Extensions of discrete valuation rings

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In the following two lectures, we review extensions of discrete valuation rings.

1 Simple extensions

We first provide some elementary means to construct extensions of discrete valuation rings. We start with the more general case of a local ring (which we do not assume to be noetherian).

Let \( A \) be a local ring with maximal ideal \( m \) and residue field \( k \), and let \( f \in A[X] \) be a monic polynomial. Then \( B := A[X]/(f) \) is a finite free ring extension of \( A \), corresponding to a finite and free morphism of schemes

\[
\text{Spec } B \to \text{Spec } A .
\]

Finite morphisms are closed, so closed points are mapped to closed points. This can be seen directly by means of Nakayama’s Lemma: Let \( \mathfrak{n} \subseteq B \) be a maximal ideal; if \( \mathfrak{n} \cap A \neq m \), then \( \mathfrak{n} + mB = B \) and hence \( \mathfrak{n} = B \), which is absurd. Thus all closed points of \( \text{Spec } B \) lie in the special fiber

\[
\text{Spec } (B \otimes_A k) \to \text{Spec } k ,
\]

and we have a natural identification

\[
\text{Spec } (B \otimes_A k) = \prod_{i \in I} \text{Spec } k[X]/(\bar{f}_i^{e_i}) ,
\]

where

\[
\bar{f} = \prod_{i \in I} \bar{f}_i^{e_i}
\]
is the prime factorization of the reduction $\bar{f}$ of $f$ in $k[X]$. Thus, if for each $i \in I$, we choose a lift $g_i \in A[X]$ of $\bar{f}_i$ and if we set $n_i := (m, g_i)$, then the $m_i$ are pairwise distinct, and we obtain a bijection

$$\{n_i ; i \in I\} \cong \text{Max } B$$

In particular, $A[X]/(f)$ is semi-local, and the residue field in $n_i$ is $k[X]/(\bar{f}_i)$.

Recall that a ring is a discrete valuation ring (DVR) if and only if it is a noetherian local ring whose maximal ideal is generated by a non-nilpotent element. The prime spectrum of a DVR has two points, precisely one of which is closed.

### 1.1 Unramified simple extensions

From now on, $A$ will always be a DVR. Let us construct unramified simple extensions:

**Lemma. 1.1.1.** Assume that $\bar{f}$ is irreducible. Then $B$ is a DVR with maximal ideal $mB$ and residue field $k[X]/(\bar{f})$

**Proof.** It suffices to remark that the free homomorphism $A \to B$ is injective, so a non-nilpotent generator maps to a non-nilpotent generator. \hfill $\square$

In the above situation, let $K$ denote the fraction field of $A$. Then $f$ is irreducible in $K[X]$, and $B$ is the integral closure of $A$ in $L := K[X]/(f)$. Indeed, $L = B \otimes_R K$ is a field, and discrete valuation rings are integrally closed (look at an integral equation for $f \in L$ over the ring of integers, and assume that $|f| > 1$).

If $\bar{f}$ is separable, then the finite extension of discretely valued fields $L/K$ which we constructed is unramified.

### 1.2 Totally ramified simple extensions

Let $A$ be DVR, and let us now assume that $f \in A[X]$ is an Eisenstein polynomial:

$$f = X^n + a_1X^{n-1} + \ldots + a_n$$

with $a_i \in m$, $a_n \notin m^2$. 

Lemma. 1.2.1. Then $B$ is a discrete valuation ring with residue field $k$, and the residue class of $X$ in $B = A[X]/(f)$ is a uniformizer.

Proof. Note that $\bar{f} = X^n$, hence the special fiber is a nilpotent thickening, of order $n$, of the closed point. By assumption, the constant term $\pi := a_n$ uniformizes $A$. Let $x$ denote the class of $X$. Then $\pi \in (x)$, hence the maximal ideal $(\pi, x)$ of $B$ is generated by $x$. Since $\pi$ is not topologically nilpotent, neither is $x$.

As before, $f$ is irreducible in $K[X]$, and $B$ is the integral closure of $A$ in $L$, where $K$ and $L$ are defined as above.

2 Integral closure and completion

Let $A$ be a discrete valuation ring with fraction field $K$. Let $L/K$ be any finite separable extension, and let $B$ denote the integral closure of $A$ in $L$.

Lemma. 2.0.2. $B$ is a finite $A$-module.

