Errata to Introduction to Lie Algebras

We would like to thank Thorsten Holm for many of the corrections below. The reprinted 1st edition (published June 2007) incorporates all corrections except those marked (†). We are grateful to Michael Andrews for the correction to the proof of Theorem 11.10. We are grateful to Jim Humphreys for pointing out a serious problem with the proof of Theorem 9.16. (See the correction in bold type below.) Fortunately there are no knock-on effects, except to make the intended solution of Exercise 9.15 unviable.

We are grateful to Matthias Franz for pointing out a gap in the proof of Proposition 11.21 and outlined a correction, which we have adopted. We thank David Tiersch for pointing out a mistake in the diagram on page 70, Hassan Azad for a correction to §15.2 and Bonita Graham for a correction to the proof of Lemma 9.8. We are grateful to Andrew Goetz for corrections to mathematical errors in Theorem 9.11, Exercise 11.12 and page 212, corrections to several other minor errors, and for supplying detailed suggestions of a number of other potential improvements to the exposition. We thank Kai Meng Tan for an important correction to §12.1. We thank James Craig for a correction to page 213 in the proof of Weyl's theorem. We thank David Hemmer and Mikko Korhonen for independently pointing out an error in Exercise 7.12(i). We thank Neal Livesay for the correction to the proof of Lemma 10.1(i) and for many corrections to minor errors.

Please send further corrections or comments to `mark.wildon@rhul.ac.uk`. This errata was last updated in July 2014.

p15, Exercise 2.7(i)	Replace $\{(x_2, 0) : x_2 \in L_1\}$ with $\{(0, x_2) : x_2 \in L_2\}$.
p22, proof of Theorem 3.2	In second display, replace $[x, w]$ with $[x, z]$ and $[y, w]$ with $[y, z]$.

p25, line 2	Replace $\lambda \in L$ with $\lambda \in \mathbf{C}$.
p30, Definition 4.6	It should perhaps be emphasised here that we are only considering finite-dimensional Lie algebras.
p32, Section 4.3 (†)	In 'as a subalgebra of a Lie algebras of upper triangular matrices.' replace 'algebras' with 'algebra'.
p33, Exercise 4.2	Replace 'Show that A belongs ...' with 'Show that x belongs ...'
p41, proof of Lemma 5.5	Replace 'For column r' with 'For column $r+1$'. In last line, replace 'column r' with 'column $r+1$'.
p44, Exercise 5.7	In last line, replace xy^{m-k} with $y^{m-k}x$.
p46, Exercise 6.1(ii)	Replace $\bar{x} : V/U \to V/W$ with $\bar{x} : V/U \to V/U$.
p48, proof of Theorem 6.3	Replace $x \in X$ with $x \in L$.
p49, proof of Theorem 6.3	Replace $\tilde{L}$ with $\bar{L}$.
p49, Remark 6.4	Delete the first 'of' in the first sentence.
p51, displayed equation	Replace $\lambda(x)$ with $\lambda(a)$.
p51, Exercise 6.4	In the last line, replace 'previous' with 'following'.
p54, Example(2) (†)	Delete 'to' in 'restriction to of the identity map'.
p57, Example 7.6	Only applies to complex Lie algebras.
p57, after Example 7.6	Replace W/V with V/W.
p58, Example 7.8	Replace A with X (twice).
p60	The definition of an L-module homomorphism should read '... is a linear map $\theta : V \to W$ such that $$\theta(x \cdot v) = x \cdot \theta(v) \quad \text{for all } v \in V \text{ and } x \in L.\text{'}$$
p60, Theorem 7.11	In last line, replace V/U with V/W.
p62, Lemma 7.13	Replace $\theta - \lambda 1_V$ with $\theta - \lambda 1_S$; replace $S = \ker(\theta - \lambda 1_V)$ with $S = \ker(\theta - \lambda 1_S)$; replace $\theta = \lambda 1_V$ with $\theta = \lambda 1_S$.
p63, Exercise 7.6(iii)	Replace 'every submodule' with 'every non-zero submodule'.

