Iterative regularization of non-smooth equations

Christian Clason       Vu Huu Nhu

Fakultät für Mathematik, Universität Duisburg-Essen

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**Motivation**

**Inverse problem:** find

- unknown parameter $u^\dagger$
  - e.g., heat source, diffusion constant, thermal conductivity, heat capacity, latent heat density, ...

given

- measurement $y$
- model $S : u \mapsto y$, e.g., solution of PDE

$\leadsto$ solve

$$S(u) = y$$

**Problem:** measurement $y = y^\delta$ noisy, range of $S$ not closed

$\leadsto$ ill-posed, needs regularization
Solve approximate, stable problem:

1. Tikhonov regularization \( \sim \) optimal control

2. Iterative regularization, e.g., Landweber iteration

\[
    u_{n+1} = u_n + w_n S'(u_n)^* (y^\delta - S(u_n)) \quad n = 1, \ldots, N
\]

- \( S'(u) \) Fréchet derivative, \( S'(u)^* \) adjoint
- Stopping index \( N = N(\delta) < \infty \) regularization parameter
- Regularization: \( N(\delta) \to \infty, u_{N(\delta)} \to u^\dagger \) for \( \delta \to 0 \)

Here: \( S \) solution operator for non-smooth PDE

\( \sim \) not Fréchet differentiable
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Non-smooth equation

\[-\Delta y + \max(0, y) = u\]

solution operator \( S : u \mapsto y (:= S(u)) \)

- well-posed (in suitable – standard – spaces)
- Lipschitz continuous
- completely continuous (\( \sim \) ill-posed)
- not Fréchet differentiable unless \(|\{x : y(x) = 0\}| = 0\)
- model for membrane under water
- can be extended to arbitrary \( f(y) \) piecewise differentiable
- simplified model for sharp phase transition (Stefan problem)
Non-smooth equation

\[-\Delta y + \max(0, y) = u\]

solution operator \( S : u \mapsto y (:= S(u)) \)

- not Fréchet differentiable unless \(|\{ x : y(x) = 0 \}| = 0\)
- but: directionally differentiable

Directional derivative \( S'(u; h) =: \eta \) solves

\[-\Delta \eta + 1_{\{y=0\}} \max(0, \eta) + 1_{\{y>0\}} \eta = h\]

not linear in \( h \sim \) not useful for algorithm
Non-smooth equation

Bouligand subdifferential

\[
\partial_B S(u) := \begin{cases} 
G \text{ linear} & \text{there is } \{u_n\} \text{ Gâteaux differentiable with } \\
& u_n \to u \text{ and } S'(u_n; h) \to G h \text{ for all } h 
\end{cases}
\]

\[-\Delta \eta + \mathbb{1}_{\{y>0\}} \eta = h\]

\(G_u : h \mapsto \eta\)

- \(G_u \in \partial_B S(u)\) Bouligand derivative [Christof/Meyer/Walter/C.]
- \(u \mapsto G_u\) uniformly bounded (in right spaces)
- linear \(\sim\) use for Landweber in place of \(S'(u)\)
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Bouligand–Landweber iteration

\[ u_{n+1}^\delta = u_n^\delta + w_n G_{u_n^\delta}^* \left( y^\delta - S(u_n^\delta) \right), \quad n = 0, 1, 2, \ldots, N(\delta) \]

- \( S: u \mapsto y \) non-smooth
- \( y^\delta \) with \( \| y^\delta - y^\dagger \| \leq \delta, \quad y^\dagger = S(u^\dagger) \) (assume unique)
- \( u_0^\delta = u_0 \) starting point
- \( w_n \) step sizes
- stopping index \( N(\delta) \) by discrepancy principle:

\[ \| y^\delta - S(u_{N(\delta)}^\delta) \|_Y \leq \tau \delta < \| y^\delta - S(u_n^\delta) \|_Y, \quad 0 \leq n < N(\delta) \]

(modified Landweber iteration [Scherzer ’95])
Well-posedness

Assume:

1. \( \{G_u\} \) uniformly bounded
2. generalized tangential cone condition (GTCC)
   
   \[ \|S(u') - S(u) - G_u(u' - u)\| \leq \mu \|S(u') - S(u)\| \quad \text{for all } u, u' \in B_\rho(u^\dagger) \]

   non-smooth PDE: satisfied for \( 1 > \mu > C(\{|x : y^\dagger(x) = 0\}|) \)

3. \( u_0 \in B_\rho(u^\dagger) \)

Then (under conditions on \( \mu, \tau, w_n \)):

- \( u^\delta_n \in B_\rho(u^\dagger) \) for all \( n \leq N(\delta) \)
- \( \delta > 0: N(\delta) < \infty \) and \( \|u^\delta_n - u^\dagger\| < \|u^\delta_{n-1} - u^\dagger\| \) for \( n \leq N(\delta) \)
- \( \delta = 0: N(\delta) = \infty \) and \( u^0_n \to u^\dagger \) for \( n \to \infty \)
Goal: show that \( u^\delta_{N(\delta)} \to u^\dagger \) for \( \delta \to 0 \)

Standard proof: combine

1. monotonicity: \( \| u^\delta_n - u^\dagger \| < \| u^\delta_{n-1} - u^\dagger \| \) for \( n \leq N(\delta) \)

