UNIVERSITÄT DUISBURG ESSEN
INSTITUT FÜR EXPERIMENTELLE MATHEMATIK

Uniformly rigid spaces and Néron models of formally finite type

Christian Kappen

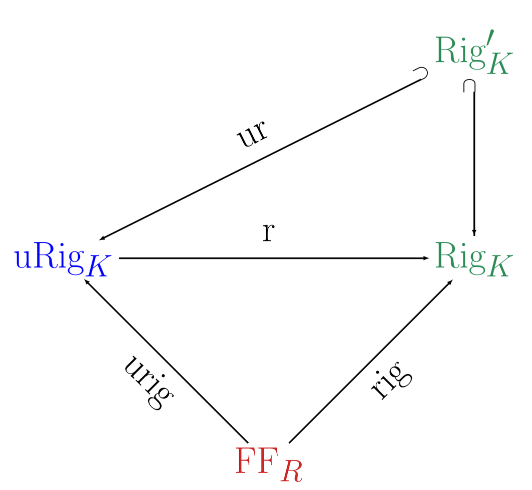
Uniformly rigid spaces: geometry of bounded functions on semi-open non-archimedean polydiscs

Generic fiber functor rig known as "Berthelot's construction", extends Raynaud's functor:

FF_R, formal schemes of locally formally finite (ff) type over cdvr R, i.e. locally $\cong \mathrm{Spf}(R[[S_1,\ldots,S_m]]\langle T_1,\ldots,T_n\rangle/\text{ideal})$	$\xrightarrow{\text{rig}}$	Rig_K, rigid spaces over $K = \mathrm{Frac}\,R$
affine object $\mathrm{Spf}(R[[S]])$	$\longmapsto$	open unit disc, not quasi-compact $\mathbb{D}^1_K = \{x \in \hat{\bar{K}} \mid \lvert x\rvert < 1\}$
Problem: $\nexists$ model $\mathfrak{D} \to \mathfrak{P}$ with $\mathfrak{D}$ and $\mathfrak{P}$ quasi-compact	$\longmapsto$	morphism $\mathbb{D}^1_K \to \mathbb{P}^1_K$ defined by unbounded function

Solution (Kappen 2010):

- Define category sAff_K of semi-affinoid K-spaces as the opposite of the category of K-algebras that are isomorphic to $(R[[S_1,\ldots,S_m]]\langle T_1,\ldots,T_n\rangle/\text{ideal}) \otimes_R K$; write $\mathrm{sSp}\,A$ for semi-affinoid K-space corresponding to K-algebra A.
- Consider $\mathrm{Max}\,A$ as set of physical points of $X = \mathrm{sSp}A$; define semi-affinoid pre-subdomains in terms of a universal property, obtain presheaf $\mathcal{O}_X$ with values in semi-affinoid K-algebras.
- Define semi-affinoid subdomains and Grothendieck- or G-topology on $\mathrm{sSp}A$ in terms of iterated admissible blowups, open immersions on completion morphisms. A formal blowup is called admissible if it can be defined by a π-adically open ideal, where π is a uniformizer of R.
- An acyclicity theorem shows that $\mathcal{O}_X$ is a sheaf; it yields a fully faithful embedding of sAff_K into the category lrGtop_K of saturated locally ringed G-topological K-spaces.
- Define category uRig_K of uniformly rigid K-spaces as full subcategory of lrGtop_K consisting of objects locally isomorphic to objects in sAff_K.
- Glueing works, and uRig_K admits fibered products. Extension of scalars $\cdot \otimes_R K$ yields a generic fiber functor $\mathrm{urig} : \mathrm{FF}_R \to \mathrm{uRig}_K$ taking affines to semi-affinoids (which are quasi-compact).
- Berthelot's functor rig yields functor $\mathrm{r} : \mathrm{uRig}_K \to \mathrm{Rig}_K$, and for each $X \in \mathrm{uRig}_K$ there is a final morphism $\mathrm{r}(X) \to X$ in lrGtop_K inducing a bijection of points and isomorphisms of stalks. Thus, we may view X as a rigid space equipped with a uniform structure.
- On the category Rig'_K of quasi-compact and quasi-separated rigid K-spaces, Raynaud's theory of models of locally topologically finite (tf) type yields a section ur of r. The following diagram of functors commutes up to isomorphism:



