Fourier, Wavelet and Monte Carlo Methods in Computational Finance

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Agenda

- Derivatives pricing, Feynman-Kac Theorem
- Fourier methods
  - Basics of COS method;
  - Basics of SWIFT method;
  - Options with early-exercise features
    - COS method for Bermudan options
    - Monte Carlo method
  - BSDEs, BCOS method (very briefly)
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  - Basics of SWIFT method;
  - Options with early-exercise features
    - COS method for Bermudan options
    - Monte Carlo method
  - BSDEs, BCOS method (very briefly)
- Joint work with
  Fang Fang, Marjon Ruijter, Luis Ortiz, Shashi Jain, Alvaro Leitao, Fei Cong, Qian Feng

Kees Oosterlee (CWI, TU Delft)  Comp. Finance  AANMPDE-9-16
Feynman-Kac Theorem

- The linear partial differential equation:

\[
\frac{\partial v(t, x)}{\partial t} + \mathcal{L}v(t, x) + g(t, x) = 0, \quad v(T, x) = h(x),
\]

with operator

\[
\mathcal{L}v(t, x) = \mu(x)Dv(t, x) + \frac{1}{2}\sigma^2(x)D^2v(t, x).
\]
Feynman-Kac Theorem

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Feynman-Kac theorem:

\[ v(t, x) = \mathbb{E}\left[ \int_t^T g(s, X_s)ds + h(X_T) \right], \]

where \( X_s \) is the solution to the FSDE

\[ dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s, \quad X_t = x. \]
Suppose we consider the Hamilton-Jacobi-Bellman (HJB) equation:

\[
\begin{align*}
\frac{\partial v(t, x)}{\partial t} &+ \sup_{a \in A}\{\mu'(x, a)Dv(t, x) + \frac{1}{2}\text{Tr}[D^2v(t, x)\sigma\sigma'(x, a)]
+ g(t, x, a)\} = 0, \\
v(T, x) &= h(x).
\end{align*}
\]

It is associated to a stochastic control problem with value function

\[
v(t, x) = \sup_{\alpha} \mathbb{E}_t^X \left[ \int_t^T g(s, X_s^\alpha, \alpha_s)ds + h(X_T^\alpha) \right],
\]

where $X_s$ is the solution to the controlled FSDE

\[
dX_s^\alpha = \mu(X_s^\alpha, \alpha_s)ds + \sigma(X_s^\alpha, \alpha_s)d\omega_s, \quad X_t^\alpha = x.
\]
The semilinear partial differential equation:

\[
\frac{\partial v(t, x)}{\partial t} + \mathcal{L}v(t, x) + g(t, x, v, \sigma(x)Dv(t, x)) = 0, \quad v(T, x) = h(x).
\]

We can solve this PDE by means of the FSDE:

\[
dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s, \quad X_t = x.
\]

and the BSDE:

\[
dY_s = -g(s, X_s, Y_s, Z_s)ds + Z_sd\omega_s, \quad Y_T = h(X_T).
\]

Theorem:

\[
Y_t = v(t, X_t), \quad Z_t = \sigma(X_t)Dv(t, X_t).
\]

is the solution to the BSDE.
Suppose you have stocks of a company, and you’d like to have cash in two years (to buy a house).

You wish at least $K$ euros for your stocks, but the stocks may drop in the coming years. How to assure $K$ euros in two years?
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You can buy insurance against falling stock prices.

This is the standard put option (i.e. the right to sell stock at a future time point).
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The uncertainty is in the stock prices.
A call option gives the holder the right to trade in the future at a previously agreed strike price, $K$. 

\[ v(T, S_T) = \max(K - S_T, 0) =: h(S_T) \]
A call option gives the holder the right to trade in the future at a previously agreed strike price, $K$.

$$\nu(T, S) = \max(K - S_T, 0) =: h(S_T)$$
Feynman-Kac Theorem (option pricing context)

Given the final condition problem

\[
\begin{aligned}
    \frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS \frac{\partial v}{\partial S} - rv &= 0, \\
    v(T, S) &= h(S_T) = \text{given}
\end{aligned}
\]

Then the value, \( v(t, S) \), is the unique solution of

\[
v(t, S) = e^{-r(T-t)} \mathbb{E}_Q \{ v(T, S_T) | \mathcal{F}_t \}
\]

with the sum of first derivatives square integrable, and \( S = S_t \) satisfies the system of SDEs:

\[
dS_t = rS_t dt + \sigma S_t d\omega_t^Q,
\]

- Similar relations also hold for (multi-D) SDEs and PDEs!
A pricing approach; European options

\[ \nu(t_0, S_0) = e^{-r(T-t_0)} \mathbb{E}^Q \{ \nu(T, S_T) | \mathcal{F}_0 \} \]

Quadrature:

\[ \nu(t_0, S_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} \nu(T, S_T) f(S_T, S_0) dS_T \]

- Trans. PDF, \( f(S_T, S_0) \), typically not available, but the characteristic function, \( \hat{f} \), often is.
A firm will have some **business in America** for several years.
- Investment may be in euros, and payment in the local currency.
- Firm’s profit can be **influenced negatively by the exchange rate.**
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Banks sell insurance against changing FX rates. The option pays out in the best currency each year.

Uncertain processes are the exchange rates, interest rate,
A firm will have some **business in America** for several years.

