Initial source recovery of the wave equation given internal boundary measurements

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Geometry + Simulation
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We study the inverse problem

Assuming that the given data \( z_d \in L^2(\Gamma_T) \), where \( \Gamma_T = \Gamma \times [0, T] \), we aim at identifying the initial condition \( u \in H^1_0(\Omega) \) which minimizes

\[
\min_{u \in H^1_0(\Omega)} f(u) = \frac{1}{2} \| y - z_d \|_{L^2(\Gamma_T)}^2 + \frac{\alpha}{2} \| u \|_{H^1_0(\Omega)}^2,
\]

where \( y \) solves the hyperbolic equation

\[
\begin{align*}
y'' - \Delta y &= 0, & (x, t) &\in \Omega \times (0, T) \\
y &= 0, & (x, t) &\in \partial \Omega \times [0, T] \\
y(0) &= u, \ y'(0) = 0, & x &\in \Omega.
\end{align*}
\]
Outline

- Solution theory
- Saddle point problem
- Robust preconditioning
- Stable discretization
- Numerical result
- Discussion
The wave equation: Solution theory

For any \((f, y_0, y_1) \in L^2(\Omega \times (0, T)) \times H^1_0(\Omega) \times L^2(\Omega)\) there exists a unique

\[
y \in \mathcal{W} = \{ y \in L^2(0, T; H^1_0(\Omega)) \mid y' \in L^2(0, T; L^2(\Omega)), y'' \in L^2(0, T; H^{-1}(\Omega)) \}
\]

which satisfies (a variational form of) the wave equation with \(y(0) = y_0, y'(0) = y_1\).
The wave equation: Solution theory

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which satisfies (a variational form of) the wave equation with \(y(0) = y_0, y'(0) = y_1\).

**Additional regularity:** \((y, y') \in C([0, T]; H^1_0(\Omega)) \times C([0, T]; L^2(\Omega)), \)

\[
\|y\|_{C([0, T]; H^1_0(\Omega))}^2 + \|y'\|_{C([0, T]; L^2(\Omega))}^2 \leq \text{const} \left( \|f\|_{L^2(\Omega \times (0, T))}^2 + \|y_0\|_{H^1_0(\Omega)}^2 + \|y_1\|_{L^2(\Omega)}^2 \right)
\]

\[
= \text{const} \left( \|Ly\|_{L^2(0, T; L^2(\Omega))}^2 + \|y(0)\|_{H^1_0(\Omega)}^2 + \|y'(0)\|_{L^2(\Omega)}^2 \right)
\]

where

\[ Ly = y'' - \Delta y. \]
The wave equation: Solution theory

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Additional regularity: \((y, y') \in C([0, T]; H^1_0(\Omega)) \times C([0, T]; L^2(\Omega))\),

\[
\|y\|_{C([0, T]; H^1_0(\Omega))}^2 + \|y'\|_{C([0, T]; L^2(\Omega))}^2 + \|L y\|_{L^2(0, T; L^2(\Omega))}^2 \\
\leq \text{const} \left( \|f\|_{L^2(\Omega \times (0, T))}^2 + \|y_0\|_{H^1_0(\Omega)}^2 + \|y_1\|_{L^2(\Omega)}^2 \right) \\
= \text{const} \left( \|L y\|_{L^2(0, T; L^2(\Omega))}^2 + \|y(0)\|_{H^1_0(\Omega)}^2 + \|y'(0)\|_{L^2(\Omega)}^2 \right)
\]

where

\[
L y = y'' - \Delta y.
\]
Definition

Let $Y \subset W$ denote the space spanned by $y$ as $(f, y_0, y_1)$ ranges over $L^2(\Omega \times (0, T)) \times H^1_0(\Omega) \times L^2(\Omega)$,

$$Y = \{ y \in W \mid (y, y') \in C([0, T]; H^1_0(\Omega)) \times C([0, T]; L^2(\Omega)), Ly \in L^2(0, T; L^2(\Omega)) \}.$$  

Endowed with the norm

$$\|y\|_Y^2 = \|Ly\|_{L^2(0, T; L^2(\Omega))}^2 + \|y(0)\|_{H^1_0(\Omega)}^2 + \|y'(0)\|_{L^2(\Omega)}^2,$$

this is a Hilbert space.

