Iterative regularization for nonsmooth inverse problems

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Motivation

Inverse problem: find
- unknown parameter $u^\dagger$
  - e.g., heat source, diffusion constant, thermal conductivity, heat capacity, latent heat density, ...

given
- measurement $y$
- model $S: u \mapsto y$, e.g., solution of PDE

$\leadsto$ solve

$$S(u) = y$$

Problem: measurement $y = y^\delta$ noisy, range of $S$ not closed
$\leadsto$ ill-posed, needs regularization
Regularization

Solve approximate, **stable** problem:

1. Tikhonov regularization $\sim \rightarrow$ optimal control
2. iterative regularization, e.g., Landweber iteration

\[ u_{n+1} = u_n + w_n S'(u_n)^*(y^\delta - S(u_n)) \quad n = 1, \ldots, N \]

- \( S'(u) \) Fréchet derivative, \( S'(u)^* \) adjoint
- stopping index \( N = N(\delta) < \infty \) regularization parameter
- regularization: \( N(\delta) \rightarrow \infty, u_{N(\delta)} \rightarrow u^\dagger \) for \( \delta \rightarrow 0 \)

Here: \( S \) solution operator for **non-smooth PDE**

$\sim \rightarrow$ **not** Fréchet differentiable
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4 Bouligand–Levenberg–Marquardt iteration

5 Numerical examples
Non-smooth equation

\[-\Delta y + \max(0, y) = u\]

solution operator \( S : u \mapsto y (:= S(u)) \)

- well-posed (in suitable – standard – spaces)
- Lipschitz continuous
- completely continuous (\(\sim\) ill-posed)
- not Fréchet differentiable unless \(|\{x : y(x) = 0\}| = 0\)
- model for membrane under water
- can be extended to arbitrary \(f(y)\) piecewise differentiable
- simplified model for sharp phase transition (Stefan problem)
Non-smooth equation

\[-\Delta y + \max(0, y) = u\]

solution operator \( S : u \mapsto y \) (\( := S(u) \))

- not Fréchet differentiable unless \( |\{x : y(x) = 0\}| = 0 \)
- but: directionally differentiable

Directional derivative \( S'(u; h) =: \eta \) solves

\[-\Delta \eta + 1_{\{y=0\}} \max(0, \eta) + 1_{\{y>0\}} \eta = h\]

not linear in \( h \mapsto \) not useful for algorithm
Non-smooth equation

Bouligand subdifferential

\[ \partial_B S(u) := \begin{cases} G \text{ linear} & \text{there is } \{u_n\} \text{ Gâteaux differentiable with } \\ u_n \to u \text{ and } S'(u_n; h) \to Gh \text{ for all } h \end{cases} \]

\[ -\Delta \eta + 1_{\{y>0\}} \eta = h \]

\[ G_u : h \mapsto \eta \]

- \( G_u \in \partial_B S(u) \) Bouligand derivative [Christof/Meyer/Walter/C.]
- \( u \mapsto G_u \) uniformly bounded (in right spaces)
- linear \( \sim \) use for Landweber in place of \( S'(u) \)
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Bouligand–Landweber iteration

\[ u_{n+1}^\delta = u_n^\delta + w_n G_{u_n^\delta}^* \left( y^\delta - S(u_n^\delta) \right), \quad n = 0, 1, 2, \ldots, N(\delta) \]

- \( S : u \mapsto y \) non-smooth
- \( y^\delta \) with \( \| y^\delta - y^\dagger \| \leq \delta, \quad y^\dagger = S(u^\dagger) \) (assume unique)
- \( u_0^\delta = u_0 \) starting point
- \( w_n \) step sizes
- stopping index \( N(\delta) \) by discrepancy principle:

\[ \| y^\delta - S(u_{N(\delta)}^\delta) \|_Y \leq \tau\delta < \| y^\delta - S(u_n^\delta) \|_Y, \quad 0 \leq n < N(\delta) \]

(modified Landweber iteration [Scherzer ’95])
Well-posedness

Assume:

1. \{G_u\} uniformly bounded
2. generalized tangential cone condition (GTCC)
   \[ \|S(u') - S(u) - G_u(u' - u)\| \leq \mu\|S(u') - S(u)\| \quad \text{for all } u, u' \in B_\rho(u^\dagger) \]
3. \( u_0 \in B_\rho(u^\dagger) \)

Then (under conditions on \( \mu, \tau, w_n \)):

- \( u_\delta^n \in B_\rho(u^\dagger) \) for all \( n \leq N(\delta) \)
- \( \delta > 0: N(\delta) < \infty \) and \( \|u_\delta^n - u^\dagger\| < \|u_\delta^{n-1} - u^\dagger\| \) for \( n \leq N(\delta) \)
- \( \delta = 0: N(\delta) = \infty \) and \( u_\delta^0 \rightarrow u^\dagger \) for \( n \rightarrow \infty \)
Goal: show that $u^\delta_{N(\delta)} \rightarrow u^\dagger$ for $\delta \rightarrow 0$

Standard proof: combine

1. monotonicity: $\|u^\delta_n - u^\dagger\| < \|u^\delta_{n-1} - u^\dagger\|$ for $n \leq N(\delta)$