Investment may be in euros, and payment in the local currency.

Firm’s profit can be **influenced negatively by the exchange rate**.

Banks sell **insurance against changing FX rates**. The option pays out in the best currency each year.

**Uncertain processes** are the exchange rates, interest rate, but also the **counterparty** of the contract may go bankrupt!
Multi-asset options belong to the class of exotic options.

$$v(S, T) = \max(\max\{S_1, \ldots, S_d\}_T - K, 0) \quad \text{(max call)}$$

$$v(S, t_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} v(S, T)f(S_T|S_0)dS$$

$$\Rightarrow \text{High-dimensional integral or a high-D PDE.}$$
The problem dimension increases if the option depends on more than one asset \( S_i \) (multiple sources of uncertainty).

If each underlying follows a geometric (lognormal) diffusion process, each additional asset is represented by an extra dimension in the problem:

\[
\frac{\partial v}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} \left[ \sigma_i \sigma_j \rho_{i,j} S_i S_j \frac{\partial^2 v}{\partial S_i \partial S_j} \right] + \sum_{i=1}^{d} \left[ r S_i \frac{\partial v}{\partial S_i} \right] - rv = 0.
\]

Required information is the volatility of each asset \( \sigma_i \) and the correlation between each pair of assets \( \rho_{i,j} \).
One can apply several numerical techniques to calculate the option price:

- Numerical integration,
- Monte Carlo simulation,
- Numerical solution of the partial-(integro) differential equation

Each of these methods has its merits and demerits.

Numerical challenges:

- The problem’s dimensionality
- Speed of solution methods
- Early exercise feature (→ free boundary problem)
Financial engineering, pricing approach:

1. Start with some financial product
2. Model asset prices involved (SDEs)
3. Calibrate the model to market data (Numerics, Opt.)
4. Model product price correspondingly (PDE, Integral)
5. Price the product of interest (Numerics, MC)
5a. Price the risk related to default (SDE, Opt.)
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5. Price the product of interest (Numerics, MC)
5a. Price the risk related to default (SDE, Opt.)
6. Understand and remove risk (Stoch., Opt., Numer.)
A characteristic function of a continuous random variable $X$, equals the Fourier transform of the density of $X$.

Derive pricing methods that

- are computationally fast
- are not restricted to Gaussian-based models
- should work as long as we have a characteristic function,

$$\hat{f}(u; x) = \int_{-\infty}^{\infty} e^{iux} f(x) dx;$$

(available for Lévy processes and also for SDE systems).
Suppose we have given a following system of SDEs:

\[ dX_t = \mu(X_t)dt + \sigma(X_t)d\omega_t + dZ_t, \]

For processes in the AJD class drift, volatility, jump intensities and interest rate components are of the affine form, i.e.

\[
\begin{align*}
\mu(X_t) & = a_0 + a_1 X_t \quad \text{for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}, \\
\lambda(X_t) & = b_0 + b_1^T X_t, \quad \text{for } (b_0, b_1) \in \mathbb{R} \times \mathbb{R}^n, \\
\sigma(X_t)\sigma(X_t)^T & = (c_0)_{ij} + (c_1)_{ij}^T X_t, \quad (c_0, c_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}, \\
r(X_t) & = r_0 + r_1^T X_t, \quad \text{for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n.
\end{align*}
\]

Duffie, Pan, Singleton (2000): For affine jump diffusion processes the discounted characteristic function can be derived!
Exponential Lévy Processes

- Lévy process \( \{X_t\}_{t \geq 0} \): process with stationary, independent increments.

- Brownian motion and Poisson processes belong to this class, as well jump processes with either finite or infinite activity.

- Asset prices can be modeled by exponential Lévy processes
  - small jumps describe the day-to-day "noise";
  - big jumps describe large stock price movements.

- The characteristic function of a Lévy process is known:

\[
\hat{f}(u; X_t) = \mathbb{E}[\exp(iuX_t)] = \exp(t(i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy1_{|y|<1}\nu(dy))),
\]

the celebrated Lévy-Khinchine formula.
The **COS method**:
- Exponential convergence;
- Greeks (derivatives) are obtained at no additional cost.

All based on the availability of a characteristic function.

**The basic idea:**
- Replace the density by its Fourier-cosine series expansion;
- Coefficients have simple relation to characteristic function.
Series Coefficients of the Density and the ChF

- Fourier-Cosine expansion of density function on interval $[a, b]$

$$f(x) = \sum_{n=0}^{\infty} F_n \cos \left( n\pi \frac{x - a}{b - a} \right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b - a} \int_{a}^{b} f(x) \cos \left( n\pi \frac{x - a}{b - a} \right) dx.$$
Fourier-Cosine expansion of density function on interval \([a, b]:\)

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\[
F_n := \frac{2}{b - a} \int_{a}^{b} f(x) \cos \left( n\pi \frac{x - a}{b - a} \right) dx.
\]

\(F_n\) has direct relation to the ChF, \(\hat{f}(u) := \int_{\mathbb{R}} f(x)e^{iux} dx\)

