Space-time methods for optimal control models in pedestrian dynamics

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Empirical studies of human crowds started about 50 years ago. Nowadays there is a large literature on different micro- and macroscopic approaches available. Challenges: microscopic interactions not clearly defined, multiscale effects, finite size effects,.....
Microscopic and macroscopic models

Microscopic models (model every particle):
- force based models: position of a particle is determined by forces acting on it
- stochastic optimal control models: each agent wants to minimize a stochastic cost functional
- lattice based models: domain divided into cells, particles may (or may not) jump from one cell to another with a certain transition probability

Macroscopic models: number of individuals goes to infinity, nonlinear transport-diffusion equation based on conservation of mass
Nonlinear diffusion transportation models

Intuitive assumption: total number of individuals is conserved in time and the speed of individuals is linked to the density of the surrounding pedestrian flow.

Conservation law:

$$\partial_t \rho(x, t) + \text{div}(F(\rho(x, t))v(x, t)) = 0,$$

where $x \in \Omega \subset \mathbb{R}^d$ with $d = \{1, 2, 3\}$ is the position in space, $t \in (0, T]$ the time, $\rho(x, t)$ the pedestrian density, $v(x, t)$ the velocity and $F(\rho)$ the mobility/penalization function for high densities such as $F(\rho) = \rho_{\max} - \rho$ or $F(\rho) = \rho(\rho_{\max} - \rho)^2$ with $\rho_{\max}$ being the maximal density, e.g., we will choose $\rho_{\max} = 1$ later. See also

Hughes’ model for pedestrian flow

- Pedestrians have a common sense/drive of the task described via a potential $\phi$, where $-\nabla \phi$ gives the direction.
- Pedestrians try to minimize the travel time.
- Pedestrians try to avoid high densities, speed depends on the density of the surrounding pedestrian flow.

$$\rho_t - \text{div}(\rho f^2(\rho) \nabla \phi) = 0,$$

$$|\nabla \phi| = \frac{1}{f(\rho)},$$

where $f(\rho)$ provides a weighting or cost wrt high densities, i.e., saturation for $\rho \to \rho_{\text{max}}$. More detailed discussion can be found in

An optimal control approach for fast exit scenarios

Let us consider an evacuation/fast exit scenario, i.e., a room with one or several exits from which a group wants to leave as fast as possible. Let $\Omega \subset \mathbb{R}^2$, $\Gamma = \partial \Omega = \Gamma_E \cup \Gamma_N$, and $(0, T)$ be the time interval. The minimization reads as: $\min_{\rho, \nu} \mathcal{J}(\rho, \nu)$ with

$$\mathcal{J}(\rho, \nu) = \frac{1}{2} \int_0^T \int_{\Omega} \rho(x, t)|\nu(x, t)|^2 \, dx \, dt + \frac{\alpha}{2} \int_0^T \int_{\Omega} \rho(x, t) \, dx \, dt,$$

where

- kinetic energy
- exit time

subject to

$$\partial_t \rho + \text{div}(\rho \nu) = \frac{\sigma^2}{2} \Delta \rho, \quad \text{in } \Omega \times (0, T),$$

$$\left(-\frac{\sigma^2}{2} \nabla \rho + \rho \nu\right) \cdot n = 0, \quad \text{on } \Gamma_N \times (0, T),$$

$$\left(-\frac{\sigma^2}{2} \nabla \rho + \rho \nu\right) \cdot n = \beta \rho, \quad \text{on } \Gamma_E \times (0, T),$$

$$\rho(\cdot, 0) = \rho_0(\cdot), \quad \text{in } \Omega.$$
Two different approaches to solve the problem

Target: evacuate the group of people as fast as possible

Approach 1: group has to leave the room via the exit(s)