p65, Exercise 7.12(i) (†)	The 'only if' part of the claimed result is false: a simple counterexample is given by the natural 2-dimensional representation of $\mathsf{sl}_2(\mathbf{C})$. A correct result is as follows: 'Prove more generally that V is isomorphic to $V^\star$ if and only if there is a linear isomorphism $P : V \to V$ such that $$P\varphi(x)P^{-1} = -\varphi(x)^{tr}$$ for all $x \in L$, where $\varphi(x) : V \to V$ is the matrix representing the action of x on V.'
p65, Exercise 7.12	Part (iii) should be labelled (ii).
p70, Diagram (†)	Replace $X^{d-2}Y$ with $X^{d-2}Y^2$.
p73, after Corollary 8.6 (†)	Replace 'a vector v of the type considered ...' with 'a vector w of the type considered ...'.
p74, after Theorem 8.7	Replace 'Appendix C' with 'Appendix B'.
p75, Exercise 8.4	Replace all appearances of W with V.
p75, Exercise 8.6	In first line, replace L with $\mathsf{sl}(2, \mathbf{C})$. Displayed equation should end 'for $v \in M$', not 'for $v \in V$'.
p81, proof of Lemma 9.8 (†)	Replace $\operatorname{tr}(A_x B_x)$ with $tr(A_x A_y)$ in final displayed equation.
p82, line 10	Replace $\dim W \cap \dim W^\perp \neq 0$ with $W \cap W^\perp \neq 0$.
p83, proof of Theorem 9.11 (†)	It should be justified that I is simple. If J is an ideal of I then $[I^\perp, J] \subseteq I^\perp \cap I = 0$, so $[L, J] = [I, J] = J$. Hence J is an ideal of L, so since I has minimum non-zero dimension, $J = I$.
p84, first displayed equation	Replace $\ldots, L_r$ with $\ldots \oplus L_r$.
p84, line 3	The argument becomes clearer if one replaces $[I, L_i] \subseteq I \cap L_i = 0$ with $[I, L_i] \subseteq I \cap I^\perp = 0$.
p85, proof of Prop. 9.13	In 3rd paragraph, delete the sentence starting 'We have ... ' and replace with 'If M is properly contained in $\operatorname{Der} L$ then $M^\perp \neq 0$, so it is sufficient to prove that $M^\perp = 0$.'
p85, before §9.6	Replace 'is a direct sum of semisimple Lie algebras' with 'is a direct sum of simple Lie algebras'.

p87, Theorem 9.16($\star$) **The argument given before this theorem that the abstract Jordan decomposition of an element x of a semisimple Lie subalgebra $L \subseteq \mathsf{gl}(V)$ agrees with its usual Jordan decomposition as an element of $\mathsf{gl}(V)$ is flawed. If the usual Jordan decomposition is $x = d + n$, then our argument shows that $\operatorname{ad} d : \mathsf{gl}(V) \to \mathsf{gl}(V)$ and $\operatorname{ad} n : \mathsf{gl}(V) \to \mathsf{gl}(V)$ restrict to endomorphisms of L, and $\operatorname{ad} d = \operatorname{ad} d'$, $\operatorname{ad} n = \operatorname{ad} n'$, where $x = d' + n'$ is the abstract Jordan decomposition of x. However, we do not know that $d \in L$, so we cannot use the uniqueness of the abstract Jordan decomposition to deduce that $d = d'$.**

The proof of Theorem 9.16 is therefore invalid. The only use made of this theorem later in the book is in the solution to Exercise 9.15.

p93, line 3 Replace $\alpha : H \to L$ with $\alpha : H \to \mathbf{C}$.

p93, line 12 Replace $\alpha \in L^\star$ with $\alpha \in H^\star$.

p93, proof of Lemma 10.1(i) (†) Replace 'is an eigenvector for each $\operatorname{ad} h \in H$' with 'is an eigenvector of $\operatorname{ad} h$ for each $h \in H$'.

p94, penultimate paragraph Replace 'thus H is not contained in any larger abelian subalgebra of H' with 'thus H is not contained in any larger abelian subalgebra of $\mathsf{sl}(3, \mathbf{C})$.

p98, line 3 (†) Replace $\kappa(x, y)$ with $\kappa(x, w)$.