Lemma

There exists a positive constant $C = C(\Omega, \Omega_s, T)$ such that $\|y\|_{L^2(\Gamma_T)} \leq C \|y\|_Y$ for all $y \in Y$.

Corollary

For every $\alpha > 0$, the norm

$$\|y\|_{Y_\alpha}^2 = \|y\|_{L^2(\Gamma_T)}^2 + \|Ly\|_{L^2(0, T; L^2(\Omega))}^2 + \alpha \|y(0)\|_{H^1_0(\Omega)}^2 + \|y'(0)\|_{L^2(\Omega)}^2$$

is equivalent to $\| \cdot \|_Y$ on $Y$.  

The saddle point problem

Set \( \Lambda = L^2(0, T; L^2(\Omega)) \times L^2(\Omega) \). The Lagrangian reads

\[
\mathcal{L} : Y \times \Lambda \to \mathbb{R}, \quad \mathcal{L}(y, \lambda) = \frac{1}{2} a(y, y) + b(y, \lambda) - l(y),
\]

where the (continuous) bilinear forms \( a : Y \times Y \to \mathbb{R} \) and \( b : Y \times \Lambda \to \mathbb{R} \) are given by

\[
a(y, \bar{y}) = (y, \bar{y})_{L^2(\Gamma_T)} + \alpha (y(0), \bar{y}(0))_{H^1_0(\Omega)},
\]

\[
b(y, \lambda) = (Ly, w)_{L^2(0, T; L^2(\Omega))} + (y'(0), \phi)_{L^2(\Omega)}, \quad \lambda = (w, \phi),
\]

together with the linear form \( l : Y \to \mathbb{R}, \ l(y) = (y, z_d)_{L^2(\Gamma_T)} \).

Find \((y, \lambda) \in Y \times \Lambda\) such that

\[
\begin{cases}
    a(y, \bar{y}) + b(\bar{y}, \lambda) = l(\bar{y}) & \text{for all } \bar{y} \in Y, \\
    b(y, \bar{\lambda}) = 0 & \text{for all } \bar{\lambda} \in \Lambda.
\end{cases}
\]
Well-posedness of the saddle point problem

Existence and uniqueness under the Brezzi conditions:

1. The bilinear forms $a : Y_\alpha \times Y_\alpha \to \mathbb{R}$ and $b : Y_\alpha \times \Lambda \to \mathbb{R}$ are bounded,

   $$|a(y, \overline{y})| \leq C_a \|y\|_{Y_\alpha} \|\overline{y}\|_{Y_\alpha}, \quad |b(y, \lambda)| \leq C_b \|y\|_{Y_\alpha} \|\lambda\|_{\Lambda}$$

   for all $y, \overline{y} \in Y_\alpha, \lambda \in \Lambda$.

2. The bilinear form $a$ is coercive on $\mathcal{N}(b) = \{y \in Y_\alpha \mid b(y, \lambda) = 0 \text{ for all } \lambda \in \Lambda\}$. There exists a constant $k_0 > 0$ such that

   $$a(y, y) \geq k_0 \|y\|_{Y_\alpha}^2$$

   for all $y \in \mathcal{N}(b)$.

3. The bilinear form $b$ satisfies the inf-sup condition. There exists a constant $\beta_0 > 0$ such that

   $$\sup_{0 \neq y \in Y_\alpha} \frac{b(y, \lambda)}{\|y\|_{Y_\alpha}} \geq \beta_0 \|\lambda\|_{\Lambda}$$

   for all $\lambda \in \Lambda$.

