [\theta_B, \frac{\pi}{2}] \), in which case it may be independent of \( \theta \)).
In addition, problem (4.2) also has, in $\theta_B, \frac{\pi}{2}$, a solution $\varepsilon_p^2 > 1$, given by

$$\varepsilon_p^2 = \frac{1}{2 \lambda_p^2 \cos^2 \theta} \left( 1 - \sqrt{1 - \lambda_p^2 \sin^2(2\theta)} \right). \quad (8.7)$$

This solution satisfies the condition

$$\varepsilon_p^2 \leq 2 \sin^2 \theta \quad \forall \theta \in [\theta_B, \frac{\pi}{2}]. \quad (8.8)$$

Condition (8.4) is necessary for the existence of a solution $\varepsilon > 1$ of (4.2) (not necessarily independent of $\theta$) in $[0, \frac{\pi}{2}]$. 

\[\square\]
9 Constant Real Solutions to (4.2).

- If $\tan^2 \theta_B \geq 2$, define

  \[
  \varepsilon_p = \begin{cases} 
  \varepsilon_0 & \text{if } 0 \leq \theta \leq \theta_B, \\
  \varepsilon_1 & \text{if } \theta_B \leq \theta < \frac{\pi}{2}.
  \end{cases} \tag{9.1}
  \]

- If $1 < \tan^2 \theta_B < 2$, define $\theta_1$ by the identity

  \[
  2 \sin^2 \theta_1 = \tan^2 \theta_B \tag{9.2}
  \]

  (See figure 9), and

  \[
  \varepsilon_p = \begin{cases} 
  \varepsilon_0 & \text{if } 0 \leq \theta \leq \theta_B, \\
  \varepsilon_1 & \text{if } \theta_B \leq \theta \leq \theta_1, \\
  \varepsilon_2 & \text{if } \theta_1 < \theta < \frac{\pi}{2}.
  \end{cases} \tag{9.3}
  \]
Theorem 9.1 Define $\varepsilon_p : [0, \frac{\pi}{2}] \rightarrow [1, +\infty]$ by (9.1) if $\tan^2 \theta_B \geq 2$, or by (9.3) if $1 < \tan^2 \theta_B < 2$. Then:

1) $\varepsilon_p$ is a continuous solution of (4.2);
2) $\varepsilon_p$ is independent of $\theta$ if and only if the measured values of $\gamma_p$ satisfy the following condition: There is $\beta > 1$ such that

$$
\gamma_p(\theta) = \left| \frac{\beta \cos \theta - \sqrt{\beta - \sin^2 \theta}}{\beta \cos \theta + \sqrt{\beta - \sin^2 \theta}} \right| \tag{9.4}
$$

for all $\theta \in [0, \frac{\pi}{2}]$ (compare to (2.2)). In this case, $\varepsilon_p(\theta) \equiv \beta = \tan^2 \theta_B$. \(\diamondsuit\)
10 Common Real Solutions to (4.1) and (4.2).

Theorem 10.1 Let $\varepsilon_n$ and $\varepsilon_p$ be as in theorems 6.1 and 8.1. Then:

1) $\varepsilon_n(\theta) = \varepsilon_p(\theta)$ on all of $[0, \frac{\pi}{2}]$ if and only if the compatibility condition

$$\lambda_n^2 \cos^2 \theta + \sin^2 \theta = \begin{cases} 
\lambda_n \lambda_p & \text{if } 0 \leq \theta \leq \theta_B, \\
\frac{\lambda_n}{\lambda_p} & \text{if } \theta_B \leq \theta < \frac{\pi}{2},
\end{cases}$$

(10.1)

holds in $[0, \frac{\pi}{2}]$, together with the additional conditions $\gamma_p \leq \gamma_n^2$ if $\theta_B \leq \theta \leq \theta_1$, or $\gamma_n^2 \leq \gamma_p$ if $\theta_1 \leq \theta < \frac{\pi}{2}$. In this case, the common solution $\varepsilon_n = \varepsilon_p =: \varepsilon_c$ is given by

$$\varepsilon_c = \begin{cases} 
\lambda_n \lambda_p & \text{if } 0 \leq \theta \leq \theta_B, \\
\frac{\lambda_n}{\lambda_p} & \text{if } \theta_B \leq \theta < \frac{\pi}{2},
\end{cases}$$

(10.2)

2) This common solution $\varepsilon_c$ is independent of $\theta$ if and only if $\gamma_n$ and $\gamma_p$ are of the form (6.2) and (9.4), with $\alpha = \beta$; in this case, $\varepsilon_c = \tan^2 \theta_B$. $\diamond$
11 Complex Solutions of \((4.1) + (4.2)\).