\((\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0),\)

\[
F_n \approx P_n := \frac{2}{b - a} \int_{\mathbb{R}} f(x) \cos \left( n\pi \frac{x - a}{b - a} \right) dx
= \frac{2}{b - a} \Re \left\{ \hat{f} \left( \frac{n\pi}{b - a} \right) \exp \left( -i \frac{na\pi}{b - a} \right) \right\}.
\]
Replace $F_n$ by $P_n$, and truncate the summation:

$$f(x) \approx \frac{2}{b - a} \sum_{n=0}^{N-1} \Re \left\{ \hat{f} \left( \frac{n \pi}{b - a} \right) \exp \left( in \pi \frac{-a}{b - a} \right) \right\} \cos \left( n \pi \frac{x - a}{b - a} \right).$$

**Example:** $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a, b] = [-10, 10]$ and $x = \{-5, -4, \ldots, 4, 5\}$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>error</td>
<td>0.2538</td>
<td>0.1075</td>
<td>0.0072</td>
<td>4.04e-07</td>
<td>3.33e-16</td>
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<tr>
<td>cpu time (sec.)</td>
<td>0.0025</td>
<td>0.0028</td>
<td>0.0025</td>
<td>0.0031</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

**Exponential error convergence in $N$.**
Pricing European Options

- Start from the risk-neutral valuation formula:

\[ v(t_0, x) = e^{-r\Delta t} \mathbb{E}^Q [v(T, y)|\mathcal{F}_0] = e^{-r\Delta t} \int_{\mathbb{R}} v(T, y)f(y, x)dy. \]

- Truncate the integration range:

\[ v(t_0, x) = e^{-r\Delta t} \int_{[a,b]} v(T, y)f(y, x)dy + \varepsilon. \]

- Replace the density by the COS approximation, and interchange summation and integration:

\[ \hat{v}(t_0, x) = e^{-r\Delta t} \sum_{n=0}^{N-1} \Re \left\{ \hat{f} \left( \frac{n\pi}{b-a}; x \right) e^{-in\pi \frac{a}{b-a}} \right\} \mathcal{H}_n, \]

where the series coefficients of the payoff, \( \mathcal{H}_n \), are analytic.
Pricing European Options

- Log-asset prices: \( x := \log(S_0/K) \) and \( y := \log(S_T/K) \).
- The payoff for European call options reads
  \[
  v(T, y) \equiv \max(K(e^y - 1), 0).
  \]
- For a call option, we obtain
  \[
  H_{\text{call}}^k = \frac{2}{b-a} \int_0^b K(e^y - 1) \cos \left( k\pi \frac{y-a}{b-a} \right) dy
  \]
  \[
  = \frac{2}{b-a} K \left( \chi_k(0, b) - \psi_k(0, b) \right).
  \]
- For a vanilla put, we find
  \[
  H_{\text{put}}^k = \frac{2}{b-a} K \left( -\chi_k(a, 0) + \psi_k(a, 0) \right).
  \]
Results, Heston stochastic volatility PDE

\[
\frac{\partial v}{\partial t} = \frac{1}{2} S^2 y \frac{\partial^2 v}{\partial S^2} + \rho \gamma S y \frac{\partial^2 v}{\partial S \partial y} + \frac{1}{2} \gamma^2 y \frac{\partial v}{\partial y^2} + rS \frac{\partial v}{\partial S} + \kappa (\bar{\sigma} - y) \frac{\partial v}{\partial y} - rv.
\]

- **GPU computing**: Multiple strikes for parallelism, 21 IC’s.

<table>
<thead>
<tr>
<th></th>
<th>Heston model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
</tr>
<tr>
<td>MATLAB</td>
<td>msec</td>
</tr>
<tr>
<td></td>
<td>max.abs.err</td>
</tr>
<tr>
<td>GPU</td>
<td>msec</td>
</tr>
</tbody>
</table>

**Table 1**: Maximum absolute error when pricing a vector of 21 strikes.

- **Exponential convergence**, Error analysis in our papers.
- Also work with wavelets instead of cosines.
Approximate density $f$ by a finite combination of Shannon scaling functions and recovering the coefficients from its Fourier transform.

$$\varphi_{m,k}(x) = 2^{m/2} \frac{\sin(\pi(2^m x - k))}{\pi(2^m x - k)}, \quad k \in \mathbb{Z}. \quad (1)$$

For $m = k = 0$, we have the father wavelet,

$$\varphi(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Wavelets can be moved (by $k$), stretched or compressed (by $m$) to accurately represent local properties of a function. Shannon wavelets have a slow decay in the time domain but a very sharp compact support in the frequency (Fourier) domain.
Using the classical *Vieta formula*, the cardinal sine can be approximated:

\[
sinc(t) \approx \prod_{j=1}^{J} \cos \left( \frac{\pi t}{2j} \right) = \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos \left( \frac{2j - 1}{2^J} \pi t \right). \tag{2}
\]

Following the wavelets theory, function \( f \) can be approximated at a level of resolution \( m \), i.e.,

\[
f(x) \approx \mathcal{P}_m f(x) = \sum_{k \in \mathbb{Z}} c_{m,k} \varphi_{m,k}(x) \approx f_m(x) := \sum_{k=k_1}^{k_2} c_{m,k} \varphi_{m,k}(x), \tag{3}
\]

for certain accurately chosen values \( k_1 \) and \( k_2 \).
\[ v(x, t) = e^{-r(T-t)} \int_{\mathbb{R}} v(y, T)f(y|x) \, dy, \]

\[ f(y|x) \approx \sum_{k=k_1}^{k_2} c_{m,k}(x) \varphi_{m,k}(y), \]

