Albert-Ludwigs-Universität Freiburg
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Holomorphic Invariant Theory and Differential Forms in Higher-dimensional Algebraic Geometry

Zusammenfassende Darstellung der wissenschaftlichen Veröffentlichungen

vorgelegt von

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Contents

Introduction

My research is focussed on complex algebraic geometry and Kähler geometry. My main interests lie in the study of group actions, and in the birational geometry of higher-dimensional algebraic varieties and complex manifolds. It is one of the fascinating aspects of the modern developments in algebraic geometry that these two subjects are closely interrelated. This is also reflected in the research projects presented here, in which I study problems from both areas using an interplay of methods from algebraic and Kählerian geometry, complex analysis, algebra and representation theory.

The first chapter discusses my research on group actions in complex algebraic and analytic geometry. Section 1.2, summarising my paper [Gre10a], focusses on algebraicity properties of complex-analytic quotients of algebraic varieties on which a complex-reductive group acts in a Hamiltonian fashion. In the proof of the projectivity results described in this section it is important to control the singularities of the quotients under consideration as well as to study invariant meromorphic functions on complex G-spaces and their relation to meromorphic functions on quotient spaces. These two aspects of equivariant geometry are studied further in Sections 1.3 and 1.4: It is a classical result of invariant theory, proven by Boutot [Bou87], that the class of varieties with rational singularities is stable under taking good quotients. In my article [Gre11], whose contents are presented in Section 1.3, I generalise Boutot's theorem to a larger class of singular G-varieties and prove an analogous result for quotients in the analytic category. Section 1.4, summarising my paper [GM11], is devoted to the investigation of invariant meromorphic functions on Stein spaces, discussing an analytic version of Rosenlicht's Theorem [Ros63] and the construction of meromorphic quotients for actions of reductive and certain non-reductive groups.

Besides quotients by reductive group actions, other special important classes of rational singularities, e.g., canonical singularities [Rei87], play a prominent role in higher-dimensional algebraic geometry, especially in the Minimal Model Program. The research presented in Chapter 2 focusses on the investigation of the geometry of varieties with canonical singularities. In the Gorenstein case, these singularities can be characterised as those for which regular n-forms defined on the smooth locus of the variety under consideration extend to any resolution. The natural question whether or not differential forms of arbitrary degree extend to any resolution of a given variety with at worst canonical singularities is answered positively by the articles [GKK10, GKKP11], surveyed in Sections 2.2 and 2.3. Through these extension results one is naturally lead to study the sheaf of so-called reflexive differentials. The project [GR11], summarised in Section 2.4, studies algebraic properties of this sheaf on cone singularities, comparing it to the sheaf of Kähler differentials. Finally, some of these results are used in [GKP11] to study the geometry of varieties with trivial canonical bundle, generalising the fundamental Decomposition Theorem of Beauville [Bea83] and Bogomolov [Bog74] from the smooth to a singular setup; cf. the discussion in Section 2.5.

By the Decomposition Theorem just mentioned, one of the basic building blocks for compact Kähler manifolds with trivial canonical bundle are hyperkähler, or irreducible holomorphic-symplectic, manifolds. In general, it is an important question (motivated for example by mirror symmetry) to determine conditions under which a given hyperkähler manifold admits a so-called Lagrangian fibration. The research project discussed in the third chapter approaches this question under a geometric condition proposed by Beauville [Bea11, Sect. 1.6]. In the paper [GLR11a], summarised in Section 3.2, it is shown that any non-projective hyperkähler manifold containing a Lagrangian torus admits a Lagrangian fibration. Moreover, rather explicit conditions for the existence of fibrations on projective hyperkähler manifolds are given. As discussed in Section 3.3, the four-dimensional projective case is settled in [GLR11b].

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Technical Remarks

- Copies of published articles and latest `arXiv`-Versions of preprints are provided in the appendices.
- In the following

  Theorem (Example theorem, Thm. 2.3)**.** ...

  refers to Theorem 2.3 in the paper under discussion.
- If not mentioned otherwise, we work over the field $\mathbb{C}$ of complex numbers.

Chapter 1

Geometric Invariant Theory on complex spaces

1.1 Introduction

For a finite-dimensional regular representation $\rho : G \to GL_{\mathbb{C}}(V)$ of a complex reductive Lie group G, it is a classical result due to Hilbert that the algebra $\mathbb{C}[V]^G$ of G-invariant polynomials is finitely generated. It defines an affine algebraic variety via $V /\!\!/ G := \mathrm{Spec}(\mathbb{C}[V]^G)$ and a G-invariant surjective affine morphism $\pi : V \to V /\!\!/ G$. If $\mathscr{O}_Y$ denotes the structure sheaf of a given algebraic variety Y, the map π fulfills $(\pi_* \mathscr{O}_V)^G = \mathscr{O}_{V /\!\!/ G}$. The algebraic variety $V /\!\!/ G$ parametrises the closed G-orbits in V. We call $\pi : V \to V /\!\!/ G$ the *algebraic Hilbert quotient* or *good quotient* of V by the action of G. More generally, algebraic Hilbert quotients always exist for algebraic actions of complex reductive Lie groups on affine varieties.