**Theorem**

The bilinear forms $a$ and $b$ satisfy the Brezzi conditions. Moreover, $C_a, C_b, k_0, \beta_0$ can be chosen independent of $\alpha$ for

$$a : Y_\alpha \times Y_\alpha \to \mathbb{R} \quad \text{and} \quad b : Y_\alpha \times \Lambda \to \mathbb{R}.$$
The saddle point problem in operator notation

Let \( A : \mathcal{Y}_\alpha \to \mathcal{Y}'_\alpha \) and \( B : \mathcal{Y}_\alpha \to \Lambda' \) be given by

\[
\langle Ay, \overline{y} \rangle_{\mathcal{Y}'_\alpha \times \mathcal{Y}_\alpha} = a(y, \overline{y}) \quad \text{and} \quad \langle By, \lambda \rangle_{\Lambda' \times \Lambda} = b(y, \lambda)
\]

for all \( y, \overline{y} \in \mathcal{Y}_\alpha, \lambda \in \Lambda \).

Using this operator notation, the saddle point problem can be written as

\[
\mathcal{A}_\alpha : \mathcal{Y}_\alpha \times \Lambda \to \mathcal{Y}'_\alpha \times \Lambda', \quad \mathcal{A}_\alpha \left( \begin{array}{c} y \\ \lambda \end{array} \right) = \left( \begin{array}{cc} A & B' \\ B & 0 \end{array} \right) \left( \begin{array}{c} y \\ \lambda \end{array} \right) = \left( \begin{array}{c} I \\ 0 \end{array} \right).
\]

**Corollary**

*For every \( \alpha > 0 \) the linear self-adjoint operator \( \mathcal{A}_\alpha \) is bounded and continuously invertible. Moreover, there exist positive constants \( \overline{c}, \underline{c} \), both independent of \( \alpha \), such that

\[
\| \mathcal{A}_\alpha \|_{\mathcal{L}(\mathcal{Y}_\alpha \times \Lambda, (\mathcal{Y}_\alpha \times \Lambda)')} \leq \overline{c} \quad \text{and} \quad \| \mathcal{A}_\alpha^{-1} \|_{\mathcal{L}((\mathcal{Y}_\alpha \times \Lambda)', \mathcal{Y}_\alpha \times \Lambda)} \leq \underline{c}^{-1}.
\]
Robust preconditioning

For the operator $\mathcal{A}_\alpha$ we define the \textit{preconditioner}

$$
\mathcal{B}_\alpha : Y_\alpha \times \Lambda \rightarrow Y'_\alpha \times \Lambda', \quad \mathcal{B}_\alpha = \begin{pmatrix} P_{Y_\alpha} & 0 \\ 0 & P_\Lambda \end{pmatrix}.
$$

Thus, $\mathcal{B}_\alpha^{-1} \mathcal{A}_\alpha : Y_\alpha \times \Lambda \rightarrow Y_\alpha \times \Lambda$ is an isomorphism and self-adjoint with respect to the inner product on $Y_\alpha \times \Lambda$. Moreover,

$$
\| \mathcal{B}_\alpha^{-1} \mathcal{A}_\alpha \|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)} = \| \mathcal{A}_\alpha \|_{\mathcal{L}(Y_\alpha \times \Lambda, (Y_\alpha \times \Lambda)')} \quad \text{and} \\
\| (\mathcal{B}_\alpha^{-1} \mathcal{A}_\alpha)^{-1} \|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)} = \| \mathcal{A}_\alpha^{-1} \|_{\mathcal{L}((Y_\alpha \times \Lambda)', Y_\alpha \times \Lambda)}.
$$

\textbf{Corollary}

\textit{The condition number of $\mathcal{B}_\alpha^{-1} \mathcal{A}_\alpha$ is uniformly bounded with respect to $\alpha$. In other words}

$$
\kappa(\mathcal{B}_\alpha^{-1} \mathcal{A}_\alpha) := \| \mathcal{B}_\alpha^{-1} \mathcal{A}_\alpha \|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)} \| (\mathcal{B}_\alpha^{-1} \mathcal{A}_\alpha)^{-1} \|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)} \leq \frac{1}{c}.
$$
Discretized problem