- A coherent ideal on $\mathrm{sSp}\,A$ in sAff_K is associated to an ideal in A. There is a good notion of closed uniformly rigid subspaces.
- If $\mathfrak{X}$ and $\mathfrak{Y}$ are objects in FF_R with uniformly rigid generic fibers X and Y such that X is separated, then the schematic closure of the graph of any morphism $Y \to X$ in $\mathfrak{Y} \times \mathfrak{X}$ exists.
- The above theory complements "orthogonally" the theories of Berkovich and Huber.

Néron models of formally finite type

Recall: $\mathfrak{X} \in \mathrm{FF}_R$ is of locally topologically finite (tf) type if it is locally isomorphic to $\mathrm{Spf}(R\langle T_1,\ldots,T_n\rangle/\text{ideal})$. We let K^{sh} denotes the strict henselization of K.

Definitions (Bosch-Schlöter 1995, Kappen 2010):

Let X be a smooth rigid K-space. The Néron model of X is a universal pair $(\mathfrak{X}, u)$, where $\mathfrak{X}$ is a smooth formal R-scheme of locally topologically finite (tf) type and where $u : \mathfrak{X}^{\mathrm{rig}} \to X$ is a morphism in Rig_K.

Let X be a smooth uniformly rigid K-space. The Néron model of X is a universal pair $(\mathfrak{X}, u)$, where $\mathfrak{X}$ is a smooth formal R-scheme of locally formally finite (ff) type and where $u : \mathfrak{X}^{\mathrm{urig}} \to X$ is a morphism in uRig_K.

Considering formal Néron models of ff type for rigid K-spaces does not yield a good theory, which is due to the phenomenon explained above in the column on the left.

Theorem (Bosch and Schlöter 1995):
Let G be a rigid K-group. If $G(K^{\mathrm{sh}})$ is bounded, then the Néron model $(\mathfrak{G}, u)$ of G exists, and $\mathfrak{G}$ is quasi-compact. If G is the analytification of an algebraic K-group whose Néron model $\mathcal{G}$ exists where $\mathcal{G}$ is quasi-compact or G is commutative, then the Néron model $(\mathfrak{G}, u)$ of G exists as well, and it is obtained from $\mathcal{G}$ via π-adic completion. In both cases, u is a retrocompact open immersion. □

Theorem (Wegel 1997):
In the above situation, $\mathfrak{G}^{\mathrm{rig}}$ contains all formally unramified points of G. □

Theorem (Kappen 2010):
Let G be a quasi-paracompact smooth rigid K-group, let $\mathfrak{G}$ be a quasi-paracompact smooth formal R-group scheme of locally tf type, and let $u : \mathfrak{G}^{\mathrm{rig}} \hookrightarrow G$ be a retrocompact open immersion respecting the group structures. If $\mathfrak{G}^{\mathrm{rig}}$ contains all formally unramified points of G, then $(\mathfrak{G}, u^{\mathrm{ur}})$ is the Néron model of G^{ur}. □

We conclude that for any reasonable rigid K-group G, the Néron model $\mathfrak{G}$ of G, which is of locally tf type, satisfies the stronger universal property of the Néron model of G^{ur}. Using completion and descent techniques, one can now prove the existence of Néron models for uniformly rigid groups which do not lie in the image of ur; the resulting Néron models can be of ff type (cf. the column on the right).

Methods needed in the proof of the above Theorem:

- Uniformly rigid spaces.
- Higher direct images of coherent sheaves under proper morphisms in locally noetherian formal geometry: behavior with respect to flat base change and uniformly rigid generic fibers.
- Grothendieck's formal existence theorem applied to reductions modulo powers of a uniformizer π makes it possibly pass to π-adic formal schemes (not of locally tf type) in local situations. This leads to a theory of local envelopes for uniformly rigid spaces.
- Formally unramified points appear on the 'compactifying boundaries' of envelopes of formally smooth uniformly rigid spaces. These points become visible in the framework of Huber's adic spaces.