In the complex-analytic category consider the holomorphic action of a complex reductive Lie group G on a Stein space X. In analogy to the affine algebraic situation, there exists a Stein space $X /\!\!/ G$ and a G-invariant surjective holomorphic Stein map $\pi : X \to X /\!\!/ G$ with $(\pi_* \mathscr{H}_X)^G = \mathscr{H}_{X /\!\!/ G}$, see [Sno82] and [Hei91]. Here, $\mathscr{H}_Y$ denotes the sheaf of holomorphic functions on a given complex space Y. We call $\pi : X \to X /\!\!/ G$ an *analytic Hilbert quotient*. In analogy to the algebraic category, the space $X /\!\!/ G$ parametrises the closed G-orbits in X. More precisely, it is the quotient of X by the equivalence relation

$$x \sim y \text{ if and only if } \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset.$$

In this chapter closely interrelated geometric, cohomological and function-theoretic aspects of holomorphic group actions and analytic Hilbert quotients are studied. Section 1.2 studies algebraicity properties of analytic Hilbert quotients of algebraic varieties, in Section 1.3 the singularities of quotient spaces are investigated, and Section 1.4 focusses on invariant meromorphic functions and their relation to meromorphic functions on analytic Hilbert quotients.

1.2 Compact Kähler quotients of algebraic varieties and Geometric Invariant Theory

[Gre10a] **Daniel Greb**, *Compact Kähler quotients of algebraic varieties and Geometric Invariant Theory*, Advances in Mathematics **224** (2010), 401-431.

Given an arbitrary algebraic G-variety X for a complex reductive group G, it is, in general, not possible to find a good quotient $\pi : X \to X /\!\!/ G$ for the action of G on X. Hence, one looks for open G-invariant subsets of X for which a good quotient exists. To this end, Mumford introduced the notion of *semistability* with respect to a *linearisation*, a lifting of the G-action to an ample line bundle L on X, cf. [MFK94]. This yields a Zariski-open subset $X(L)$ of semistable points for which there exists a good quotient $\pi : X(L) \to X(L) /\!\!/ G$ with quasi-projective $X(L) /\!\!/ G$. The study of this construction is the subject of Geometric Invariant Theory (GIT).

As in the algebraic case, for a general holomorphic G-space an analytic Hilbert quotient does not necessarily exist. The fundamental idea of Guillemin and Sternberg, Kirwan, and Mumford is to use symplectic geometry in order to find G-invariant open subsets that admit an analytic Hilbert quotient. Hereby, the role of a linearisation is played by a *momentum map* for the action of a maximal compact subgroup K of G. Given a complex Kähler manifold (or Kähler space) X with K-invariant Kähler form ω, a momentum map is a smooth map $\mu : X \to \mathrm{Lie}(K)^*$ that is equivariant with respect to the action of K on X and the coadjoint representation on $\mathrm{Lie}(K)^*$, and whose components $\mu^\xi = \mu(\cdot)(\xi)$ fulfill the Hamiltonian equations

$$d\mu^\xi = \imath_{\xi_X}\omega \qquad \text{for all } \xi \in \mathrm{Lie}(K).$$

Here, $\imath_{\xi_X}$ denotes contraction with the vector field ξ_X on X that is induced by the K-action. In the above situation, the set of *μ-semistable points*

$$X(\mu) := \{x \in X \mid \overline{G \bullet x} \cap \mu^{-1}(0) \neq \emptyset\}$$

admits an analytic Hilbert quotient, see [HL94], [HHL94], and [Sja95]. If X is projective algebraic, and if the Kähler form ω as well as the momentum map μ are induced by an embedding of X into some projective space, both the set $X(\mu)$ of semistable points and the quotient $X(\mu) /\!\!/ G$ can also be constructed via Geometric Invariant Theory. In particular, the complex space $X(\mu) /\!\!/ G$ is projective algebraic, and the map $\pi : X(\mu) \to X(\mu) /\!\!/ G$ is a good quotient in the sense of GIT, cf. [MFK94].

However, already on projective manifolds there exist many Kähler forms which do not arise as curvature forms of ample line bundles. From the point of view of complex and symplectic geometry it is therefore natural to study also the semistability conditions induced by these forms and their relation to the algebraic geometry of the underlying G-variety. Furthermore, especially when studying questions related to the variation of GIT quotients and applications, e.g. to moduli problems, one encounters interesting phenomena related to non-integral Kähler forms, even if one is a priori interested in ample line bundles. See [Sch00, Ex. 1.1.5] for an example related to moduli spaces of semistable sheaves on higher-dimensional projective manifolds. The above discussion motivates the following definition: an *algebraic Hamiltonian G-variety* is a complex algebraic G-variety X together with a (not necessarily integral) K-invariant Kähler form ω and a K-equivariant momentum map $\mu : X \to \mathrm{Lie}(K)^*$ with respect to ω.

Under certain mild assumptions on the singularities, it was shown in [Gre10b] (which contains the results obtained in my PhD thesis) that compact momentum map quotients of algebraic Hamiltonian G-varieties are projective algebraic and that the corresponding sets of semistable points are algebraically Zariski-open. This contrasts with examples of compact non-projective geometric quotients of smooth projective algebraic $\mathbb{C}^*$-varieties constructed by Białynicki-Birula and Święcicka [BBŚ89].

In the paper under discussion I show that momentum map quotients of algebraic Hamiltonian G-varieties have even stronger algebraicity properties and we give an essentially complete picture of the relation between momentum geometry and Geometric Invariant Theory for Hamiltonian actions on algebraic varieties.

We conclude this section with a summary of the main results and an outline of the paper.

