Implementation of functional approach to a posteriori error control for curvilinear Timoshenko beams

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A posteriori error estimates

A posteriori error estimates provide an opportunity to evaluate a norm of difference between exact and given approximate solutions without using the exact solution itself.

Functional approach is developed by Prof. S. Repin and his colleagues. It is reliable and can be used for a wide range of approximations.

In applied mechanics there are several beam models of various complexity. Known results for functional type error estimates:

- Straight Timoshenko beam: M. Frolov, 2010
Beam models and underlying assumptions

- Plane cross sections remain plane after bending

- All longitudinal sections resist bending independently

- **Euler-Bernoulli beam**: plane cross sections remain orthogonal to the deformed axis (works fine for thin beams and small deflections)

- **Timoshenko beam**: shear effects considered, plane cross sections may not remain orthogonal to the deformed axis (can be applied to beams of moderate thickness and even composite beams)
Curvilinear Timoshenko beam:
Notation

\( u_{tr} \) – traverse displacement,
\( F_{tr} \) – traverse force,
\( u_{ax} \) – axial displacement,
\( F_{ax} \) – axial force,
\( \theta \) – rotation of beam cross-section,
\( m \) – bending moment,
\( k \) – shear correction factor,
\( A \) – area of the cross-section,
\( E \) – Young’s modulus,
\( G \) – shear modulus,
\( I_S \) – sectional moment of inertia,
\( R \) – curvature radius;
′ denotes the derivative \( \frac{d}{dx} \) where \( x \in \mathcal{I} = (0, L) \), \( L \) – the beam’s length.
Curvilinear Timoshenko beam:
Problem statement

Equilibrium equations:

\[
\begin{align*}
\left(\frac{EA}{R} \left( u'_{ax} + \frac{u_{tr}}{R} \right) \right)' &+ \frac{kGA}{R} \left( u'_{tr} + \theta - \frac{u_{ax}}{R} \right) + F_{ax} = 0, \\
- \frac{EA}{R} \left( u'_{ax} + \frac{u_{tr}}{R} \right) + \left( kGA \left( u'_{tr} + \theta - \frac{u_{ax}}{R} \right) \right)' + F_{tr} &= 0, \\
\left( EI_{S} \theta' \right)' - kGA \left( u'_{tr} + \theta - \frac{u_{ax}}{R} \right) + m &= 0,
\end{align*}
\]

(1)
Curvilinear Timoshenko beam: 
Problem statement

A clamped beam is assumed:

\[
\begin{aligned}
    u_{tr}(0) &= u_{tr}(L) = 0, \\
    u_{ax}(0) &= u_{ax}(L) = 0, \\
    \theta(0) &= \theta(L) = 0,
\end{aligned}
\]

where \( L \) is the beam’s length.

For any conforming approximate solution \((\tilde{u}_{ax}, \tilde{u}_{tr}, \tilde{\theta})\) the following error measure is estimated:

\[
\epsilon^2 = 2 \left( J(\tilde{u}_{ax}, \tilde{u}_{tr}, \tilde{\theta}) - J(u_{ax}, u_{tr}, \theta) \right) = 
\|
\sqrt{EA} \left( e'_{ax} + \frac{e_{tr}}{R} \right) \|^2 + \|
\sqrt{EI}e'_{\theta} \|^2 + \|
\sqrt{kGA} \left( e'_{tr} + e_{\theta} - \frac{e_{ax}}{R} \right) \|^2
\]

where \( e_{ax} = u_{ax} - \tilde{u}_{ax}, \ e_{tr} = u_{tr} - \tilde{u}_{tr}, \ e_{\theta} = \theta - \tilde{\theta} \).
Curvilinear Timoshenko beam:
Simplifying notation

The error measure can be rewritten as

\[
\epsilon^2 = \left\| \frac{1}{\sqrt{EA}} e_\gamma \right\|^2 + \left\| \frac{1}{\sqrt{EI_s}} e_\psi \right\|^2 + \left\| \frac{1}{\sqrt{kGA}} e_\xi \right\|^2,
\]

using the following notation:

\[
\begin{align*}
\gamma &= EA \left( u'_{ax} + \frac{u_{tr}}{R} \right), \quad \tilde{\gamma} = EA \left( \tilde{u}'_{ax} + \frac{\tilde{u}_{tr}}{R} \right), \quad e_\gamma = \gamma - \tilde{\gamma}; \\
\xi &= kGA \left( u'_{tr} + \theta - \frac{u_{ax}}{R} \right), \quad \tilde{\xi} = kGA \left( \tilde{u}'_{tr} + \tilde{\theta} - \frac{\tilde{u}_{ax}}{R} \right), \quad e_\xi = \xi - \tilde{\xi}; \\
\psi &= EI_s \theta', \quad \tilde{\psi} = EI_s \tilde{\theta}', \quad e_\psi = \psi - \tilde{\psi}.
\end{align*}
\]
Curvilinear Timoshenko beam: Free variables