p98, 3rd paragraph Replace $[h, y] = -\alpha(h)x = 0$ with $[h, y] = -\alpha(h)y = 0$.

p98, Exercise 10.3(ii) Replace S_α with $\mathsf{sl}(\alpha)$.

p100, proof of Prop. 10.9 Replace the sentence starting 'We have $\alpha(v) = 0 \ldots$' with 'The zero-eigenspace of h_α on M is H, which is contained in $K \oplus \mathsf{sl}(\alpha)$. Hence $v \in (K \oplus \mathsf{sl}(\alpha)) \cap V = 0$, which is a contradiction.'

p101, Prop. 10.10 Replace (ii) with 'There are integers $r, q \geq 0$ such that *if* $k \in \mathbf{Z}$ then $\beta + k\alpha \in \Phi$ if and only if $-r \leq k \leq q$. Moreover $r - q = \beta(h_\alpha)$.' (The original statement is true, but only the weaker version given above is proved.)

p102, Proof of Prop. 10.10 (†)	Related to previous: M should not be called the root string through β because we are only allowing $k \in \mathbf{Z}$.
p105, proof of Lemma 10.14	The matrix should be transposed; i.e. replace (α_1, α_ℓ) with (α_ℓ, α_1) and (α_ℓ, α_1) with (α_1, α_ℓ).
p106, Exercise 10.5	The formula should read $\dim L = \dim H + \lvert\Phi\rvert$, not $\dim L = \dim H + 2\lvert\Phi\rvert$.
p106, Exercise 10.6 (†)	Comma missing in displayed equation.
p106, Exercise 10.7(i)	Replace 'span H' with 'span $H^\star$'.
p110, Definition 11.1	Replace 'A subset R of a real vector space E ...' with 'A subset R of a real inner-product space E ...'.
page 110, line −2 (†)	(t_α, t_α) should be $\kappa(t_\alpha, t_\alpha)$.
p111, Exercise 11.1	Change E to $\mathbf{R}^{\ell+1}$.
p112, Prop. 11.5(b)	The hypothesis 'and $(\beta, \beta) > (\alpha, \alpha)$' is unnecessary. (To prove the more general version, note that if β is shorter than α then, swapping α and β, we get that $\alpha - \beta \in R$. Then $\beta - \alpha \in R$ by (R2).)
p113, first diagram	The lower root labelled β should be labelled $-\beta$.
p113, Example 11.6(b)	After the diagram, replace 'root space' with 'root system'.
p115, §11.3	In second line, replace 'vector space basis for R' with 'vector space basis for E'.
p117, top line	Replace 'elements of α' with 'elements of B'.
p117, proof of Thoerem 11.10 (†)	In line 3, as we do not yet know that B is a base, it is circular to use Exercise to show that the angle between α and β is obtuse. Instead note that if the angle is acute, then by Proposition 11.5, $\alpha - \beta$ is a root, and so either $\alpha - \beta$, or $\beta - \alpha$ lies in R^+. In the first case $\alpha = (\alpha - \beta) + \beta$ is the sum of two elements of R^+, and similarly in the second case for β. This contradicts the definition of B.
p122, proof of Proposition 11.21 (†)	It is not clear that φ satisfies Condition 11.19(b), and in fact it takes some work to show this. **Step 1**: If $\alpha \in R$ then we can write $\alpha = \sum_i k_i \alpha_i$ for some $k_i \in \mathbf{Z}$. Then since $\langle\ ,\ \rangle$ is linear in its

first component we have $\langle \varphi(\alpha), \varphi\alpha_j) \rangle = \langle \alpha, \alpha_j \rangle$. This shows that Condition 11.19(b) holds when $\beta \in B$.

Step 2: Now observe that for any $\alpha_i, \alpha_j \in B$ we have

$$\begin{aligned} \varphi(s_{\alpha_i}(\alpha_j)) &= \varphi(\alpha_j - \langle \alpha_j, \alpha_i \rangle \alpha_i) \\ &= \varphi(\alpha_j) - \langle \alpha_j, \alpha_i \rangle \varphi(\alpha_i) \\ &= \alpha'_j - \langle \alpha'_j, \alpha'_i \rangle \alpha'_i \\ &= s_{\alpha'_i}(\alpha'_j). \end{aligned}$$

By linearity of φ it follows that $\varphi(s_{\alpha_i}(v)) = s_{\alpha'_i}\varphi(v)$ for all $v \in E$.