With: \( c_{m,k} = \langle f, \varphi_{m,k} \rangle = \int_{\mathbb{R}} f(x) \varphi(2^m x - k) \, dx. \)

\[ c_{m,k} \approx c_{m,k}^* := \frac{2^m/2}{2J-1} \sum_{j=1}^{2^J-1} \int_{\mathbb{R}} f(x) \cos \left( \frac{2j - 1}{2^j} \pi (2^m x - k) \right) \, dx. \quad (4) \]

or:

\[ c_{m,k} \approx c_{m,k}^* = \frac{2^m/2}{2J-1} \sum_{j=1}^{2^J-1} \Re \left\{ \hat{f} \left( \frac{(2j - 1)\pi 2^m}{2^j} \right) e^{\frac{ik\pi (2j-1)}{2^j}} \right\}. \quad (5) \]
Example: Fat-tailed densities:

We can control density values and the mass as a byproduct.

\[
f\left(\frac{h}{2^m}\right) \approx \mathcal{P}_m f\left(\frac{h}{2^m}\right) = 2^{\frac{m}{2}} \sum_{k \in \mathbb{Z}} c_{m,k} \delta_{k,h} = 2^{\frac{m}{2}} c_{m,h},
\]

...plus...

\[
\mathcal{A} = \frac{1}{2^{m/2}} \left( \frac{c_{m,k_1}}{2} + \sum_{k_1 < k < k_2} c_{m,k} + \frac{c_{m,k_2}}{2} \right),
\]

| $[a, b]$ | $|f\left(\frac{k_1}{2^m}\right)|$ | $|f\left(\frac{k_2}{2^m}\right)|$ | Error (area) |
|---------|-----------------|-----------------|--------------|
| $[-1, 1]$ | $5.97e-02$ | $4.20e-02$ | $9.82e-01$ |
| $[-5, 5]$ | $1.30e-01$ | $1.06e-03$ | $3.40e-01$ |
| $[-20, 20]$ | $1.10e-08$ | $1.82e-15$ | $7.05e-09$ |

$\Rightarrow$ Difficult to accurately determine $[a, b]$ by cumulants.
Replace \( f \) by its approximation \( f_m \):

\[
v(x, t) \approx e^{-r(T-t)} \sum_{k=k_1}^{k_2} c_{m,k}(x) \cdot \mathcal{H}_{m,k},
\]

with the pay-off coefficients,

\[
\mathcal{H}_{m,k} := \int_{I_k} v(y, T) \varphi_{m,k}(y) \, dy.
\]

With \( \bar{k}_1 := \max(k_1, 0) \), pay-off coefficients, \( \mathcal{H}_{m,k} \), for a European call option are approximated by,

\[
\mathcal{H}^*_{m,k} := \begin{cases} 
\frac{K_2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[ l_1 \left( \frac{\bar{k}_1}{2^m}, \frac{\bar{k}_2}{2^m} \right) \right. & - l_2 \left( \frac{\bar{k}_1}{2^m}, \frac{\bar{k}_2}{2^m} \right), \\
0, & \text{if } k_2 \leq 0,
\end{cases}
\]

where \( l_1 \) and \( l_2 \) are closed formulae. FFT can be applied to compute the pay-off coefficients.
Example: **Multiple strikes valuation:**

<table>
<thead>
<tr>
<th></th>
<th>scale $m$</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>21 strikes</strong></td>
<td>CPU time (milli-seconds)</td>
<td>3.04</td>
<td>3.84</td>
<td>6.62</td>
</tr>
<tr>
<td></td>
<td>Max. absolute error</td>
<td>$2.04e-02$</td>
<td>$5.63e-05$</td>
<td>$3.63e-06$</td>
</tr>
<tr>
<td><strong>1 strike</strong></td>
<td>CPU time (milli-seconds)</td>
<td>0.32</td>
<td>0.52</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>Absolute error</td>
<td>$4.78e-03$</td>
<td>$1.61e-05$</td>
<td>$6.56e-07$</td>
</tr>
</tbody>
</table>

**Table 2:** Simultaneous valuation of 21 call options with $K$ from 50 to 150, and of only one European call option under the Heston dynamics with parameters $S_0 = 100, \mu = 0, \lambda = 1.5768, \eta = 0.5751, \bar{u} = 0.0398, u_0 = 0.0175, \rho = -0.5711$. 
The controller only has control over his terminal time.

\[ dX_t = \mu(X_t)dt + \sigma(X_t)d\omega_t, \]

With \( t \in [0, T] \), and stopping times \( T_{t,T} \), the finite horizon optimal stopping problem is formulated as

\[
\nu(t, x) = \sup_{\tau \in T_{t,T}} \mathbb{E}\left[ \int_{t}^{\tau} e^{-r(T-\tau)}g(s, X_s) \, ds + e^{-r(\tau-t)}h(X_\tau) \right].
\]

The value function \( \nu \) is related to the HJB variational inequality:

\[
\min\left[ -\frac{\partial \nu}{\partial t} - \mathcal{L}\nu - g, \nu - h \right] = 0,
\]

The problem is called a free boundary problem.