Introducing a triple of free elements $\hat{\gamma} \in H^1(I)$, $\hat{\xi} \in H^1(I)$, $\hat{\psi} \in H^1(I)$, we get:

$$
\varepsilon^2 = \int_0^L \frac{1}{EA} (\hat{\gamma} - \tilde{\gamma}) e_{\gamma} dx + \int_0^L \left( \hat{\gamma}' + \frac{\hat{\xi}}{R} + F_{ax} \right) e_{ax} dx + \\
+ \int_0^L \frac{1}{kGA} (\hat{\xi} - \tilde{\xi}) e_{\xi} dx + \int_0^L \left( -\frac{\hat{\gamma}}{R} + \hat{\xi}' + F_{tr} \right) e_{tr} dx + \\
+ \int_0^L \frac{1}{EI_S} (\hat{\psi} - \tilde{\psi}) e_{\psi} dx + \int_0^L \left( \hat{\psi}' - \hat{\xi} + m \right) e_{\theta} dx.
$$
Curvilinear Timoshenko beam: Estimating $\epsilon^2$ terms

Estimating the encircled terms is straightforward. For estimating terms that contain elements of equilibrium equations, an extra inequality needs to be utilized:

$$
\frac{1}{C^2} \leq \inf_{(v_0, w_0, \phi_0)} \frac{\chi^2(v_0, w_0, \phi_0)}{\Delta^2(v_0, w_0, \phi_0)},
$$

where $(v_0, w_0, \phi_0) \in H^1_0(I) \times H^1_0(I) \times H^1_0(I)$,

$$
\chi^2 = \left\| \sqrt{EA} \left( v_0' + w_0 \frac{v_0}{R} \right) \right\|^2 + \left\| \sqrt{EI_S} \phi_0' \right\|^2 + \left\| \sqrt{kGA} \left( w_0' + \phi_0 - \frac{v_0}{R} \right) \right\|^2,
$$

$$
\Delta^2 = \left\| C_{ax}^{1/2} v_0 \right\|^2 + \left\| C_{tr}^{1/2} w_0 \right\|^2 + \left\| C_{\theta}^{1/2} \phi_0 \right\|^2.
$$

Constants $C_{ax}, C_{tr}, C_{\theta}$ are chosen as normalized combinations of material and shape parameters: $C_{tr} = kGA/L^2$, $C_{ax} = EA/L^2$, $C_{\theta} = EI_S/L^2$. 
Curvilinear Timoshenko beam: A posteriori error estimate

Finally,

$$\epsilon \leq M = M_1 + M_2 \quad \text{or} \quad \epsilon^2 \leq (1 + \beta) M_1^2 + (1 + \beta^{-1}) M_2^2$$

where

$$
\begin{align*}
M_1^2 &= \left\| \frac{1}{\sqrt{EA}} (\hat{\gamma} - \tilde{\gamma}) \right\|^2 + \left\| \frac{1}{\sqrt{kGA}} (\hat{\xi} - \tilde{\xi}) \right\|^2 + \left\| \frac{1}{\sqrt{EI_S}} (\hat{\psi} - \tilde{\psi}) \right\|^2 , \\
M_2^2 &= C^2 R^2 , \\
R^2 &= \left\| C_{ax}^{-1/2} \left( \hat{\gamma}' + \frac{\hat{\xi}}{R} + F_{ax} \right) \right\|^2 + \left\| C_{tr}^{-1/2} \left( -\frac{\hat{\gamma}}{R} + \hat{\xi}' + F_{tr} \right) \right\|^2 + \\
&\quad + \left\| C_{\theta}^{-1/2} \left( \hat{\psi}' - \hat{\xi} + m \right) \right\|^2
\end{align*}
$$

and $\beta$ is an arbitrary positive parameter.
Numerical example 1: Thin straight beam
M. Frolov, Russ. J. NAMM, 2010

<table>
<thead>
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<th>$M$</th>
<th>ratio</th>
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<td>$1.04 \times 10^{-2}$</td>
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### Numerical example 2: Thin curvilinear beam

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Numerical example 3:
Curvilinear beam of moderate thickness

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Conclusions

- A new functional a posteriori error estimate is derived for curvilinear Timoshenko beams.

- The estimate is implemented and numerically tested on a number of examples with ANSYS.

- The estimate is reliable, can be applied to any conforming approximate solution and appears to be a generalization of a functional a posteriori error estimate for straight Timoshenko beams known from the literature.

- In all considered cases overestimation of the true error is sufficiently small to conclude that the estimate is efficient and may become a practically valuable tool for engineering computations with black-box commercial software for CAE.
Thank you for your attention!