Theorem 1.2.1 (Algebraicity of momentum map quotients, Thm. 1.1)**.** *Let $G = K^{\mathbb{C}}$ be a complex reductive Lie group and let X be a G-irreducible algebraic Hamiltonian G-variety with at worst 1-rational singularities. Assume that the zero fibre $\mu^{-1}(0)$ of the momentum map $\mu : X \to \mathrm{Lie}(K)^*$ is nonempty and compact. Then*

1. *the analytic Hilbert quotient $X(\mu) /\!\!/ G$ is a projective algebraic variety,*
2. *the set $X(\mu)$ of μ-semistable points is algebraically Zariski-open in X,*
3. *the map $\pi : X(\mu) \to X(\mu) /\!\!/ G$ is a good quotient,*
4. *there exists a G-linearised Weil divisor D (in the sense of Hausen [Hau04]) such that $X(\mu)$ coincides with the set $X(D, G)$ of semistable points with respect to D.*

Parts (1) and (2) are already contained in [Gre10b], and we have included them here in order to convey a complete picture of the situation; the main contribution of the paper under discussion is part (3). Theorem 1.2.1 generalises results proven by Heinzner and Migliorini [HM01] (for smooth projective Hamiltonian G-varieties) and Sjamaar [Sja95] (for non-compact smooth varieties with integral Kähler form and proper moment map) to a singular, non-projective and non-compact setup. Note that Theorem 1.2.1 applies in particular to Hamiltonian G-varieties with proper momentum map.

A variety or complex space X is said to have only *1-rational singularities*, if for any resolution of singularities $f : \widetilde{X} \to X$ the sheaf $R^1 f_* \mathscr{O}_{\widetilde{X}}$ vanishes. The class of complex spaces with 1-rational singularities is the natural class of singular spaces to which projectivity results for Kähler Moishezon manifolds generalise, cf. [Nam02]. The fundamental properties of 1-rational singularities that allow to carry over the projectivity results from the smooth to the singular case were described in [KM92, §12.1]. Furthermore, this class is stable under taking good quotients (in the algebraic category [Gre09]) and analytic Hilbert quotients (in the analytic category [Gre11]). For a detailed discussion of these results the reader is referred to Section 1.3.

In Section 11 of the paper a Kählerian non-projective proper algebraic surface is constructed. This shows the necessity of the assumption on the singularities in Theorem 1.2.1. To the author's knowledge, this surface is the first explicit example of a Kählerian non-projective proper algebraic variety in the literature.

Theorem 1.2.1 above is deduced from the following main result of this paper:

Theorem 1.2.2 (Thm. 1.2)**.** *Let G be a complex reductive Lie group, let X be a G-irreducible normal algebraic G-variety, and let $U \subset X$ be a G-invariant analytically Zariski-open subset of X such that the analytic Hilbert quotient $\pi : U \to U/\!\!/G$ exists. If $U/\!\!/G$ is a projective algebraic variety, then*

1. *the set U is algebraically Zariski-open in X,*
2. *the map $\pi : U \to U/\!\!/G$ is is a good quotient.*

In fact, large parts of the statement of Theorem 1.2.2 still hold under the weaker assumption that the quotient $U/\!\!/G$ is a complete algebraic variety, as will become clear in the following outline of its proof. As a first step we prove

Theorem 1.2.3 (Openness Theorem, Thm. 6.1)**.** *Let G be a connected complex reductive Lie group and let X be an irreducible normal algebraic G-variety. Let U be a G-invariant analytically Zariski-open subset of X such that the analytic Hilbert quotient $\pi : U \to U/\!\!/G$ exists. If $U/\!\!/G$ is a complete algebraic variety, then U is Zariski-open in X.*

The main tool used in this section is the Rosenlicht quotient of X by G, which parametrises generic G-orbits in X and is a model for the field of G-invariant rational functions on X. For more information on Rosenlicht quotients and invariant meromorphic functions on complex spaces, we refer the reader to Section 1.4, where Rosenlicht's results are generalised to the holomorphic setup and several applications are derived.

As a second step we prove algebraicity of the quotient map:

Theorem 1.2.4 (Algebraicity Theorem, Thm. 7.1)**.** *Let G be a connected complex reductive Lie group, let X be an irreducible normal algebraic G-variety, and let U be a G-invariant Zariski-open subset of X such that the analytic Hilbert quotient $\pi : U \to Q$ exists. If Q is a complete algebraic variety, then the quotient map π is algebraic.*

This result is new even for Hamiltonian actions and quotients of semistable points with respect to some momentum map. For the proof we construct an algebraic family of compact cycles from the group action and investigate the interplay between the associated map (to an auxiliary Rosenlicht quotient) and the quotient map π.