\( C \) is the continuation region, the complement set is the stopping or exercise region (receive the reward \( g \)).
The pricing formulas for a Bermudan option with $M$ exercise dates reads, for $m = M - 1, \ldots, 1$:

$$
\begin{align*}
\left\{ \begin{array}{l}
c(S, t_m) = e^{-r \Delta t_m} \mathbb{E} [v(S, t_{m+1})|S_{t_m}], \\
v(S, t_m) = \max (h(S, t_m), c(S, t_m))
\end{array} \right.
\end{align*}
$$

and $v(S, t_0) = e^{-r \Delta t_0} \mathbb{E} [v(S, t_1)|S_{t_0}]$

- Use Newton’s method to determine early exercise point $x^*_m$, i.e. the root of $h(x, t_{m+1}) - c(x, t_{m+1}) = 0$.
- Use the COS formula for $v(x, t_0)$. 

COS pricing method for early–exercise options

The pricing formulas, for $m = M - 1, \ldots, 1$:

$$
\begin{align*}
  c(x, t_m) &= e^{-r \Delta t} \int_{\mathbb{R}} v(y, t_{m+1}) f(y | x) dy, \\
  v(x, t_m) &= \max (h(x, t_m), c(x, t_m)),
\end{align*}
$$

followed by

$$
v(x, t_0) = e^{-r \Delta t} \int_{\mathbb{R}} v(y, t_1) f(y | x) dy.
$$

Here $x, y$ are state variables of consecutive exercise dates. In order to get the option value, a **backward recursion procedure** is performed on the Fourier–Cosine coefficient $\mathcal{H}_k(t_{m+1})$:

$$
\mathcal{H}_k(t_{m+1}) = \frac{2}{b - a} \int_{a}^{b} \max (c(x, t_{m+1}), h(x, t_{m+1})) \cos (k \pi \frac{y - a}{b - a}) dy,
$$

and then use $\mathcal{H}_k(t_1)$ in the COS formula.
Bermudan puts with 10 early-exercise dates

Table 3: Test parameters for pricing Bermudan options

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Model</th>
<th>$S_0$</th>
<th>$K$</th>
<th>$T$</th>
<th>$r$</th>
<th>$\sigma$</th>
<th>Other Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>BS</td>
<td>100</td>
<td>110</td>
<td>1</td>
<td>0.1</td>
<td>0.2</td>
<td>—</td>
</tr>
<tr>
<td>3</td>
<td>CGMY</td>
<td>100</td>
<td>80</td>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>$C = 1, G = 5, M = 5, Y = 1.5$</td>
</tr>
</tbody>
</table>

BS

\[ \text{log}_{10}|\text{error}| \]

milliseconds

CGMY

\[ \text{log}_{10}|\text{error}| \]

milliseconds

Kees Oosterlee (CWI, TU Delft)
Towards higher dimensions: Monte Carlo Formulation

- Dynamic Programming:
  - COS approximation is fine for up to 3, 4 dimensions! SWIFT works as well for these options.
  - Higher D: Monte Carlo simulation techniques, like SGBM.
  - The Bermudan option at time $t_m$ and state $S_{t_m}$ is given by
    \[
    v_{t_m}(S_{t_m}) = \max(h(S_{t_m}), c_{t_m}(S_{t_m})).
    \] (8)

- The continuation value $c_{t_m}$, is:
  \[
  c_{t_m}(S_{t_m}) = e^{-r\Delta t_m} \mathbb{E}[v_{t_{m+1}}(S_{t_{m+1}})|S_{t_m}] .
  \]
Towards higher dimensions: Monte Carlo Formulation

- Dynamic Programming:

- COS approximation is fine for up to 3, 4 dimensions! SWIFT works as well for these options.

  **Higher D:** Monte Carlo simulation techniques, like SGBM.

- The Bermudan option at time $t_m$ and state $S_{t_m}$ is given by

  $$v_{t_m}(S_{t_m}) = \max(h(S_{t_m}), c_{t_m}(S_{t_m})).$$

- The continuation value $c_{t_m}$, is:

  $$c_{t_m}(S_{t_m}) = e^{-r\Delta t_m} \mathbb{E} \left[ v_{t_{m+1}}(S_{t_{m+1}}) | S_{t_m} \right].$$
Step 1: The grid points in SGBM are generated by simulation,
\{S_{t_0}(n), \ldots, S_{t_M}(n)\}, \ n = 1, \ldots, N,

Step 2: Compute the option value at terminal time.

Step 3: Bundle the grid points at \( t_m \) into \( B_{t_m}(1), \ldots, B_{t_m}(\nu) \)
non-overlapping bundles.

Step 4: For \( B_{t_m}(\beta), \ \beta = 1, \ldots, \nu \), compute \( Z(S_{t_{m+1}}, \alpha_{t_{m+1}}^\beta) \).

\( Z : \mathbb{R}^d \times \mathbb{R}^K \rightarrow \mathbb{R} \), is a parametrized function which assigns values to states \( S_{t_{m+1}} \).

Step 5: The continuation values for grid points in
\( B_{t_m}(\beta), \ \beta = 1, \ldots, \nu \), are approximated by

\[ \widehat{c}_{t_m}(S_{t_m}(n)) = \mathbb{E}[Z(S_{t_{m+1}}, \alpha_{t_{m+1}}^\beta)|S_{t_m}(n)] \]
The objective is to choose, corresponding to each bundle $\beta$ at $t_m$, a parameter vector $\alpha_{tm+1}^{\beta}$ so that, $V_{tm+1}(S_{tm+1}) \approx Z(S_{tm+1}, \alpha_{tm+1}^{\beta})$.