Step 3: To deal with a general $\beta \in R$ we use Step 2 and the Weyl group to get around the fact that $\langle\ ,\ \rangle$ is not linear in its second component. By Proposition 11.14 there exist $\alpha_{i_1}, \ldots, \alpha_{i_r} \in B$ and $\alpha_j \in B$ such that $s_{\alpha_{i_1}} \ldots s_{\alpha_{i_r}} \alpha_j = \beta$. By induction on r it follows from Step 2 that

$$\varphi(\beta) = \varphi(s_{\alpha_{i_1}} \ldots s_{\alpha_{i_r}}(\alpha_j)) = s_{\alpha'_{i_1}} \ldots s_{\alpha'_{i_r}}(\alpha'_j).$$

Hence for any $\alpha \in R$ we have

$$\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \varphi(\alpha), s_{\alpha'_{i_1}} \ldots s_{\alpha'_{i_r}}(\alpha'_j) \rangle.$$

For any $u, v \in E$ and $\alpha_j \in E$ we have

$$\begin{aligned} \langle u, s_{\alpha_j}(v) \rangle &= \frac{2(u, s_{\alpha_j}(v))}{(s_{\alpha_j}}(v), s_{\alpha_j}(v)) \\ &= \frac{2(s_{\alpha_j}(u), v)}{(v, v)} \\ &= \langle s_{\alpha_j}(u), v \rangle. \end{aligned}$$

Hence $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle s_{\alpha'_{i_r}} \ldots s_{\alpha'_{i_1}} \varphi(\alpha), \alpha'_j \rangle$. It now follows from Step 2 that

$$\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \varphi(s_{\alpha_{i_r}} \ldots s_{\alpha'_{i_1}}(\alpha)), \alpha'_j \rangle.$$

By Step 1 we get $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle s_{\alpha_{i_r}} \ldots s_{\alpha_{i_1}}(\alpha), \alpha_j \rangle$. By moving the reflections back to the right-hand side we get

$$\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, s_{\alpha_{i_1}} \ldots s_{\alpha_{i_r}}(\alpha_j) \rangle = \langle \alpha, \beta \rangle$$

as required.

The rest of the proof proceeds as before. The displayed equation on page 123 has already been established in Step 2.

p123, Exercise 11.12 (†) Add the hypotheses that $n \geq 2$ and that $U_1, U_2, \ldots, U_n$ are distinct.

p126, Before first displayed equation Insert 'A case by case analysis shows that the weight 0 does not appear in the subspace of off-diagonal matrices' before 'Let' on line 7.

p126, 2nd displayed equation Should read $\Phi = \{\alpha \in H^\star : \alpha \neq 0, L_\alpha \neq 0\}$.

p126, Equation ($\star$) (†) Replace L_0 with H.

p126, Lemma 12.2 Replace 'Suppose that for all $h \in H$ ...' with 'so for all non-zero $h \in H$ ...'. (This follows from ($\star$), since by assumption L is a classical Lie algebra, and so the off-diagonal part of L is, by ($\star$), a sum of non-zero weight spaces.)

p127, Prop. 12.3 Replace '...where Φ is the set of $\alpha \in H^\star$ such that ...' with '...where Φ is the set of non-zero $\alpha \in H^\star$ such that ...'.

p131, Step (2c) Change $h_{ij} = \ldots$ to $k_{ij} := \ldots$ (twice).

p135, Step (1) The definitions of p_{ii} and q_{ii} should be given separately, as $p_{ii} = e_{i,\ell+i}$, $q_{ii} = e_{\ell+i,i}$.

p135, Step (2) Replace 'If $\alpha = \varepsilon_i + \varepsilon_j$ ...' with 'If $\alpha = \varepsilon_i + \varepsilon_j$, then $x_\alpha = p_{ij}$ and $x_{-\alpha} = q_{ji}$ and

$$h = (e_{ii} - e_{\ell+i,\ell+i}) + (e_{jj} - e_{\ell+j,\ell+j})$$

if $i \neq j$, and $h = e_{ii} - e_{\ell+i,\ell+i}$ if $i = j$. Hence $[h, x_\alpha] = 2x_\alpha$ in both cases.'