Proof. Let $\text{Tr} : L \to K$ denote the trace map. By separability, $\text{Tr}(xy)$ is a nondegenerate symmetric $K$-bilinear form on $L$. Let $\{e_i\}$ be a basis for $L$ over $K$, with all $e_i \in B$. Let $V$ denote the free $A$-module spanned by the $e_i$. For every sub-$A$-module $M$ of $L$, we let $M^*$ denote those $x \in L$ such that $\text{Tr}(xy) \in A$ for all $y \in M$. Then one has $B \subseteq B^*$: Indeed, conjugates of integral elements (in a suitable extension of $L$) are integral. Hence:

$$V \subseteq B \subseteq B^* \subseteq V^*$$

However, $V^*$ is finite free over $A$, so $B$ also is, the ring $A$ being noetherian.

Since $A$ is a DVR and since $B$ has no $A$-torsion, the $A$-algebra $B$ is necessarily finite and free. Furthermore, it follows that $B$ is a one-dimensional normal domain, that is, a Dedekind domain.

Let us remark that the statement of the lemma is false in general without separatedness assumption on $L/K$; however, separatedness can be dropped when $R$ is complete or, more generally, excellent.
Inverse limits commute with finite direct sums; hence

\[ \hat{B} = B \otimes_A \hat{A}, \]

where \( \hat{B} \) is the completion of \( B \) with respect \( mB \) or, equivalently, with respect to the Jacobson radical

\[ \text{jac } B = \bigcap_{n \in \text{Max } B} n \]

of \( B \). In the above notation,

\[ \hat{B} = \bigoplus_{i \in I} B_i, \]

where

\[ B_i = \lim_{\rightarrow n} B/n_i^n \]

is the \( n_i \)-adic completion of \( B \), which is a complete discrete valuation ring (CDVR). We see that while \( B \) is a domain, its completion \( \hat{B} \) splits as a finite direct sum of cdvrs, according to the prime factorization of \( mB \).

**Lemma. 2.0.3.** The following holds:

(i) If \( A \) is complete, then so is \( B \), and \( B \) is a DVR. That is, finite extensions are complete and unsplit.

(ii) If \( B \) is a DVR, then \( \hat{L} = L \otimes_K \hat{K} = L \hat{K} \), and \( L \cap \hat{K} = K \) in \( \hat{L} \).

**Proof.** Statement (i) is trivial in view of the above discussion: If \( A = \hat{A} \), then \( \hat{B} = B \otimes_A \hat{A} = B \) which is a domain; since \( \hat{B} \) is a direct product of CDVRs, we see that \( B \) is a CDVR, as claimed.

Let’s prove (ii). Clearly \( \hat{L} = \hat{B} \otimes_A K \), which is \( (B \otimes_A \hat{A}) \otimes_A K \). As for the last statement, let \( K' \) denote \( L \cap \hat{K} \), and let \( f \in K[X] \) be the minimal polynomial of a generating primitive element for \( K'/K \). If it had degree > 1, then \( L \otimes_K K' \) would split nontrivially as a direct sum. Then \( L \otimes_K \hat{K} \) would split as well, contradiction.

The situation considered in (ii) actually appears in practice: If \( K \) is complete and if \( E \) is an unramified separable extension of \( K \) that is not finite, then the integral closure \( \mathcal{O}_E \) of \( A \) in \( E \) is a non-complete DVR, and for any finite extension \( L \) over \( E \), the integral closure of \( \mathcal{O}_E \) in \( L \) (which coincides with the integral closure of \( A \) in \( L \)) is a DVR.
3 Unramified extensions and reduction

Let us now consider the situation where \( L/K \) is Galois, where \( k \) is perfect and where \( mB \) does not split. An element of \( \text{Gal}(L/K) \) preserves the property of elements of \( L \) of being integral over \( A \), so it induces an automorphism of \( B \) over \( A \), which must necessarily preserve the maximal ideal. Hence, it induces an automorphism of \( k_L \) over \( k \), where \( k_L \) denotes the residue field of \( B \). We easily check that we obtain a homomorphism \( \text{Gal}(L/K) \to \text{Aut}(k_L/k) \).

**Proposition.** 3.0.4. The extension \( k_L/k \) is normal, hence Galois, and the homomorphism just described is surjective.

**Proof.** Normality is seen as follows. Consider an element \( \bar{x} \in k_L \), and lift it to an element \( x \in B \). Let \( \bar{g} \) be the minimal polynomial of \( \bar{x} \) over \( k \), and let \( f \) be the minimal polynomial of \( x \) over \( K \). Then \( f \) splits completely in \( L[X] \), and since Galois conjugation preserves valuations, we see that \( f \in A[X] \). Hence, \( \bar{g} \) divides \( \bar{f} \), which splits completely in \( k_L[X] \), and it follows that \( \bar{g} \) splits completely in \( k_L[X] \). It follows that the extension \( k_L/k \) is normal.