We use OLS, to define

$$Z(S_{tm+1}, \hat{\alpha}_{tm+1}^{\beta}) = \sum_{k=1}^{K} \hat{\alpha}_{tm+1}^{\beta}(k) \phi_k(S_{tm+1}).$$

(9)
Computing the continuation value

- The continuation value is computed as:

\[
\hat{c}_{tm}(S_{tm}(n)) = \mathbb{E}[Z(S_{tm+1}, \hat{\alpha}_{tm+1}^\beta)|S_{tm} = S_{tm}(n)],
\]

where \(S_{tm}(n) \in B_{tm}(\beta)\).

- This can be written as:

\[
\hat{c}_{tm}(S_{tm}(n)) = \mathbb{E} \left[ \left( \sum_{k=1}^{K} \hat{\alpha}_{tm+1}^\beta(k) \phi_k(S_{tm+1}) \right) | S_{tm} = S_{tm}(n) \right]
\]

\[
= \sum_{k=1}^{K} \hat{\alpha}_{tm+1}^\beta(k) \mathbb{E} \left[ \phi_k(S_{tm+1}) | S_{tm} = S_{tm}(n) \right].
\]

- Choose basis functions \(\phi\) so that \(\mathbb{E} \left[ \phi_k(S_{tm+1}) | S_{tm} = X \right]\) has an analytic solution.
Arithmetic Basket Option on 15 assets

- Convergence and computational time:
Arithmetic Basket Option on 15 assets

- Convergence and computational time:

![Graph showing convergence and computational time for different methods.](image)

Legend:
- k–means direct est
- k–means path est.
- RB2 direct est.
- RB2 path est.
Arithmetic Basket Option on 15 assets

- Convergence and computational time:
Expected exposure in CVA (joint work with Qian Feng)
Exposure and Counterparty risk

- Exposure at a path at time $t_m$ with stock $S_{t_m}$

$$E(S_{t_m}, t_m) = \begin{cases} 
  c_{t_m}(S_{t_m}), & \text{not exercised} \\
  0, & \text{exercised}
\end{cases} \quad (11)$$

- Expected Exposure (EE) at time $t_m$

$$EE(t_m) = \mathbb{E} [E(S_{t_m}, t_m)] \approx \frac{1}{N} \sum_{n=1}^{N} E(S_{t_m}(n), t_m) \quad (12)$$

$\Rightarrow c_{t_m}(S_{t_m})$ is calculated at each time step, along the path, by SGBM.
The semilinear partial differential equation:

\[
\frac{\partial v(t,x)}{\partial t} + \mathcal{L}v(t,x) + g(t,x,v,\sigma(x)Dv(t,x)) = 0, \quad v(T,x) = h(x),
\]

We can solve this PDE by means of the FSDE:

\[
dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s, \quad X_t = x.
\]

and the BSDE:

\[
dY_s = -g(s,X_s,Y_s,Z_s)ds + Z_sd\omega_s, \quad Y_T = h(X_T).
\]

Theorem:

\[
Y_t = v(t,X_t), \quad Z_t = \sigma(X_t)Dv(t,X_t).
\]

is the solution to the decoupled FBSDE.
Backward SDE

\[ dY_t = -g(t, Y_t, Z_t)dt + Z_t d\omega_t, \quad Y_T = \xi. \]

\( g \) is the driver function. \( \xi \) is \( \mathcal{F}_T \)-measurable random variable.

A solution is a pair adapted processes \((Y, Z)\) satisfying

\[ Y_t = \xi + \int_{t}^{T} g(s, Y_s, Z_s) ds - \int_{t}^{T} Z_s d\omega_s. \]

\( Y \) is adapted if and only if, for every realization and every \( t \), \( Y_t \) is known at time \( t \). Adapted process cannot “see into the future”.

A BSDE is not a time-revered FSDE: at time \( t \) \((Y_t, Z_t)\) is \( \mathcal{F}_t \)-measurable and the process does not “know” the terminal condition yet.
Backward SDE

\[ dY_t = -g(t, Y_t, Z_t)dt + Z_t d\omega_t, \quad Y_T = \xi. \]

\(g\) is the driver function. \(\xi\) is \(\mathcal{F}_T\)-measurable random variable. A solution is a pair adapted processes \((Y, Z)\) satisfying

\[ Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s d\omega_s. \]

\(Y\) is adapted if and only if, for every realization and every \(t\), \(Y_t\) is known at time \(t\). Adapted process cannot “see into the future”.