Next, we study coherence properties of sheaves of invariants:

Theorem 1.2.5 (Coherence Theorem, Thm. 8.1). *Let G be a connected complex reductive Lie group. Let X be an irreducible algebraic G-variety with analytic Hilbert quotient $\pi : X \to Q := X /\!/ G$ where Q is a complete algebraic variety. If $\mathscr{F}$ is any coherent algebraic G-sheaf on X, then $(\pi_*\mathscr{F})^G$ is a coherent algebraic sheaf on Q.*

Using these preparatory steps, the proof of Theorem 1.2.2 proceeds as follows: Assuming projectivity of the quotient $U /\!/ G$, Theorem 3 of [Gre10b] provides us with a G-equivariant biholomorphic map $\varphi : U \to Z$ to a quasi-projective algebraic G-variety with good quotient $\pi_Z : Z \to U /\!/ G$. Note that the proof of [Gre10b, Thm. 3] heavily uses the assumption on the projectivity (and not just completeness) of $U /\!/ G$. The map φ is induced from a holomorphic section in $(\pi_*\mathscr{F}^h)^G$ for some coherent algebraic G-sheaf $\mathscr{F}$ on U. Here, $\cdot^h$ denotes analytification of algebraic sheaves. Using the Coherence Theorem, an étale slice theorem (also proven in the paper), and GAGA properties of the quotient $U /\!/ G$ we show that generically on $U /\!/ G$ the sheaf $(\pi_*\mathscr{F}^h)^G$ is algebraically isomorphic to the corresponding sheaf $(\pi_*\mathscr{F}^G)^h$ of equivariant *algebraic* maps from U to Z. Algebraicity of φ follows. Consequently, π is just a model of π_Z, hence itself a good quotient.

Several examples showing the necessity of the completeness and projectivity assumptions on the quotient $U /\!/ G$ are given in the individual sections.

With the help of our main result a second important question in GIT is approached: how many different open subsets U with good quotient or analytic Hilbert quotient exist in a given G-variety? Using Theorem 1.2.1 as well as a finiteness result of Białynicki-Birula [BB98] we prove

Corollary 1.2.6 (Finiteness of momentum map quotients, Cor. 11.4). *Let $G = K^{\mathbb{C}}$ be a complex reductive Lie group and let X be a G-irreducible algebraic G-variety with at worst 1-rational singularities. Then there exist only finitely many (necessarily Zariski-open) subsets of X that can be realised as the set of μ-semistable points with respect to some K-invariant Kähler structure and some momentum map $\mu : X \to \mathrm{Lie}(K)^*$ with compact zero fibre $\mu^{-1}(0)$.*

Finally, we refine the statement of Theorem 1.2.1 in the case of semisimple group actions on projective varieties: using arguments of Hilbert-Mumford type and an ampleness criterion of Hausen [Hau04, Thm. 5.4] and Białynicki-Birula/Święcicka we show the following result:

Corollary 1.2.7 (Thm. 11.6). *Let G be a connected semisimple complex Lie group with maximal compact subgroup K, and let X be an irreducible projective algebraic Hamiltonian G-variety with at worst 1-rational singularities with momentum map $\mu : X \to \mathrm{Lie}(K)^*$. Then there exists an ample G-linearised line bundle L on X such that $X(\mu)$ coincides with the set $X(L, G)$ of semistable points with respect to L.*

1.3 Rational singularities and quotients by holomorphic group actions

[Gre11] **Daniel Greb**, *Rational singularities and quotients by holomorphic group actions*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (5), Vol. **X** (2011), issue 2, 413-426.

As we have seen in the previous section, when studying algebraic or holomorphic group actions, knowledge about the singularities of associated quotient varieties often turns out to be a crucial step in the arguments. In a fundamental paper [Bou87], Jean-François Boutot proved that the class of varieties with *rational singularities* is stable under taking good quotients by reductive groups. This important result generalises the Hochster-Roberts Theorem [HR74] and has found various applications in algebraic geometry, symplectic geometry, and representation theory.

In the paper under discussion I study singularities of analytic Hilbert quotients of complex spaces by complex reductive and compact Lie groups and prove the following refined analytic version of Boutot's result.

Theorem 1.3.1 (Thm. 1.1)**.** *Let G be a complex reductive Lie group acting holomorphically on a complex space X. Assume that the analytic Hilbert quotient $\pi : X \to X/\!/G$ exists.*

1. *If X has rational singularities, then $X/\!/G$ has rational singularities.*
2. *If X has 1-rational singularities, then $X/\!/G$ has 1-rational singularities.*

Recall from Section 1.2 that the notion of *1-rational singularity* gains its importance from the fact that it provides the natural setup in which projectivity results for Kähler Moishezon manifolds extend to singular spaces, see [Nam02, Thm. 1.6]. Moreover, as we have also seen in Section 1.2, complex spaces with 1-rational singularities play a decisice role in the algebraicity results for momentum map quotients of algebraic varieties obtained in my papers [Gre10a, Gre10b].

As a corollary of Theorem 1.3.1 the following result for spaces on which there are a priori no actions of complex groups, e.g. bounded domains, is proven.

Theorem 1.3.2 (Thm. 1.2)**.** *Let K be a compact real Lie group acting holomorphically on a complex space X. Assume that the analytic Hilbert quotient $\pi : X \to X/\!/K$ exists.*

1. *If X has rational singularities, then $X/\!/K$ has rational singularities.*
2. *If X has 1-rational singularities, then $X/\!/K$ has 1-rational singularities.*

Theorem 1.3.2 follows from Theorem 1.3.1 via an application of Heinzner's results [Hei91, HI97] on the existence of *universal complexifications* for Stein complex K-spaces.

Furthermore, we obtain the following refinement of Boutot's result in the algebraic category as a byproduct of our proof of Theorem 1.3.1.