p151, Exercise 13.1 Replace 'That is, find a linear map between the vector spaces ...' with 'That is, find a linear map between the underlying vector spaces of root systems of types B_2 and C_2 ...'.

p154, proof of Prop. 14.2(†) In penultimate paragraph, replace 'we only have to show $[x, a] \in L$...' with 'we only have to show $[x, a] \in I$...'.

p164, line 7 of §15.1	Replace 'with respect to Φ' with 'with respect to Π'.
p166, proof of Lemma 15.3	Here $\{\alpha_1, \ldots, \alpha_\ell\}$ is a base of a root system for the Lie algebra L. The subspace W should be defined as the span of elements of the form $$f_{\alpha_{i_1}} f_{\alpha_{i_2}} \ldots f_{\alpha_{i_k}} \cdot v$$ and not as the span of the single element $$f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_k} \cdot v.$$ Similar changes must then be made in the following lines. See also the next correction.
p166, proof of Lemma 15.3	In the second displayed equation replace $[e_\alpha, f_{\alpha_1}]$ with $[e_\alpha, f_{\alpha_{i_1}}]$.
p168, 2nd displayed equation	Change $v_i \wedge w_j$ to $v_i \wedge v_j$.
p170, §15.1.3	Replace 'we introduce a symbol $v_i otimes w_j$' with 'we introduce a symbol $v_i \otimes w_j$'.
p174, §15.2.2 (†)	Line 8 of §15.2.2: replace $\alpha \in \Phi$ with $\alpha \in \Phi^+$.
p178, before Exercise 15.4	The definition of the Chevalley group should read '$G_F(L) := \langle \tilde{A}_\alpha(t) : \alpha \in \Phi, t \in F \rangle$'.
p183, penultimate paragraph	Replace 'prime characteristic 0' with 'prime characteristic p'.
p184, diagram	There is an extra '3' in the diagram of the quiver.
p187, Exercise 15.7	Make the following addition: '... if we replace ι with ι' *and $U(L)$ with V in the commutative diagram above*, then V has the universal property ...'.
p191, Theorem 16.1(a)	Replace $V/\ker\alpha \cong W$ with $V/\ker\alpha \cong \operatorname{im}\alpha$.
p196, line 5	Replace $a(X)(x-\lambda_i)^{a_i} v$ with $a(x)(x-\lambda_i)^{a_i} v$. Replace $f(X)$ with $p(X)$ in last line.
p196, line 7 (†)	Insert a comma after $(X - \lambda_1)^{a_1}$.
p198, diagram	The vectors spanning the axes must be swapped.
p212, lines 2 and 3 (†)	Replace a_{ji} with a_{ij}.
p213, penultimate line (†)	The identity map on V does not have image in W, so is not an admissible choice. Instead take a subspace U of W, so that $V = W \oplus U$ as vector spaces, and

take the map $g : V \to W$ that is the identity on W and has kernel U. The coset $g+M_0$ is then a non-zero element of M_S/M_0.

p231, Line −3 (†) Replace 1.13 with 1.12.

p233, 2.11 In first line replace PxP^{-1} with $P^{-1}xP$.

p234, 3.2 Replace $\varphi[y_1 z_1]$ with $\varphi[y_1, z_1]$.

p236, 6.5 Replace 'there is a basis of $\operatorname{ad} L$ ...' with 'there is a basis of L ...'

p238, 8.6(i) In second display, the second line should end $\frac{1}{2}h(eh+2e) \cdot v$, not $\frac{1}{2}h(he+2e) \cdot v$.

p240, 9.15 (†) In fourth paragraph of solution, change 'generated by w', to 'generated by w, where w is any vector such that $w + M = v$'.

p241, 10.6 Displayed equation should end $\ldots = -1 \times -1 = 1$.

p246, 19.1 (†) In second line q_{1i_1} should read $q_{i_1 1}$.