2. stability: \( u^\delta_n \to u^0_n \) for all \( n = 1, \ldots \)

Problem:

- stability requires continuity of \( u \mapsto G_u \)
- \( u \mapsto G_u \) not continuous for \( S \) non-smooth
- \( \sim \) use asymptotic stability
Asymptotic stability

Definition

Iterative method generating \( \{u^\delta_n\}_{n \leq N(\delta)} \) asymptotically stable for \( \delta \to 0 \) if exists subsequence \( \{\delta_k\} \) with:

- For all \( 0 \leq n \leq \overline{N} := \lim_{k \to \infty} N(\delta_k) \in \mathbb{N} \cup \{\infty\} \)
  
  \[ u^\delta_k \to \tilde{u}_n \quad \text{as} \quad k \to \infty \]
  
  for some \( \tilde{u}_n \in \overline{B}_U(u^\dagger, \rho) \)

- If \( \overline{N} = \infty \),
  
  \[ \tilde{u}_n \to u^\dagger \quad \text{as} \quad n \to \infty \]

- \( \tilde{u}_n \) generated by perturbed noise-free iteration

- perturbation needs to vanish for \( n \to \infty \)
Asymptotic stability

Bouligand–Landweber iteration

\[ u_{n+1}^\delta = u_n^\delta + w_n G_{u_n^\delta}^* \left( y^\delta - S(u_n^\delta) \right), \quad n = 0, 1, 2, \ldots, N(\delta) \]

- asymptotically stable for non-smooth PDE
  (proof uses GTCC and compact embedding for \( \mathcal{R}(G_u^*) \))
- \( \sim \) regularization (under conditions on \( \mu, \tau, w_n \)):

\[ u_{N(\delta)}^\delta \rightarrow u^\dagger \quad \text{for} \quad \delta \rightarrow 0 \]
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Levenberg–Marquardt iteration

\[ u_{n+1}^\delta = \arg\min_{u \in D(S)} \| S'(u_n^\delta)(u - u_n^\delta) - y^\delta - S(u_n^\delta) \|^2 + \alpha_n \| u - u_n^\delta \|^2 \]

\[ = u_n^\delta + \left( \alpha_n I + S'(u_n^\delta)^* S'(u_n^\delta) \right)^{-1} S'(u_n^\delta)^* \left( y^\delta - S(u_n^\delta) \right) \]

\[ \alpha_n = \alpha_0 r^n, \; r < 1 \]

- stopping by discrepancy principle
- TCC + transfer operator property

\[ S'(u_2) = Q(u_1, u_2)S'(u_1) \quad Q \text{ linear, near identity} \]

- stable, convergent regularization
- \( N(\delta) = O(1 + | \log \delta |) \)
Bouligand–Levenberg–Marquardt iteration

\[ u_{n+1}^\delta = \arg\min_{u \in D(S)} \| G_{u_n}^\delta (u - u_n^\delta) - y^\delta - S(u_n^\delta) \|^2 + \alpha_n \| u - u_n^\delta \|^2 \]

\[ = u_n^\delta + \left( \alpha_n I + G_{u_n}^{\delta*} G_{u_n}^\delta \right)^{-1} G_{u_n}^{\delta*} \left( y^\delta - S(u_n^\delta) \right) \]

- \( \alpha_n = \alpha_0 r^n, \ r < 1 \)
- stopping by discrepancy principle
- GTCC + transfer operator property (holds for non-smooth PDE)

\[ G_{u_2} = Q(u_1, u_2) G_{u_1} \quad Q \text{ linear, near identity} \]

- asymptotically stable, convergent regularization
- \( N(\delta) = O(1 + | \log \delta |) \)
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Numerical example

\[
-\Delta y + \max(0, y) = u
\]

- finite element discretization
- semismooth Newton method for solution (evaluation of \( S \))
- constructed exact solution \( u^\dagger \) with \(| \{ x : y^\dagger(x) = 0 \} | > 0 \)
- random Gaussian noise: \( \| y^\delta - y^\dagger \| = \delta \)
- \( \mu = 0.1, \quad \tau = 1.4, \quad \rho = 5, \quad w_n = \frac{2-2\mu}{\bar{L}^2}, \quad \bar{L} = 5 \times 10^{-2} \)
- compare two starting points:
  1. \( u_0 \equiv 0 \)
  2. \( \tilde{u}_0 \) satisfying \( u^\dagger - u_0 \in \mathcal{R}(G_{{u^\dagger}}^*) \) (generalized source condition)
Numerical example: results with $u_0$

(a) Bouligand–Landweber

(b) Bouligand–Levenberg–Marquardt
Numerical example: results with $\bar{u}_0$

(a) Bouligand–Landweber

(b) Bouligand–Levenberg–Marquardt

Motivation  Non-smooth equation  Landweber  Levenberg–Marquardt  Examples
Conclusion

Summary

- iterative regularization using Bouligand derivatives
- inverse source problems for non-smooth PDEs
- convergence under asymptotic stability

Outlook

- convergence rates under source condition
- accelerated Landweber iteration
- other non-smooth equations, variational inequalities
- coefficient inverse problems

Preprint, Python/Julia codes:
http://www.uni-due.de/mathematik/agclason/clason_pubs.php