Corollary 1.3.3 (Thm. 4.3). *Let X be a normal algebraic G-variety with good quotient $\pi : X \to X /\!/ G$. Let $f : \widetilde{X} \to X$ be a resolution of X, and let $g : Z \to X /\!/ G$ a resolution of $X /\!/ G$. Assume that $R^j f_* \mathscr{O}_{\widetilde{X}} = 0$ for $1 \leq j \leq q$. Then also $R^j g_* \mathscr{O}_Z = 0$ for $1 \leq j \leq q$.*

This corollary generalises the main result of my unpublished preprint [Gre09] and can be used to study quotients of G-varieties with worse than rational singularities; see for example [Kov99, Lemma 3.3] for cohomological conditions imposed by Cohen-Macaulay singularities.

Although the main result refers to complex spaces with group action, one of the key technical results of this paper does not refer to the equivariant setup at hand and may also be of independent interest.

Theorem 1.3.4 (Vanishing for cohomology with support I, Thm. 3.6). *Let $f : \widetilde{X} \to X$ be a resolution of an irreducible normal complex space X of dimension $n \geq 2$. Let $x \in X$, $F = f^{-1}(x)_{\mathrm{red}}$, and assume that $\bigcup_{k \geq 1} \operatorname{supp} R^k f_* \mathscr{O}_{\widetilde{X}} \subset \{x\}$. Then, we have*

$$H^q_F(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = \{0\} \quad \text{for all } q < n.$$

This vanishing result follows from the analytic version of Grauert-Riemenschneider vanishing (see e.g. [Sil77, Lem. A.2]) via an application of relative duality. An analogous result has already been used by Boutot in his proof of the rationality of quotient singularities, see the proposition on p. 66 of [Bou87]. However, while the derivation of Theorem 1.3.4 from Grauert-Riemenschneider vanishing is straightforward in the algebraic category, in the analytic category a detailed comparison of algebraic and analytic cohomology groups with support is needed. This part of the article generalises work of Karras [Kar86, Sect. 1] from isolated to certain non-isolated singularities.

1.4 Invariant meromorphic functions on Stein manifolds

[GM11] **Daniel Greb** and Christian Miebach, *Invariant meromorphic functions on Stein manifolds*, `arXiv:1010.2935`, will appear in Annales de l'Institut Fourier (Grenoble) **61** (2011).

One of the fundamental results relating invariant theory and the geometry of algebraic group actions is Rosenlicht's Theorem [Ros56, Thm. 2]: for any action of a linear algebraic group on an algebraic variety there exists a finite set of invariant rational functions that separate orbits in general position. Moreover, there exists a rational quotient, i.e., a Zariski-open invariant subset on which the action admits a geometric quotient. In Section 1.2 we have seen one example how this result can be used to study the geometry of algebraic group actions.

The purpose of the paper under discussion is to study meromorphic functions invariant under holomorphic group actions and to construct quotients of Rosenlicht-type in the analytic category.

Examples of non-algebraic holomorphic actions of $\mathbb{C}^*$ on abelian varieties with nowhere Hausdorff orbit space show that even in the compact Kähler case an analogue of Rosenlicht's Theorem does not hold without further assumptions. On the positive side, if a complex reductive group acts *meromorphically* on a compact Kähler space (and more generally a compact complex space of class $\mathscr{C}$), the existence of meromorphic quotients was shown by Lieberman [Lie78] and Fujiki [Fuj78].

As a natural starting point in the non-compact case we consider group actions on spaces with rich function theory such as Stein spaces. Actions of reductive groups and their subgroups on these spaces are known to possess many features of algebraic group actions. However, while the holomorphic invariant theory in this setup is well understood, cf. [Hei91, Sno82] as well as the discussion in Section 1.1, invariant meromorphic functions until now have been much less studied.

In the paper we develop fundamental tools to study meromorphic functions in an equivariant setup. We use these tools to prove the following result, which provides a natural generalisation of Rosenlicht's Theorem to Stein spaces with actions of complex reductive groups.

Theorem 1.4.1 (Main Theorem)**.** *Let $H < G$ be an algebraic subgroup of a complex reductive Lie group G and let X be a Stein G-space. Then, there exists an H-invariant Zariski-open dense subset Ω in X and a holomorphic map $p\colon \Omega \to Q$ to a Stein space Q such that*

1. *the map p is a geometric quotient for the H-action on Ω,*
2. *the map p is universal with respect to H-stable analytic subsets of Ω,*
3. *the map p is a submersion and realises Ω as a topological fibre bundle over Q,*
4. *the map p extends to a weakly meromorphic map (in the sense of Stoll[1]) from X to Q,*
5. *for every H-invariant meromorphic function $f \in \mathscr{M}_X(X)^H$, there exists a unique meromorphic function $\bar{f} \in \mathscr{M}_Q(Q)$ such that $f|_\Omega = \bar{f} \circ p$, i.e., the map $p\colon \Omega \to Q$ induces an isomorphism between $\mathscr{M}_X(X)^H$ and $\mathscr{M}_Q(Q)$, and*
6. *the H-invariant meromorphic functions on X separate the H-orbits in Ω.*

The idea of proof is to first establish a weak equivariant analogue of Remmert's and Narasimhan's embedding theorem for Stein spaces [Nar60]. More precisely, given a G-irreducible Stein G-space we prove the existence of a G-equivariant holomorphic map into a finite-dimensional G-representation space V that is a proper embedding when restricted to a big Zariski-open G-invariant subset. Since the G-action on V is algebraic, we may then apply Rosenlicht's Theorem to this linear action. Subsequently, a careful

[1]We refer the reader to [Sto58a], [Sto58b], and Section 2.4 of the paper for the precise definition.

comparison of algebraic and holomorphic geometric quotients allows us to carry over the existence of a Rosenlicht-type quotient from V to X.