Let us now choose \( \bar{x} \) to be a primitive element, and let \( \bar{\sigma} \) be any element of \( \text{Gal}(k_L/k) \). Then \( \bar{\sigma}(\bar{x}) \) is a zero of \( \bar{g} \), hence of \( \bar{f} \), which lifts to a zero \( x' \in B \) of \( f \). The Galois action on the set of roots of an irreducible polynomial is transitive, so we find \( \sigma \in \text{Gal}(L/K) \) such that \( \sigma x = x' \). Since \( k_L \) is generated by \( \bar{x} \), this \( \sigma \) lifts \( \bar{\sigma} \).

The order of \( \text{Gal}(k_L/k) \) being equal to the residue degree \( f = [k_L : k] \), we see that the kernel \( I \) of the above homomorphism, the inertia subgroup of \( \text{Gal}(L/K) \), has order equal to the ramification index of \( L \) over \( K \), by the formula \( [L : K] = ef \).

4 Some examples

Let’s consider cyclotomic extensions. We look at \( \mu_n \), the \( n \)-th roots of unity, zeros of the polynomial \( X^n - 1 \), in a fixed algebraic closure of \( K \). Let us first consider the case where \( n \) is prime to the characteristic of \( k \); then certainly \( n \) is prime to the characteristic of \( K \), so \( \mu_n \) has \( n \) elements. Let \( \zeta \) be a generator of \( \mu_n \), that is, a primitive \( n \)-th root of unity. We set \( L = K(\mu_n) = K(\zeta) \). Let
us assume that $B$ is a DVR (which is automatic when $K$ is complete) and that $k$ is perfect. Observe that $\zeta \in B^*$.

**Proposition. 4.0.5.** The extension $L/K$ is unramified.

**Proof.** We must show that the surjective reduction homomorphism

$$\text{Gal}(L/K) \to \text{Gal}(k_L/k)$$

is in fact bijective. Hence it suffices to prove that if $\zeta^i \equiv \zeta^j$ modulo $m_L$, then $\zeta^i = \zeta^j$. So assume that $\zeta^i - \zeta^j \in m_L$; it follows that $\zeta^{i-j} - 1 \in m_L$, so we must show is that

$$\zeta^i - 1 \notin m_L \quad \text{for} \quad 1 \leq i \leq n - 1 \ .$$

However, substituting $X = 1$ in the identities

$$\prod_{i=1}^{n-1} (X - \zeta^i) = \frac{X^n - 1}{X - 1} = X^{n-1} + \ldots + X + 1$$

yields $\prod_{i=1}^{n-1} (1 - \zeta^i) = n$, which is nonzero in $k_L$. \hfill \Box

Let us now see what happens if we adjoin $p$-power roots of unity. We stick to the case $K = \mathbb{Q}_p$, for any prime $p$. Let, again, $\zeta$ be a primitive $n$-th root of unity, where now $n = p^m$ for some $m$. In this case, we have:

**Proposition. 4.0.6.** The extension $L/K$ is totally ramified of degree $\varphi(n) = (p-1)p^{m-1}$, and its Galois group is naturally identified with $(\mathbb{Z}/n\mathbb{Z})^*$. Moreover, $\zeta - 1$ is a uniformizer of $B$, and $B = A[\zeta]$.

**Proof.** Let us write $\pi = \zeta - 1$. Clearly $\zeta^{p^{m-1}}$ is a primitive $p$-th root of unity, hence $\zeta$ is a root of

$$F(X) := X^{p^{m-1}(p-1)} + X^{p^{m-1}(p-2)} + \ldots + 1 \ .$$

In other words, $\pi$ is a root of $F(X + 1)$. This polynomial is Eisenstein of degree $p^{m-1}(p-1)$. Indeed: Its constant term is $F(1) = p$, and it reduces to $X^{p^{m-1}(p-1)} = 0$ modulo $p$. The last statement is seen as follows: We know that

$$F(X) = \frac{X^{p^n} - 1}{X^{p^{m-1}} - 1} \ .$$
Modulo \( p \), this is
\[
\frac{(X-1)^{p^n}}{(X-1)^{p^n-1}} = (X-1)^{p^n-p^{n-1}} = (X-1)^{(p-1)p^{n-1}}.
\]

The statement concerning \( F(X+1) \) is now clear, and so are the remaining assertions of the proposition. \( \square \)

5 Hensel’s Lemma

A nonzero polynomial in \( A[X] \) is called \textit{primitive} if it has Gauss norm one; in particular it is primitive whenever it is monic. Hensel’s Lemma says that if the reduction \( \bar{f} \) admits a factorization into relatively prime factors, then this factorization lifts to a factorization in \( A[X] \) such the degree of one factor is preserved. In particular, consider the case where \( \bar{f} \) has a simple zero.