A BSDE is not a time-revered FSDE: at time \(t\) \((Y_t, Z_t)\) is \(\mathcal{F}_t\)-measurable and the process does not “know” the terminal condition yet.
\[ Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \]  

(13)

\[ \mathcal{X}_t := (X_t, Y_t, Z_t). \]
\[ Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s)ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \] (13)

\[ \mathcal{X}_t := (X_t, Y_t, Z_t). \]

Taking conditional expectation \( \mathbb{E}_m[.] = \mathbb{E}[.|\mathcal{F}_{t_m}] \):

\[
Y_m = \mathbb{E}_m[Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[g(s, \mathcal{X}_s)]ds - \mathbb{E}_m \left[ \int_{t_m}^{t_{m+1}} Z_s d\omega_s \right]
\]

(14)
\[ Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathbf{x}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \] (13)

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\[ Y_m = \mathbb{E}_m[Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[g(s, \mathbf{x}_s)] ds - \mathbb{E}_m \left[ \int_{t_m}^{t_{m+1}} Z_s d\omega_s \right]. \] (14)
\( Y \) component

\[
Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \tag{13}
\]

\( \mathcal{X}_t := (X_t, Y_t, Z_t) \).

Taking conditional expectation \( \mathbb{E}_m[.] = \mathbb{E}[.|\mathcal{F}_{t_m}] \):

\[
Y_m = \mathbb{E}_m[Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[g(s, \mathcal{X}_s)] ds - 0 \tag{14}
\]
\[ Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \]  

(13)

\[ \mathcal{X}_t := (X_t, Y_t, Z_t). \]

Taking conditional expectation \( \mathbb{E}_m[.] = \mathbb{E}[.|\mathcal{F}_{t_m}] : \)

\[ Y_m = \mathbb{E}_m[Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[g(s, \mathcal{X}_s)] ds \]

\[ \approx \mathbb{E}_m[Y_{m+1}] + \Delta t g(t_m, \mathcal{X}_m). \]

(14)
\[ Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) \, ds - \int_{t_m}^{t_{m+1}} Z_s \, d\omega_s. \] (15)

Multiplying by \( \Delta \omega_{m+1} \), taking the conditional expectation gives
\[ Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \] (15)

Multiplying by \( \Delta \omega_{m+1} \), taking the conditional expectation gives

\[ Z_m \approx \frac{1}{\Delta t} E_m [Y_{m+1} \Delta \omega_{m+1}] \]
Discretization scheme FBSDE

\( X_0^\Delta = x_0, \)

for \( m = 0, \ldots, M - 1 : \)

\( X_{m+1}^\Delta = X_m^\Delta + \mu(X_m^\Delta) \Delta t + \sigma(X_m^\Delta) \Delta \omega_{m+1}, \)

\( Y_M^\Delta = h(X_M^\Delta), \)

for \( m = M - 1, \ldots, 0 : \)

\( Z_m^\Delta = \frac{1}{\Delta t} \mathbb{E}_m[Y_{m+1}^\Delta \Delta \omega_{m+1}], \)

\( Y_m^\Delta = \mathbb{E}_m[Y_{m+1}^\Delta] + g(t_m, X_m^\Delta) \Delta t. \)

Ingredients:

- Characteristic function \( X_{m+1}^\Delta \), given \( X_m^\Delta = x \)
- \( \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)] \)
- \( \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta) \Delta \omega_{m+1}] \)
- Recover Fourier cosine coefficients backward in time
- FFT algorithm
Discretization scheme FBSDE

\[ X_0^\Delta = x_0, \]

for \( m = 0, \ldots, M - 1 : \)

\[ X_{m+1}^\Delta = X_m^\Delta + \mu(X_m^\Delta)\Delta t + \sigma(X_m^\Delta)\Delta \omega_{m+1}, \]

\[ Y_M^\Delta = h(X_M^\Delta), \]

for \( m = M - 1, \ldots, 0 : \)

\[ Z_m^\Delta = \frac{1}{\Delta t} \mathbb{E}_m[Y_{m+1}^\Delta \Delta \omega_{m+1}], \]

\[ Y_m^\Delta = \mathbb{E}_m[Y_{m+1}^\Delta] + g(t_m, X_m^\Delta)\Delta t. \]

Ingredients:

- Characteristic function \( X_{m+1}^\Delta \), given \( X_m^\Delta = x \)
- \( \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)] \)
- \( \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)\Delta \omega_{m+1}] \)
- Recover Fourier cosine coefficients backward in time
- FFT algorithm
The COS formula:

\[ I := \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)] = \int_{\mathbb{R}} h(t_{m+1}, \zeta) f(\zeta|x) d\zeta \]

\[ \approx \int_a^b h(t_{m+1}, \zeta) f(\zeta|x) d\zeta. \quad (16) \]

Replace the density function and function \( h \) by their Fourier cosine series expansions.

\[ I \approx \frac{b-a}{2} \sum_{k=0}^{N-1} \mathcal{H}_k(t_{m+1}) P_k(x). \quad (17) \]

The COS formula:

\[ \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)] \approx \sum_{k=0}^{N-1} \mathcal{H}_k(t_{m+1}) \Re \left( \hat{f}_{X_{m+1}^\Delta} \left( \frac{k\pi}{b-a} | X_m^\Delta = x \right) e^{ik\pi \frac{a}{b-a}} \right). \]
Asset price: \( dS_t = \bar{\mu}S_t \, dt + \bar{\sigma}S_t \, d\omega_t \).

Hedge portfolio \( Y_t \) with: \( a_t \) assets \( S_t \) and \( Y_t - a_tS_t \) bonds

\[
dY_t = r(Y_t - a_tS_t) \, dt + a_t \, dS_t
\]
Example: European Call Option - GBM - $P$-Measure

Asset price: $dS_t = \mu S_t dt + \sigma S_t d\omega_t$.