The Base Change conductor of an abelian variety with potentially multiplicative reduction

This is work in progress with C.-L. Chai. The residue field of K is assumed to be perfect of characteristic $p > 0$; for simplicity, we will even assume it to be algebraically closed. We use a superscript NM to indicate the formation of the Néron model. Let G be an semi-abelian K-variety, and let L/K be a finite Galois extension such $G \otimes_K L$ has semi-abelian reduction; then the formation of the Néron model of $G \otimes_K L$ commutes with further base change.

Definition (base change conductor):

$$c(G/K) := \frac{1}{e_{L/K}} \mathrm{length}_{\mathcal{O}_L} \operatorname{coker} \mathrm{Lie}\left((G^{\mathrm{NM}} \otimes_{\mathcal{O}_K} \mathcal{O}_L) \to (G \otimes_K L)^{\mathrm{NM}}\right)$$

The number $c(G/K) \in \mathbb{Q}_{\geq 0}$ is an important arithmetic invariant of G/K. It is in general not invariant under isogenies, but it is an isogeny invariant on the category of algebraic tori and on the category of abelian K-varieties with potentially multiplicative reduction. This follows from work of Chai, Yu and de Shalit. One of their main results is the following;

Theorem (Chai, Yu, de Shalit 2001):
Let T/K be an algebraic torus, let L/K be a Galois splitting field for K, let $X^*(T/K)$ be the cocharacter group of T/K, viewed as a $\mathrm{Gal}(L/K)$-module, and let χ denote the character of $X^*(T/K) \otimes_{\mathbb{Z}} \mathbb{Q}$; then

$$c(T/K) = \frac{1}{2}(\mathrm{Ar}_{L/K}, \chi)\,,$$

where $\mathrm{Ar}_{L/K}$ denotes the Artin character of L/K. □

If A is an abelian K-variety with potentially multiplicative reduction and if L/K is a stabilizing Galois extension for A, then the formal completion of $(A \otimes_K L)^{\mathrm{NR}}$ along its unit section is a formal $\mathcal{O}_L$-torus whose cocharacter group $X^*(A)$ is a finite free $\mathbb{Z}_p$-module equipped with a $\mathrm{Gal}(L/K)$-action. Let χ denote the character of $X^*(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Chai has defined a class function $\mathrm{bAr}_{L/K}$ with values in cyclotomic extensions of $\mathbb{Q}_p$ such that

$$\mathrm{bAr}_{L/K} + \overline{\mathrm{bAr}}_{L/K} = \mathrm{Ar}_{L/K}\,,$$

where a bar denotes conjugation, and he conjectured that

$$c(A/K) = (\mathrm{bAr}_{L/K}, \chi)\,.$$

Chai outlined a strategy to derive this formula from the above result for tori; his arguments require a theory of Néron models of formally finite type for non-archimedean analytic groups. The strategy for the proof of the above formula is now the following:

- Using Galois descent, define a category $\mathcal{C}_K$ of strongly concordant groups in uRig_K.
- Elementary calculations show that $\mathcal{C}_K$ is the pseudo-abelian envelope of $\mathrm{Tori}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$.
- Using the theory on the left, one shows that objects in $\mathcal{C}_K$ admit Néron models and that their base change conductor can be defined.
- One has natural localization functors from Tori_K and from the category of abelian K-varieties with potentially multiplicative reduction to $\mathcal{C}_K$. On the level of Néron models, they correspond to formal completion along closed subgroups of special fibers. In particular, they preserve base change conductors.
- Using the formula for tori above, one proves the formula $c(G/K) = (\mathrm{bAr}_{L/K}, \chi)$ for objects G in $\mathcal{C}_K$. One thereby obtains the desired formula for abelian K-varieties with potentially multiplicative reduction.