Since \( X^{p-1} - 1 \) splits over \( \mathbb{F}_p \) into \( p - 1 \) distinct factors, it follows that it already splits in \( \mathbb{Z}_p \); that is, \( \mathbb{Q}_p \) contains all \( (p-1) \)-st roots of unity. More generally, it follows:

\textbf{Lemma. 5.0.7.} Let \( K \) be a local field with residue field \( \mathbb{F}_q \) of characteristic \( p > 0 \), and let \( n \) be an integer prime to \( p \). Then \( \mu_n \subseteq k \) if and only if \( \mu_n \subseteq K \).

We give a direct proof: If \( \mu_n \subseteq K \), then \( \mu_n \subseteq A \), and the inclusion of \( \mu_n \) into \( k \) follows by reduction. Let us prove the converse implication. Let \( \zeta_1 \in A \) be any lift of a primitive \( n \)-th root of unity. It suffices to construct a sequence \( (\zeta_m)_m \) in \( A \) such that \( \zeta_m^n \equiv 1 \) modulo \( m^m \) and such that \( \zeta_{m+1}^n \equiv \zeta_m \) modulo \( m^m \). The last condition will guarantee convergence, while the first condition will imply that the limit is an \( n \)-root of unity. (The limit is then necessarily a primitive \( n \)-th root of unity, for if a lower power was equal to one, then the same would hold for the reduction.) We proceed by induction on \( m \). Let us assume that \( \zeta_m \) has been defined. Then
\[
\zeta_m^n \equiv 1 + \alpha \pi^m \text{ modulo } m^{m+1},
\]
and we attempt to find \( \zeta_{m+1} \) in the form
\[
\zeta_{m+1} = \zeta_m + \beta \pi^m,
\]
where modulo $m^{m+1}$ the term
\[
\zeta_{m+1}^n \equiv \zeta_m^n + n\zeta_m^{n-1}\beta\pi^m \\
\equiv 1 + (\alpha + n\zeta_m^{n-1}\beta)\pi^m
\]
needs to be 1. Hence, it suffices to take $\beta = -\alpha/(n\zeta_m^{n-1})$, which is possible since $\zeta_m$ is a unit in $A$ by construction and since the same holds for $n$ by assumption.

6 Infinite extensions

In the following, we write $\mathcal{O}_K$ instead of $A$ and $m_K$ instead of $m$, similarly for all other valued fields that we will encounter.

Let $K$ be the fraction field of a CDVR $\mathcal{O}_K$ with perfect residue field $k$. We fix a separable algebraic closure $K^{\text{sep}}$ of $K$; all separable algebraic extensions of $K$ will be considered within $K^{\text{sep}}$. Let $E/K$ be a separated algebraic extension, and let $\mathcal{O}_E \subseteq E$ denote the integral closure of $\mathcal{O}_K$ in $E$.

Let $I$ denote the filtered set of finite separable sub-extensions $K_i$ of $K$ in $E$. Then $\mathcal{O}_E = \bigcup \mathcal{O}_{K_i}$, for this union is the set of elements in $E$ which are integral over $\mathcal{O}_K$ and contained in some $K_i$, the second condition being vacuous. The inclusions $\mathcal{O}_{K_i} \subseteq \mathcal{O}_{K_j}$ are local, so obtain a filtered union $p_E = \bigcup p_i$ in $\mathcal{O}_E$, which is easily checked to be an ideal. Moreover,
\[
\mathcal{O}_E^* = \bigcup_{i \in I} \mathcal{O}_{K_i}^* = \mathcal{O}_E \setminus p_E.
\]

Hence $\mathcal{O}_E$ is a local ring (even a valuation ring) with maximal ideal $p_E$; its fraction field is clearly $E$.