The geometric quotient constructed in this paper provides us with a new and effective tool to investigate invariant meromorphic functions on Stein spaces. In the following we shortly describe two applications of our main result.

Given a Stein G-space we show that every invariant meromorphic function is a quotient of two invariant holomorphic functions precisely if the generic fibre of the analytic Hilbert quotient $\pi\colon X \to X/\!\!/G$ contains a dense orbit, see Theorem 3.5 and Corollary 3.6 of the paper. An important class of examples for this situation consists of representation spaces of semisimple groups G.

An important fundamental result of Richardson [Ric74] states that in every connected Stein G-manifold there exists an open and dense subset on which all isotropy groups are conjugate in G. As a further application of the Main Theorem, we prove the following sharpening of Richardson's result.

Proposition 1.4.2 (Prop. 3.11)**.** *Let G be a complex reductive Lie group and let X be a G-connected Stein G-manifold. Assume that the principal orbit type is reductive. Then,*

1. *in the statement of the Main Theorem the set Ω can be chosen in such a way that for all $x, y \in \Omega$ there exists a $g \in G$ with $G_y = gG_xg^{-1}$. In particular, there exists a G-invariant Zariski-open dense subset of X consisting of orbits of principal orbit type;*
2. *additionally, Ω can be chosen such that $p\colon \Omega \to Q$ is an analytic Hilbert quotient. In particular, Ω is Stein and $p\colon \Omega \to Q$ is a holomorphic fibre bundle with typical fibre G/G_x.*

As an immediate consequence, we obtain

Corollary 1.4.3 (Cor. 3.13)**.** *Let G be a complex reductive Lie group and let X be a G-connected Stein G-manifold. Assume that the action is generically free. Then, in the statement of the Main Theorem the set Ω can be chosen such that $p\colon \Omega \to Q$ is a G-principal fibre bundle.*

In particular, for every effective action of $(\mathbb{C}^*)^k$ on a Stein manifold we find a Zariski-open subset that is a principal fibre bundle over the meromorphic quotient.

Chapter 2

Differential forms on singular spaces

2.1 Introduction

Holomorphic differential forms are an indispensable tool to study the global geometry of non-singular projective varieties and compact Kähler manifolds. In the presence of singularities, with the exception of forms of degree one and forms of top degree, the influence of differential forms on the geometry of a variety is much less explored. At the same time, in higher-dimensional algebraic geometry one is naturally lead to consider singular varieties in a given birational equivalence class, even if one is primarily interested in projective manifolds. Hence, the need for a comprehensive geometric and cohomological theory of differential forms on singular spaces arises.

On singular varieties there are various approaches to define the right analogue of the sheaves of differential forms on a smooth variety. One candidate is the sheaf of *reflexive differentials* $\Omega_X^{[p]}$, i.e., the push-forward of the sheaf of differential forms on the smooth locus X_{reg} of X, which coincides with the double dual of the sheaf of Kähler differentials $\Omega_X^p = \bigwedge^p \Omega_X^1$; a further candidate is the push-forward $\pi_*\Omega_{\widetilde{X}}^p$ of the sheaf of differential forms on a desingularisation $\widetilde{X}$ of X (which is in fact independent of the chosen resolution π). In general, these two sheaves do not coincide. It was observed by Grauert and Riemenschneider in [GR70] that on a normal variety Serre duality holds for the sheaf $\Omega_X^{[n]} = \omega_X$ of reflexive n-forms while Kodaira vanishing holds for $\pi_*\Omega_{\widetilde{X}}^n$, $n = \dim X$. Consequently, it is a natural first question to ask when the two sheaves $\pi_*\Omega_{\widetilde{X}}^n$ and $\Omega_X^{[n]}$ coincide.

To answer this question we first focus on the case in which X has locally free dualising sheaf ω_X. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities. If $U \subseteq X$ is any open subset, to give a section $\tau \in \omega_X(U)$, it is equivalent to give an n-form on the smooth locus of U. In other words, to give a section $\tau \in \omega_X(U)$, it is equivalent to give an n-form $\tau' \in \omega_{\widetilde{X}}(\pi^{-1}(U) \setminus E)$, where $E \subset \widetilde{X}$ is the exceptional locus of the resolution map π. In contrast, a section $\sigma \in \widetilde{\omega}_X(U)$ is, by definition, an n-form $\sigma' \in \omega_{\widetilde{X}}(\pi^{-1}(U))$.

In summary, we obtain the following equivalent reformulation of our question:

Question 2.1.1. *When is it true that any n-form, defined on an open set of the form $\pi^{-1}(U) \setminus E \subset \widetilde{X}$ extends across E, to give a form on $\pi^{-1}(U)$?*

Recall for example from [KM98, Chap. 2.3] that a variety is said to have *canonical* singularities if the canonical sheaf ω_X (or equivalently the canonical divisor K_X) is $\mathbb{Q}$-Cartier, and if for any resolution $\pi : \widetilde{X} \to X$ with corresponding expression

$$K_{\widetilde{X}} \equiv \pi^*(K_X) + \sum_{E\ \pi-\text{exc.}} a(E, X) \cdot E$$

all the discrepancies $a(E, X)$ are greater than or equal to zero. Hence, in the situation considered earlier (ω_X locally free) we see that, by definition, n-forms extend to any resolution if and only if X has canonical singularities. Furthermore, still under the assumption of a locally free dualising sheaf, a variety has canonical singularities if and only if it has rational singularities, a class of singularities studied in detail in my work on group actions surveyed in Chapter 1 above.