[Work in progress!] \(\implies\) Preliminary, partial results.

- Given \(\gamma_n, \gamma_p \in ]0, 1[\) as measured, and \(\theta \in [0, \frac{\pi}{2}[,\) define

\[
\mu_n := \frac{1 + \gamma_n^2}{1 - \gamma_n^2}, \quad \mu_p := \frac{1 + \gamma_p^2}{1 - \gamma_p^2}, \quad (11.1)
\]

\[
C = C(\theta) := (\mu_n^2 - 1) \cos^2 \theta - \mu_n \mu_p + 1. \quad (11.2)
\]

- Note: \(1 < \mu_p \leq \mu_n, \quad \mu_n < \lambda_n < 2 \mu_n, \quad \mu_p < \lambda_p < 2 \mu_p.\)

- Note: \(C(0) > 0 > C(\frac{\pi}{2});\) so, there is \(\theta_0 \in ]0, \frac{\pi}{2}[^{\text{ such that }}\ C'(\theta_0) = 0.\)

- Note:

\[
\theta_0 < [\ =, >] \frac{\pi}{4} \iff \gamma_p > [\ =, <] \gamma_n^2. \quad (11.3)
\]
(Complex Solutions to \((4.1)+(4.2), \text{cont.}\))

Strategy:

- Set \(\varepsilon = x + jy\), \(\sqrt{\varepsilon - \sin^2\theta} = u + jv\).

- Replacing into \((4.1)+(4.2)\) obtain

\[
v^2 = -(u^2 - 2(\mu_n \cos\theta)u + \cos^2\theta) =: -P(u), \quad (11.4)
\]
\[
x = u^2 - v^2 + \sin^2\theta, \quad (11.5)
\]
\[
y = 2u|v|. \quad (11.6)
\]

- Note: \(\varepsilon\) is a solution \(\iff\) \(\bar{\varepsilon}\) is a solution.

- Thus, determine \(u\). Need \(P(u) < 0; \iff\) compatibility conditions

\[
\frac{1}{\lambda_n} \cos\theta < u < \lambda_n \cos\theta \quad (11.7)
\]

\(\iff u > 0; \text{ in fact, } [\mu_n \cos\theta < u < \lambda_n \cos\theta] \text{ to have } \Re(\varepsilon) > 1).\)
Theorem 11.1 1) If $\gamma_n^2 = \gamma_p$, let $\theta_\ast := \arccos \sqrt{\frac{\lambda_n}{2\mu_n}}$. Then, problem (4.1)+(4.2) has a solution $\varepsilon \in \mathbb{C}$ for all $\theta \in \left[ \theta_\ast, \frac{\pi}{2} - \theta_\ast \right]$, given by

$$u = \frac{1}{2 \mu_n \cos \theta}.$$  \hfill (11.8)

In addition, if $\theta = \frac{\pi}{4}$, problem (4.1)+(4.2) has also infinitely many complex conjugate solutions

$$\varepsilon = \varepsilon(r) = (r^2 - \lambda_p r + 1) \pm j r \sqrt{2 \lambda_p r - r^2 - 1},$$  \hfill (11.9)

parametrized by $r \in \left[ \frac{1}{\lambda_n}, \lambda_n \right]$. In particular, the solution (11.8) corresponds to the value $r = \frac{1}{\lambda_n}$ in (11.9).
(Theorem 11.1, cont.)