2. stability: $u^\delta_n \rightarrow u^0_n$ for all $n = 1, \ldots$

Problem:

- stability requires continuity of $u \mapsto G_u$
- $u \mapsto G_u$ is not continuous for $S$ non-smooth
- $\sim$ use asymptotic stability
Asymptotic stability

Definition

Iterative method generating \( \{u_{\delta}^n\}_{n \leq N(\delta)} \) asymptotically stable for \( \delta \to 0 \) if exists subsequence \( \{\delta_k\} \) with:

- For all \( 0 \leq n \leq \overline{N} := \lim_{k \to \infty} N(\delta_k) \in \mathbb{N} \cup \{\infty\} \)
  \[
  u_{\delta_k}^n \to \tilde{u}_n \quad \text{as} \quad k \to \infty
  \]
  for some \( \tilde{u}_n \in \overline{B}_U(u^\dagger, \rho) \)

- If \( \overline{N} = \infty \),
  \[
  \tilde{u}_n \to u^\dagger \quad \text{as} \quad n \to \infty
  \]

- \( \tilde{u}_n \) generated by perturbed noise-free iteration

- Perturbation needs to vanish for \( n \to \infty \)
Asymptotic stability

Bouligand–Landweber iteration

\[ u_{n+1}^{\delta} = u_n^{\delta} + w_n G_d^{*} \left( y^{\delta} - S(u_n^{\delta}) \right), \quad n = 0, 1, 2, \ldots, N(\delta) \]

- asymptotically stable for non-smooth PDE
  (proof uses GTCC and compact embedding for \( R(G_d^*) \))
- \( \sim \) regularization (under conditions on \( \mu, \tau, w_n \)):

\[ u_{N(\delta)}^{\delta} \to u^{\dagger} \quad \text{for} \quad \delta \to 0 \]
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Levenberg–Marquardt iteration

\[ u_{n+1}^\delta = \arg\min_{u \in \mathcal{D}(S)} \| S'(u_n^\delta)(u - u_n^\delta) - y^\delta - S(u_n^\delta) \|^2 + \alpha_n \| u - u_n^\delta \|^2 \]

\[ = u_n^\delta + \left( \alpha_n I + S'(u_n^\delta)^* S'(u_n^\delta) \right)^{-1} S'(u_n^\delta)^* \left( y^\delta - S(u_n^\delta) \right) \]

- \( \alpha_n = \alpha_0 r^n, \ r < 1 \)
- stopping by discrepancy principle
- TCC + transfer operator property

\[ S'(u_2) = Q(u_1, u_2) S'(u_1) \quad Q \text{ linear, near identity} \]

- \( \leadsto \) stable, convergent regularization
- \( N(\delta) = O(1 + | \log \delta |) \)
Bouligand–Levenberg–Marquardt iteration

\[ u_{n+1}^{\delta} = \arg\min_{u \in D(S)} \| G_{u_n^{\delta}}(u - u_{n}^{\delta}) - y^{\delta} - S(u_{n}^{\delta}) \|^2 + \alpha_n \| u - u_{n}^{\delta} \|^2 \]

\[ = u_{n}^{\delta} + \left( \alpha_n I + G_{u_n}^{*} G_{u_n}^{\delta} \right)^{-1} G_{u_n}^{*} \left( y^{\delta} - S(u_{n}^{\delta}) \right) \]

- \( \alpha_n = \alpha_0 r^n, \; r < 1 \)
- stopping by discrepancy principle
- GTCC + transfer operator property (holds for non-smooth PDE)

\[ G_{u_2} = Q(u_1, u_2)G_{u_1} \quad Q \text{ linear, near identity} \]

- \( \sim \) asymptotically stable, convergent regularization
- \( N(\delta) = O(1 + | \log \delta |) \)
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Numerical example

\[-\Delta y + \max(0, y) = u\]

- finite element discretization
- semismooth Newton method for solution (evaluation of $S$)
- constructed exact solution $u^\dagger$ with $|\{x : y^\dagger(x) = 0\}| > 0$
- random Gaussian noise: $\|y^\delta - y^\dagger\| = \delta$
- $\mu = 0.1$, $\tau = 1.4$, $\rho = 5$, $w_n = \frac{2-2\mu}{L^2}$, $\bar{L} = 5 \times 10^{-2}$
- compare two starting points:
  1. $u_0 \equiv 0$
  2. $\bar{u}_0$ satisfying $u^\dagger - u_0 \in \mathcal{R}(G^*_u)$ (generalized source condition)
Numerical example: results with $\mathcal{U}_0$

Motivation

Non-smooth equation

Landweber

Levenberg–Marquardt

Examples
Numerical example: results with $\bar{u}_0$

(a) Bouligand–Landweber

(b) Bouligand–Levenberg–Marquardt
Conclusion

Summary

- iterative regularization using Bouligand derivatives
- inverse source problems for non-smooth PDEs
- convergence under asymptotic stability

Outlook

- convergence rates under source condition
- accelerated Landweber iteration
- other non-smooth equations, variational inequalities
- coefficient inverse problems

Preprint, Python/Julia codes:

http://www.uni-due.de/mathematik/agclason/clason_pubs.php