Approach 2: group has to get from one place to the other
→ transport problem
Space-time approach to optimal mass transport


proposed to reset the $L^2$ Monge-Kantorovich mass transfer problem in a fluid mechanics framework: $\min_{\rho, \mathbf{v}} \mathcal{J}(\rho, \mathbf{v})$ with

$$\mathcal{J}(\rho, \mathbf{v}) = \int_0^T \int_{\Omega} \rho(\mathbf{x}, t)|\mathbf{v}(\mathbf{x}, t)|^2 d\mathbf{x} dt,$$

subject to

$$\partial_t \rho + \text{div}(\rho \mathbf{v}) = 0, \quad \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \quad \rho(\mathbf{x}, T) = \rho_T(\mathbf{x}).$$

More efficient numerical treatment proposed in


by setting the momentum $\mathbf{m} = \rho \mathbf{v}$.
Generalization of the optimal control problem

Introducing the function $F(\rho)$ describing the nonlinear mobilities, we consider the following optimization problem:

$$
\min_{\rho,v} \mathcal{J}(\rho, v) = \min_{\rho,v} \frac{1}{2} \int_{Q_T} F(\rho(x, t))|v(x, t)|^2 \, dx \, dt + \frac{\alpha}{2} \int_{Q_T} \rho(x, t) \, dx \, dt,
$$

such that

$$
\begin{align*}
\partial_t \rho(x, t) + \text{div}(F(\rho(x, t))v(x, t)) &= \frac{\sigma^2}{2} \Delta \rho(x, t), & \text{in } \Omega \times (0, T), \\
(F(\rho)v - \frac{\sigma^2}{2} \nabla \rho) \cdot n &= 0, & \text{on } \Gamma_N \times (0, T), \\
(F(\rho)v - \frac{\sigma^2}{2} \nabla \rho) \cdot n &= \beta \rho, & \text{on } \Gamma_E \times (0, T), \\
\rho(x, 0) &= \rho_0(x), & \text{in } \Omega.
\end{align*}
$$
Momentum formulation

Denoting the momentum \( \mathbf{m} = F(\rho) \mathbf{v} \), we can rewrite the minimization functional as

\[
\mathcal{J}(\rho, \mathbf{m}) = \frac{1}{2} \int_0^T \int_\Omega \frac{|\mathbf{m}(\mathbf{x}, t)|^2}{F(\rho(\mathbf{x}, t))} d\mathbf{x} \, dt + \frac{\alpha}{2} \int_0^T \int_\Omega \rho(\mathbf{x}, t) d\mathbf{x} \, dt,
\]

s.t.

\[
\partial_t \rho + \text{div}(\mathbf{m}) = \frac{\sigma^2}{2} \Delta \rho, \quad \text{in } \Omega \times (0, T),
\]

\[
(\mathbf{m} - \frac{\sigma^2}{2} \nabla \rho) \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_N \times (0, T),
\]

\[
(\mathbf{m} - \frac{\sigma^2}{2} \nabla \rho) \cdot \mathbf{n} = \beta \rho, \quad \text{on } \Gamma_E \times (0, T),
\]

\[
\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \quad \text{in } \Omega.
\]
Optimality system

Find the solution \((\rho, m, \lambda)\), where \(\lambda\) is the Lagrange multiplier, of

\[
\begin{align*}
\partial_t \rho + \text{div}(m) - \frac{\sigma^2}{2} \Delta \rho &= 0, \\
\frac{m}{F(\rho)} - \nabla \lambda &= 0, \\
\partial_t \lambda + \frac{F'(\rho)|m|^2}{2F^2(\rho)} + \frac{\sigma^2}{2} \Delta \lambda &= \frac{\alpha}{2}, \\
(m - \frac{\sigma^2}{2} \nabla \rho) \cdot n &= 0, \\
\frac{\sigma^2}{2} \nabla \lambda \cdot n &= 0, \\
(m - \frac{\sigma^2}{2} \nabla \rho) \cdot n &= \beta \rho, \\
\frac{\sigma^2}{2} \nabla \lambda \cdot n + \beta \lambda &= 0, \\
\rho(x, 0) &= \rho_0(x), \\
\lambda(x, T) &= 0,
\end{align*}
\]

in \(\Omega \times (0, T)\),

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in \(\Omega \times (0, T)\),

in \(\Omega \times (0, T)\),

on \(\Gamma_N \times (0, T)\),

on \(\Gamma_E \times (0, T)\),
in \(\Omega\).
A priori estimate for the forward problem

\[ V = L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)), \quad Q = [L^2(\Omega \times (0, T))]^d \]