2) If $\gamma_n^2 \neq \gamma_p$, there exist angles $\theta_i \in ]0, \frac{\pi}{2} [$, $1 \leq i \leq 4$, with

$$0 < \theta_1 < \theta_2 < \frac{\pi}{4} < \theta_3 < \theta_4 < \frac{\pi}{2} , \quad (11.10)$$

such that problem (4.1)+(4.2) has a solution $\varepsilon \in \mathbb{C}$ if and only if $\theta \in ]\theta_1, \theta_2[$ or $\theta \in ]\theta_3, \theta_4[$. This solution is given by

$$u = \left( \frac{(\mu_n - \mu_p) \cos(2\theta)}{2C \cos \theta} \right), \quad (11.11)$$

where $C$ is as in (11.2).

• Note: $v = 0 \implies y = 0 \implies \varepsilon = \varepsilon_n$ of (6.1).

• Note: $\theta = 0 \implies u = \frac{1}{2\mu_n} \implies P(u) = \frac{1}{4\mu_n^2} > 0 \implies v = \Im(\varepsilon) \notin \mathbb{R}$: no.
12 How good are the above results?


- Soil is sand, for which $\varepsilon = 3 + 0.05 j$ is known.

- In their paper, $\Im(\varepsilon)$ is neglected ( $|\Im(\varepsilon)| \ll \Re(\varepsilon)$ ).

- Brewster angle would be $\theta = \arctan(\sqrt{3}) = 60^0$; in fact, in paper, $\tilde{\theta}_B \approx 60^0$. 
(How good are the above results?, cont.)

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(How good are the above results?, cont.)

\[ \text{eps from Fig. 5;} \\
\text{"True" epsilon} = 3 + 0.05 j; \\
\theta = \text{angle in degrees}; \\
\gamma_n \text{ and } \gamma_p \text{ from table (in dB);} \\
\text{complex epsilon from formula (11.11) for } u; \\
\% \text{ error} = \sqrt{\text{(Re(eps)-3)}^2 + \text{Im(eps)} - 0.05}^2 / 3. \]

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<td>2.79198</td>
<td>0.12589</td>
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A few open questions.

- Find additional conditions for $\varepsilon \in \mathbb{C}$ to be independent of $\theta$.

- By theorem 9.1, optimal strategy to solve (4.1)+(4.2) for $\varepsilon \in \mathbb{R}_{>1}$ is to “wait” till position $\theta = \theta_B$: $\varepsilon = \tan^2 \theta_B$; but:
  1. Can one afford to wait that long, in actual measurements?
  2. $\theta_B$ is defined only for non-dispersive soils; otherwise?
  3. Even if measure $\gamma_p \approx 0$ at some $\theta_p$, can conclude $\Re(\varepsilon) \approx \tan^2 \theta_p$, $\Im(\varepsilon) \approx 0$? With what degree of confidence? $\Rightarrow$ Control of error!

- Solve analogous problems when $\gamma_n$ and $\gamma_p$ are replaced by $\frac{1}{2} |\Gamma_n - \Gamma_p|$ and ($\vdash$ or $\exists$) $\frac{1}{2} |\Gamma_n + \Gamma_p|$.

- Incorporate roughness of soil in model (typically by Root Mean Square of microscopic peaks and valleys). . . . . . . etc.
14 What we REALLY would like to do.

- Revisit the model when at least one of the media (e.g., the soil) is non-linear. That is, for instance, \( D = \varepsilon(E) \) (\( \varepsilon \) monotone ?).

- Emitted signal in air may be time-harmonic; but non-linearity of soil destroys the time-harmonicity of the reflected signal.

- **DO THE FRESNEL FORMULAS STILL HOLD?**

- At least as a 0-order approximation (of what, exactly)?

- What would replace the Fresnel formulas?

- How to give a more realistic model, which should include scattering from the terrain?

- ......
15 Conclusions.

- Equations (4.1) and (4.2) are explicitly solvable.

- (4.2) only solvable in specific ranges of $\theta$.

- For non-dispersive soils, best strategy is $\varepsilon = \tan^2 \theta_B$.

- Likewise, if it is known that soil is almost non-dispersive, and can observe an angle $\theta_p$ such that $\gamma_p \approx 0$, then $\varepsilon \approx \tan^2 \theta_p + 0 \, j$.

- Values of $\varepsilon$ from De Roo & Ulaby’s measurements via the explicit formulas match “true” values with error not exceeding 1%.

τὸ τέλος

εὐχαριστούμεν πολί