Based on these observations and the importance of differential forms of arbitrary degree in the smooth setup, as a next step it is natural to ask the following.

Question 2.1.2. *Given a variety X with at worst canonical singularities, is it true that any p-form, defined on an open set of the form $\pi^{-1}(U) \setminus E \subset \widetilde{X}$ extends across E, to give a form on $\pi^{-1}(U)$?*

Using the notation $\Omega_X^{[p]} = \left(\bigwedge^p \Omega_X^1\right)^{**} = (\imath_{X_{\text{reg}}})_* \Omega^p_{X_{\text{reg}}}$ introduced above, Question 2.1.2 is equivalent to the question whether or not

$$\pi_* \Omega^p_{\widetilde{X}} = \Omega_X^{[p]} \quad \forall 1 \le p \le n \tag{2.1}$$

for any resolution $\pi : \widetilde{X} \to X$ of singularities of a variety X with at worst canonical singularities.

Yet another reformulation will be technically convenient: recall that a sheaf $\mathscr{F}$ on a (normal) variety is called *reflexive* if $\mathscr{F}^{**} \cong \mathscr{F}$, and that this property is equivalent to $\mathscr{F}$ being torsion-free and normal, cf. [OSS80, Chap. 2, Lem. 1.1.12]. Since $\Omega_X^{[p]}$ and $\pi_*\Omega^p_{\widetilde{X}}$ agree outside of the singular set X_{sing} of X, the equality (2.1) holds if and only if $\pi_*\Omega^p_{\widetilde{X}}$ is reflexive.

Before describing the results obtained in this research project in detail, let us mention that the extension question has been studied by quite a number of authors in the past, mostly asking extension only for special values of p. For a variety X with only isolated singularities, reflexivity of $\pi_*\Omega^p_{\widetilde{X}}$ was shown by Steenbrink and van Straten for $p \le \dim X - 2$ without any further assumption on the nature of the singularities, [vSS85, Thm. 1.3]. Flenner extended these results to normal varieties, subject to the condition that $p \le \operatorname{codim} X_{\text{sing}} - 2$, [Fle88]. Namikawa proved reflexivity for $p \in \{1, 2\}$, in case X has canonical Gorenstein singularities, [Nam01, Thm. 4]. In the case of finite quotient singularities similar results were obtained in [dJS04]. A related setup involving snc pairs (X, D), and compositions of finite Galois coverings and subsequent resolutions of singularities has been studied by Esnault and Viehweg. In [EV82] these authors obtain in their special setting a result similar to the results described in Sections 2.2 and 2.3 below and additionally prove vanishing of higher direct image sheaves. We would also like to mention the article [Bar78] where differential forms are discussed even in non-normal settings.

In the following, we work in the setup and using the language of the Minimal Model Program. For the basic definitions, especially of the various classes of singularities considered, we refer the reader to [KM98] and [Kol97].

2.2 Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties

[GKK10] **Daniel Greb**, Stefan Kebekus, and Sándor J. Kovács, *Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties*, Compositio Mathematica **146** (2010), 193-219.

In this paper, we establish an extension result for forms of degree one, n, and $n-1$ on n-dimensional varieties with at worst log canonical singularities. More precisely, we prove:

Theorem 2.2.1 (Extension theorem for log canonical pairs, Thm. 1.1). *Let Z be a normal variety of dimension n and $\Delta \subset Z$ a reduced divisor such that the pair (Z, Δ) is log canonical. Let $\pi : \widetilde{Z} \to Z$ be a log resolution, and set*

$$\widetilde{\Delta}_{\mathrm{lc}} := \textit{largest reduced divisor contained in } \pi^{-1}(\Delta \cup \textit{centers of log canonicity}).$$

If $p \in \{n, n-1, 1\}$, then the sheaf $\pi_* \Omega^p_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is reflexive.

In particular, Theorem 2.2.1 implies that 1-forms defined on the smooth locus of a variety with at worst canonical singularities extend to any resolution, thus answering Question 2.1.2 postively for degree $p = 1$. For a special case of the statement for surfaces, see [Lan03, Cor. 4.3].

In order to discuss an application of Theorem 2.2.1, recall the well-known Bogomolov-Sommese vanishing theorem for snc pairs, cf. [EV92, Cor. 6.9]: If Z is a smooth projective variety, $\Delta \subset Z$ a divisor with simple normal crossings and $\mathscr{A} \subset \Omega^p_Z(\log \Delta)$ an invertible subsheaf, then the Kodaira-Iitaka dimension of $\mathscr{A}$ is not larger than p, i.e., $\kappa(\mathscr{A}) \leq p$. As a corollary to Theorem 2.2.1, we show in the paper that a similar result holds for threefold pairs with log canonical singularities.