Given conforming discretization spaces $Y_h \subset Y$ and $\Lambda_h \subset \Lambda$:

$$
\begin{pmatrix}
A_h & B_h^T \\
B_h & 0 \\
\end{pmatrix}
\begin{pmatrix}
\hat{x}_h \\
\lambda_h \\
\end{pmatrix} =
\begin{pmatrix}
\hat{z}_{d,h} \\
0 \\
\end{pmatrix}
$$

\[A_h : \quad \langle A \, y_h, \, \overline{y}_h \rangle = (y_h, \, \overline{y}_h)_{L^2(\Gamma_T)} + \alpha \, (y_h(0), \, \overline{y}_h(0))_{H^1_0(\Omega)}\]

\[B_h : \quad \langle B y_h, \, \overline{\lambda}_h \rangle = (Ly_h, \, \overline{w}_h)_{L^2(0, T; L^2(\Omega))} + (y_h'(0), \, \overline{\phi}_h)_{L^2(\Omega)}\]

with $\lambda_h = (w_h, \phi_h)$

\[
(Ly_h, \, \overline{w}_h)_{L^2(0, T; L^2(\Omega))} = \int_0^T \int_{\Omega} y_h'' \, \overline{w}_h \, dx \, dt - \int_0^T \int_{\Omega} \Delta y_h \, \overline{w}_h \, dx \, dt
\]
Stable discretization

We need to satisfy the discrete Brezzi conditions:

- **Coercivity**: We need to choose $Y_h$ s.t.,

$$\langle Ay_h, y_h \rangle \geq c_1 \| y_h \|^2_{Y_{\alpha,h}} \quad \text{for all} \quad y_h \in \ker B.$$

- **Inf-sup**: We need to choose $Y_h$ and $\Lambda_h$ s.t.

$$\inf_{0 \neq \lambda_h \in \Lambda_h} \sup_{0 \neq y_h \in Y_h} \frac{\langle By_h, \lambda_h \rangle}{\| y_h \|_{Y_{\alpha,h}} \| \lambda_h \|_{\Lambda_h}} \geq \delta_0 > 0.$$
Stable discretization: Coercivity

We circumvent the coercivity by replacing $A$ with

$$\langle A_\rho y_h, \bar{y}_h \rangle = \langle Ay_h, \bar{y}_h \rangle + \rho (Ly_h, L\bar{y}_h)_{L^2(0,T;L^2(\Omega))} + \rho (y'_h(0), \bar{y}'_h(0))_{L^2(\Omega)}$$

for $\rho > 0$.

On the continuous level this is consistence since

$$Ly = 0 \quad \text{and} \quad y'(0) = 0.$$
We circumvent the coercivity by replacing $A$ with

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On the continuous level this is consistence since

$$Ly = 0 \quad \text{and} \quad y'(0) = 0.$$ 

Note that:

$$\langle A_\rho y_h, y_h \rangle \geq c_\rho \| y_h \|_{Y_{\alpha,h}}^2 \quad \text{for all} \quad y_h \in Y_h.$$ 

This stabilization method is sometimes called augmented Lagrangian.
Stable discretization: Inf-sup

Assume we have chosen $Y_h$ such that $Y_h \subset Y$. We then choose $\Lambda_h$ to be

$$\Lambda_h := \{ \lambda_h = (Ly_h, y_h'(0)) \mid y_h(0) = 0, \ y_h \in Y_h \}.$$

Note that $\Lambda_h \subset \Lambda := L^2(0, T; L^2(\Omega)) \times L^2(\Omega)$.

A basis of $\Lambda_h$ is found by using a basis of $Y_h$.