6.1 Finitely ramified extensions

Definition. 6.1.1. Let $E/K$ be a separable algebraic extension. It is called

(i) unramified if is a union of unramified extensions,
(ii) totally ramified if it is a union of totally ramified extensions and
(iii) finitely ramified if it is a finite extension of an unramified extension.
Lemma. 6.1.2. Let $E \subseteq K^{\text{sep}}$ be finitely ramified over $K$. Then

(i) $\mathcal{O}_E$ is a DVR.

(ii) If $E'/E$ is finite separable, then $E' \hat{E} = \hat{E}'$, and $\hat{E} \cap E' = E$.

(iii) $\hat{E} \cap K^{\text{sep}} = E$.

(iv) For $E, E'$ finitely ramified over $K$, $\hat{E} = \hat{E}'$ implies $E = E'$, where $\hat{E}$ and $\hat{E}'$ are within a fixed separable closure of the completion of $K^{\text{ur}}$.

Proof. To prove (i), let us choose an intermediate field $K \subseteq \hat{E} \subseteq E$ such that $\hat{E}/K$ is unramified and such that $E/\hat{E}$ is finite. Then $p_{\hat{E}} = p_K \mathcal{O}_{\hat{E}}$ since this holds for all $K$-finite subextensions of $\hat{E}/K$, so $p_{\hat{E}}$ is generated by a non-nilpotent element, and hence $\mathcal{O}_{\hat{E}}$ is a DVR. It follows that $\mathcal{O}_E$ is a local Dedekind domain and, hence, a DVR. To see this it suffices to note that $\mathcal{O}_E$ is the integral closure of $\mathcal{O}_{\hat{E}}$ in $E$. Indeed: If $x \in E$ is integral over $\mathcal{O}_{\hat{E}}$, then it is integral over $\mathcal{O}_E$ for trivial reasons. Conversely, if it is integral over $\mathcal{O}_{\hat{E}}$, then also over $\mathcal{O}_K$ (use that finiteness implies integrality).

Statement (ii) is now a special case of Lemma 2.0.3 (ii).

Let us show (iii). The intersection $\hat{E} \cap K^{\text{sep}}$ is the union of the fields $\hat{E} \cap E'$, where $E'$ varies in the cofinal set of finite separable extensions of $K$ that contain $E$. The claim now follows from statement (ii).

To prove (iv) follows by intersecting with $K^{\text{sep}}$. ∎

6.2 Complete extensions

Definition. 6.2.1. A field extension $L/K$ is called complete if $L = \hat{E}$ for some finitely ramified extension $E$ of $K$.

Let us recall that completion leaves residue fields invariant. By Lemma 6.1.2, we have to mutually inverse bijections

$$\{E/K \text{ finitely ramified}\} \cong \{L/K \text{ complete extension}\}$$

given by completion and by intersection with $K^{\text{sep}}$ respectively. Indeed, $\hat{E} \cap K^{\text{sep}} = E$, and for $L = \hat{E}$, the completion of $E = L \cap K^{\text{sep}}$ is $L$. In particular, we see that the topology on $L$ can be recovered from the abstract field extension $L/K$ and the topology on $K$. 

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We use this correspondence to define complete unramified and complete totally ramified extensions. The complete totally ramified extensions are precisely those whose residue field extensions are trivial.

From now on, let us assume that the residue field $k$ of $K$ is finite. We define when a complete extension $L/K$ is Galois, and we define its Weil group. To motivate our definition, let us first consider finite separable extensions $E'/E/K$ where $E/K$ is unramified (and hence normal). We have a short exact sequence

$$0 \to \text{Aut}(E'/E) \to \text{Aut}(E'/K) \to \text{Gal}(E/K)$$

Lemma. 6.2.2. The extension $E'/K$ is Galois if and only if

(i) $\text{Frob}_K \in \text{Gal}(E/K)$ lifts to an element in $\text{Aut}(E'/K)$, and

(ii) $\text{Aut}(E'/E)$ has order $[E': E]$.

Proof. Since $E/K$ is finite, the group $\text{Gal}(E/K)$ is a finite cyclic group generated by $\text{Frob}_K$, so $\text{Frob}_K$ lifts to $\text{Aut}(E'/K)$ if and only if the restriction homomorphism from $\text{Aut}(E'/K)$ to $\text{Gal}(E/K)$ is surjective.

If $E'/K$ is normal, then the restriction homomorphism is surjective, and $E'/E$ is normal, so the desired implication is clear.