Lemma

Let \( \rho_0 \in L^2(\Omega) \). Let \( F(\rho) \in C^1(\mathbb{R}) \) be bounded and non-negative for \( 0 \leq \rho \leq \rho_{\text{max}} \) and let \( \sigma > 0, \beta \geq 0 \). Let \( v \in Q \) and let \( \rho \in V \) be a weak solution of

\[
\langle \partial_t \rho, \psi \rangle_{H^{-1}, H^1} + \int_{\Omega} \left( \frac{\sigma^2}{2} \nabla \rho - F(\rho)v \right) \cdot \nabla \psi \, dx = -\int_{\Gamma_E} \beta \rho \psi \, ds,
\]

for all \( \psi \in H^1(\Omega) \). Then there exist constants \( C_1, C_2 > 0 \) depending on \( F, \sigma, \Omega \) and \( T \) only, such that

\[
\| \rho \|_V \leq C_1 \| v \|_Q + C_2.
\]
Existence and uniqueness

**Theorem**

Under the assumptions of the Lemma the variational problem for
\( \min_{(\rho, v) \in V \times Q} J(\rho, v) \) subject to \( \partial_t \rho + \text{div}(F(\rho)v) = \frac{\sigma^2}{2} \Delta \rho \) has at least a weak solution in \( V \times Q \) with given \( \rho_0 \in L^2(\Omega) \).

**Theorem**

For fixed \( \rho_0 \in L^2(\Omega) \), there exists a unique weak solution
\[ (\rho, \lambda) \in L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega)) \]
to the (reduced) optimality system.
Numerical results

The code was implemented in the programming language Julia, see http://julialang.org.

Let $F(\rho) = \rho(1 - \rho)^2$ as in Hughes’ model. The optimality system is discretized by a finite volume method in space-time.

Octree mesh with $32^3$ space-time cubes (32 time slices).

In order to solve the constrained optimization problem, we apply a version of the line search sequential quadratic programming (SQP) method (Newton-type scheme).

We have two types of numerical experiments:
- with BCs, without final time condition, with $\sigma \neq 0$,
- with final time condition, without BCs, with $\sigma = 0$. 
Numerical results - with exits

Initial distribution: $\rho_0(x) = 0.4$ for $x \in [0.2, 0.8] \times [0.5, 0.7]$ and $\rho_0(x) = 0$ elsewhere; $\beta = 10$, $\sigma = 1$, $\alpha = 20$;

22 SQP iterations needed
Numerical results
2 Examples (with higher and lower initial density):

- Initial distribution: $\rho_0(x) = 0.4$ for $x \in [0.5, 0.7] \times [0.4, 0.6]$ and $\rho_0(x) = 0$ elsewhere; terminal distribution: $\rho_T(x) = 0.3$ for $x \in [0.05, 0.25] \times [0.4, 0.6]$, $\rho_T(x) = 0.1$ for $x \in [0.65, 0.85] \times [0.2, 0.4]$ and $\rho_T(x) = 0$ elsewhere.

- Initial distribution: $\rho_0(x) = 0.8$ for $x \in [0.5, 0.7] \times [0.4, 0.6]$ and $\rho_0(x) = 0$ elsewhere; terminal distribution: $\rho_T(x) = 0.5$ for $x \in [0.05, 0.25] \times [0.4, 0.6]$, $\rho_T(x) = 0.3$ for $x \in [0.65, 0.85] \times [0.2, 0.4]$ and $\rho_T(x) = 0$ elsewhere.
Numerical results
Numerical results
Summary and outlook

Summary:
- Optimal control approach to the modeling of pedestrian dynamics
- Space-time solver based on Benamou-Brenier and Haber-Horesh

Outlook:
- Improve the model as well the solver
- More numerical results, e.g., include obstacles in the domain, finer space-time meshes
- Preconditioning
- Adaptive methods in space and time
- New minimal time optimization problem for evacuation
References


Thank you for your attention!