Hedge portfolio $Y_t$ with: $a_t$ assets $S_t$ and $Y_t - a_t S_t$ bonds

$$dY_t = r(Y_t - a_t S_t) dt + a_t dS_t$$
Asset price: \( dS_t = \bar{\mu}S_t dt + \bar{\sigma}S_t d\omega_t \).

Hedge portfolio \( Y_t \) with: \( a_t \) assets \( S_t \) and \( Y_t - a_t S_t \) bonds

\[
dY_t = r(Y_t - a_t S_t) dt + a_t dS_t \\
= \left( rY_t + \frac{\bar{\mu} - r}{\bar{\sigma}}a_t S_t \right) dt + \bar{\sigma}a_t S_t d\omega_t,
\]

\( Y_T = h(S_T) = \max(S_T - K, 0) \).
Asset price: \( dS_t = \mu_S t dt + \sigma_S t d\omega_t \).

Hedge portfolio \( Y_t \) with: \( a_t \) assets \( S_t \) and \( Y_t - a_t S_t \) bonds

\[
dY_t = r(Y_t - a_t S_t) dt + a_t dS_t
= \left( rY_t + \frac{\bar{\mu} - r}{\bar{\sigma}} a_t S_t \right) dt + \bar{\sigma} a_t S_t d\omega_t,
\]

\( Y_T = h(S_T) = \max(S_T - K, 0) \).

If we set \( Z_t = \bar{\sigma} a_t S_t \), then \((Y, Z)\) solves a BSDE with driver,

\[
g(t, x, y, z) = -ry - \frac{\bar{\mu} - r}{\bar{\sigma}} z.
\]

\( Y_t \) corresponds to the value of the option and \( Z_t \) is related to the hedging strategy. The option value is given by \( v(t, S_t) = Y_t \) and
Results European call option

Exact solutions $Y_0 = \nu(0, S_0) = 3.66$ and $Z_0 = \bar{\sigma}S_0\nu_S(0, S_0) = 14.15$.

Figure 1: Euler, Milstein and order 2.0 weak Taylor scheme, $\theta = 1$ and $\theta = 0.5$. 
Summary and conclusion

- COS and SWIFT methods for pricing European options
- BCOS method for nonlinear PDEs
- Second-order convergence met 2.0 weak Taylor scheme and $\theta = 1/2$
discretization of integrals!
- Higher dimensions: Generalization to Monte Carlo method SGBM
- Challenges: Higher dimensions (MC, SGBM), stochastic control, portfolio selection

$\Rightarrow$ Towards accurate, efficient and robust solution methods.
Computational finance

- Efficient valuation of financial options
  - COS method is used at many (financial) institutions world-wide;
  - works efficient for a variety of options (barrier, Asian, multi-asset, ...)
  - High-D American options, Monte Carlo simulation, SGBM
  - Interpolation methods, Stochastic Collocation Monte Carlo methods (SCMC sampler)
  - Under recent asset price model dynamics, for different asset classes

- Risk management
  - Accurate hedge parameters;
  - Numerical estimation of tail probabilities and expected shortfall, Value-at-Risk;
  - (Counterparty) Credit risk and other types of risk;
  - Inflation options for pension funds.

- Portfolio optimization (2010-...)
  - Energy portfolio, real options analysis
  - Dynamic portfolios for pensions (target based vs time-consistent mean-variance strategy)

⇒ Challenge: Incorporate (big) financial data
We can write the Euler, Milstein, and 2.0 weak Taylor discretization schemes in the following general form

\[ X_{m+1}^\Delta = x + m(x) \Delta t + s(x) \Delta \omega_{m+1} + \kappa(x)(\Delta \omega_{m+1})^2, \quad X_m^\Delta = x. \]

For the Euler scheme:

\[ m(x) = \mu(x), \quad s(x) = \sigma(x), \quad \kappa(x) = 0. \]

For the Milstein scheme:

\[ m(x) = \mu(x) - \frac{1}{2} \sigma \sigma_x(x), \quad s(x) = \sigma(x), \quad \kappa(x) = \frac{1}{2} \sigma \sigma_x(x). \]

For the order 2.0 weak Taylor scheme:

\[ m(x) = \mu(x) - \frac{1}{2} \sigma \sigma_x(x) + \frac{1}{2} \left( \mu \mu_x(x) + \frac{1}{2} \mu_{xx} \sigma^2(x) \right) \Delta t, \]
\[ s(x) = \sigma(x) + \frac{1}{2} \left( \mu_x \sigma(x) + \mu \sigma_x(x) + \frac{1}{2} \sigma_{xx} \sigma^2(x) \right) \Delta t, \]
\[ \kappa(x) = \frac{1}{2} \sigma \sigma_x(x). \]
The characteristic function of $X_{m+1}^\Delta$, given $X_m^\Delta = x$

$$\hat{f}_{X_{m+1}^\Delta} (u|X_m^\Delta = x) = \mathbb{E} \left[ \exp \left( iuX_{m+1}^\Delta \right) \bigg| X_m^\Delta = x \right]$$

$$= \exp \left( iux + ium(x)\Delta t - \frac{1}{2} u^2 s^2(x)\Delta t \frac{1}{1-2iu\kappa(x)\Delta t} \right) (1 - 2iu\kappa(x)\Delta t)^{-1/2}.$$