Conversely, let us assume that conditions (i) and (ii) hold. By (ii), the extension $E'/E$ is normal. Let $x$ be any element of $E'$, and let $f \in K[X]$ denote its minimal polynomial over $K$; we must show that $f$ splits completely in $E'[X]$. Let $x'$ be any zero of $f$ in $K^{\text{sep}}$. There exists a $K$-isomorphism $\varphi$ of $K^{\text{sep}}$ sending $x$ to $x'$. Since $E/K$ is normal, $\varphi$ restricts to an automorphism $\varphi|_E$ of $E$ over $K$. By (i), there exists an automorphism $\psi$ of $K^{\text{sep}}$ restricting to an automorphism of $E'$ over $K$ such that $\psi^{-1}\varphi$ leaves $E$ invariant. Since $E'/E$ is normal, $\psi^{-1}\varphi$ restricts to an automorphism of $E'$. It follows that $\varphi$ already restricts to an automorphism of $E'$.

Let now $E/K$ be any unramified extension, let $E'/E$ be a finite separable extension, and let $L'/L/K$ be the associated completions. By Lemma 6.1.2, $L' = E'L$ with $E' \cap L = E$, hence

$$[L': L] = [E': E]$$
We have natural injective homomorphisms

\[ \text{Gal}(E/K) \subseteq \text{Aut}(L/K) \]
\[ \text{Aut}(E'/K) \subseteq \text{Aut}(L'/K) \]

which are onto the sets of continuous automorphisms. For the following definition, we choose \( E \) such that \( E'/E \) is totally ramified; this condition determines \( E \) uniquely.

**Definition. 6.2.3.** The extension \( L'/K \) is called Galois if

(i) \( \text{Frob}_K \in \text{Aut}(L/K) \) lifts to an element of \( \text{Aut}(L'/K) \), and
(ii) \( \text{Aut}(L'/L) \) has order \( [L' : L] \).

More precisely speaking, statement (i) says that there exists a \( K \)-automorphism of \( L' \) restricting to a \( K \)-automorphism of \( L \) such that this restriction coincides with the Frobenius automorphism. Alternatively, one may consider the subgroup of all elements in \( \text{Aut}(L'/K) \) restricting to automorphisms of \( L/K \); one has a restriction homomorphism from this group to \( \text{Aut}(L/K) \), and condition (i) says that Frobenius lies in the image of this restriction homomorphism.

**Lemma. 6.2.4.** If \( E'/K \) is Galois, then \( L'/K \) is Galois as well.

**Proof.** Indeed, then Frobenius extends to a \( K \)-automorphism of \( E' \) which induces a lift of Frobenius in \( \text{Aut}(L'/K) \), thus establishing property (i), and property (ii) follows from the fact that \( L' \) is the \( E \)-linearly disjoint compositum of \( E' \) and \( L \); \( L'/L \) is then a splitting extension for a separable family of polynomials, hence Galois, and \( \text{Gal}(E'/E) \) is naturally identified with \( \text{Gal}(L'/L) \) via the base change \( \cdot \otimes_E L \). \( \square \)

If \( L'/K \) is Galois, we define the Weil group \( W(L'/K) \subseteq \text{Aut}(L'/K) \) to be the preimage of Frobenius under the restriction homomorphism (defined on the subgroup of automorphisms that restrict to automorphisms of \( L/K \)).

**Proposition. 6.2.5.** Assume that \( E'/K \) is Galois; then \( \text{Gal}(E'/K) \subseteq \text{Aut}(L'/K) \) restricts to an automorphism of Weil groups.
Proof. Indeed, consider the two short exact sequences attached to the Weil groups and the canonical homomorphism between them. We have seen on the level of kernels we have an isomorphism. On the level of cokernels, we obtain the identity on the cyclic subgroup of $\text{Gal}(E/K) \subseteq \text{Aut}(L/K)$ generated by $\text{Frob}_K$, so the statement is clear. □

More generally, we say that a totally ramified extension $L'$ of a complete unramified extension $L$ is Galois over $K$ if it is a union of finite subextensions of $L'/L$ that are Galois over $K$. Again, the Weil group of $L'/K$ is defined to be the group of $K$-automorphisms of $L'$ that restrict to elements of $\text{Aut}(L/K)$ lying in the cyclic subgroup generated by the Frobenius. It is the projective limit of the Weil groups $W(L''/K)$ with $L''/L$ finite, contained in $L'$, Galois over $K$. In the case $L' = E'L$ with $E'/E$ unramified and $E'/K$ Galois, passing to the limit again yields an isomorphism of Weil groups $W(E'/K) \cong W(L'/K)$.