The **inf-sup** condition holds with the same constant as the continuous case!
We precondition the system:

\[
B^{-1}_{\rho,h} A_{\rho,h} = \begin{pmatrix} P_{Y_h} & 0 \\ 0 & P_{\Lambda_h} \end{pmatrix}^{-1} \begin{pmatrix} A_{\rho,h} & B_h^T \\ B_h & 0 \end{pmatrix}
\]

where

\[
P_{Y_h} = A_{\rho,h} \quad \text{and} \quad P_{\Lambda_h} = \|\lambda_h\|^2_\Lambda = (\lambda_h, \lambda_h)_{L^2}.
\]

With this preconditioner the condition number \( \kappa \) is independent of \( \alpha \) and \( h \), but depend on \( \rho \)!
Discretization spaces

Our domain is a rectangle.
We consider tensor product B-splines as discretization space

\[ S_{p,\ell} = S_{p_t,\ell_t}(0, T) \otimes S_{p_x,\ell_x}((0, 1)^d). \]

Our discretization spaces are

\[ Y_h := S_{p,\ell} \cap H^1_0(0, 1) \quad \text{and} \quad \Lambda_h := \{ \lambda_h = (Ly_h, y'_h(0)) \mid y_h \in Y_{h,0} \}. \]

Our observation domain is the boundary of \((\frac{1}{4}, \frac{3}{4})^d\).
## Condition numbers

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**Table:** Condition numbers: $B_{\rho,h}^{-1}A_{\rho=1,h}$, $d = 2$ and $p = 2$

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**Table:** Iteration numbers: $B_{\rho,h}^{-1}A_{\rho=1,h}$, $d = 3$ and $p = 2$
## Iteration numbers

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**Table:** Iteration numbers: $B_h^{-1}A_{\rho,h}$, $d = 2$, $p = 2$ and $\ell = 5$
Initial source recovery in 2D

1. Choose an initial image \( y(0, x) \) on \((0, 1)^2\)
2. Calculate \( y(t, x) \) on \((0, T) \times (0, 1)^2\) by solving
   \[
   \partial_{tt} y - \Delta y = 0 \quad \text{in} \quad (0, T) \times (0, 1)^2 \\
   \partial_t y(0) = 0 \quad \text{on} \quad (0, 1)^2
   \]
3. Set \( z_d = y|_{\Gamma_T} \)
4. Use \( z_d \) in the optimal control problem and calculate \( \tilde{y} \)
Figure: Initial image
Initial $y(0)$, projection

Figure: $p = 2$ and $\ell = 6$
Recovery $\tilde{y}(0)$ with full observation

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 1.0 \quad \rho = 1.0$
Recovery $\tilde{y}(0)$ with full observation

Figure: $p = 2 \quad \ell = \quad \alpha = 10^{-7} \quad \rho = 1.0$
Recovery $\hat{y}(0)$ with full observation

Figure: $p = 2$ \hspace{0.5cm} $\ell = 6$ \hspace{0.5cm} $\alpha = 10^{-7}$ \hspace{0.5cm} $\rho = 10^{-2}$
Recovery $\tilde{y}(0)$ with full observation

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 10^{-7} \quad \rho = 10^{-5}$
Recovery $\tilde{y}(0)$ with full observation

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 10^{-7} \quad \rho = 10^{-6}$
Recovery $\tilde{y}(0)$ with full observation

Figure: $p = 2$  $\ell = 6$  $\alpha = 10^{-7}$  $\rho = 10^{-7}$
Discussion

- Well established theory + generalization
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- Stable discretization
  - We have $\alpha$ and $h$ robust preconditioner but not $\rho$ robust
  - Need low $\rho$ to recover image, but iterative methods does not converge for very low $\rho$
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  - $\Lambda_h$ might not have optimal approximation properties
- Need efficient “inversion” of the preconditioners
- System matrix is very large (3 space dimension + time) + “outer” domain. Possible to exploit tensorization!
Thank you for your attention!