Theorem 2.2.2 (Bogomolov-Sommese vanishing for log canonical threefolds and surfaces, Thm. 1.4). *Let Z be a normal variety of dimension $\dim Z \leq 3$ and let $\Delta \subset Z$ be a reduced divisor such that the pair (Z, Δ) is log canonical. Let $\mathscr{A} \subset \Omega^{[p]}_Z(\log \Delta)$ be a reflexive subsheaf of rank one. If $\mathscr{A}$ is $\mathbb{Q}$-Cartier, then $\kappa(\mathscr{A}) \leq p$.*

Let us conclude with a few comments on the proof of the main result. The extension for n-forms (after a finite covering trick) follows almost immediatly from the definitions. The proof of Theorem 2.2.1 for $(n-1)$-forms relies on universal properties of the functorial resolution of singularities and on liftings of local group actions; the necessary results concerning functorial resolutions and group actions are recalled in the paper. The proof of extension for 1-forms has two parts: Using Hodge-theoretic results of Steenbrink [Ste85] and Namikawa [Nam01] as well as local duality, extendability over isolated points is established first. This is then used as the base for an induction on dimension which is carried out using a hyperplane cutting procedure.

2.3 Differential forms on log canonical spaces

> [GKKP11] **Daniel Greb**, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, *Differential Forms on Log Canonical Spaces*, Publications Mathématiques de l'IHÉS **114** (2011), Number 1, 87-169.

This paper picks up the question of extendability of differential forms, Question 2.1.2. Given a quasi-projective variety X and a resolution of singularities $\widetilde{X} \to X$ with exceptional set $E \subset \widetilde{X}$, its main result gives conditions to guarantee that any differential p-form on $\widetilde{X} \setminus E$ extends across E. In its simplest form, it can be stated as follows:

Theorem 2.3.1 (Extension theorem for differential forms on klt varieties)**.** *Let X be a complex quasi-projective variety with at most Kawamata log terminal (klt) singularities and $\pi : \widetilde{X} \to X$ a log resolution. Then $\pi_* \Omega^p_{\widetilde{X}}$ is reflexive for all $p \leq \dim X$.*

Note that canonical singularities are klt, so this result provides a complete positive answer to Question 2.1.2 asked in the introduction to this chapter.

In fact, we prove the following more general result in the category of varieties with log canonical singularities.

Theorem 2.3.2 (Extension theorem for differential forms on lc pairs, Thm. 1.5)**.** *Let X be a complex quasi-projective variety of dimension n and let D be a $\mathbb{Q}$-divisor on X such that the pair (X, D) is log canonical. Let $\pi : \widetilde{X} \to X$ be a log resolution with π-exceptional set E and*

$$\widetilde{D} := \textit{largest reduced divisor contained in } \operatorname{supp} \pi^{-1}(\textit{non-klt locus}),$$

where the non-klt locus is the smallest closed subset $W \subset X$ such that (X, D) is klt away from W. Then the sheaves $\pi_* \Omega^p_{\widetilde{X}}(\log \widetilde{D})$ are reflexive, for all $p \leq n$.

This extends the results described in Section 2.2 to forms of arbitrary degree. A number of examples are provided in the paper in order to show that the result is sharp in many respects.

In addition to the extension results discussed above, we develop a theory of reflexive differential forms on divisorially log terminal (dlt) pairs, showing that many of the fundamental theorems and techniques known for sheaves of logarithmic differentials on smooth varieties also exist in the dlt setting. In particular, a satisfactory theory of relative differentials and a residue theory is set up; for more details, see Part III of the paper.

Furthermore, as direct applications of the main result we prove the following two results:

Theorem 2.3.3 (Pull-back map for reflexive differentials on lc pairs, Thm. 4.3)**.** *Let (X, D) be an lc pair, and let $\gamma : Z \to X$ be a morphism from a normal variety Z such that the image of Z is not contained in the reduced boundary or in the singular locus, i.e.,*

$$\gamma(Z) \not\subseteq (X, D)_{\mathrm{sing}} \cup \operatorname{supp}\lfloor D \rfloor.$$

If $1 \le p \le \dim X$ is any index and

$$\Delta := \textit{largest reduced Weil divisor contained in } \gamma^{-1}\big(\textit{non-klt locus}\big),$$

then there exists a sheaf morphism,

$$d\gamma : \gamma^* \Omega_X^{[p]}(\log\lfloor D \rfloor) \to \Omega_Z^{[p]}(\log \Delta),$$

that agrees with the usual pull-back morphism of Kähler differentials at all points $p \in Z$ where $\gamma(p) \notin (X, D)_{\mathrm{sing}} \cup \operatorname{supp}\lfloor D \rfloor$.

Theorem 2.3.4 (Reflexive differentials on rationally chain connected spaces, Thm. 5.1)**.** *Let (X, D) be a klt pair. If X is rationally chain connected, then X is rationally connected, and*

$$H^0\big(X, \Omega_X^{[p]}\big) = 0 \textit{ for all } 1 \le p \le \dim X.$$

Moreover, we generalise the Bogomolov-Sommese vanishing theorem (as already discussed in Section 2.2) to subsheaves of $\Omega_X^{[p]}$ for arbitrary p on log canical spaces and prove a version of the Lipman-Zariski conjecture for klt varieties.

Further applications of Theorem 2.3.2 (concerning pluri-differentials on rationally connected varieties with canonical singularities, Kodaira-Akizuki-Nakano vanishing for reflexive differential forms on klt varieties, and Hodge-theoretic properties of reflexive differentials) can be found in my forthcoming paper "Reflexive differential forms on singular spaces – Geometry and Cohomology" [GKP12], which is joint work with Stefan Kebekus and Thomas Peternell.

The proof of Theorem 2.3.2 is technically quite involved and proceeds in two major steps. First, extension with logarithmic poles is established. For this the following new vanishing theorem for direct image sheaves on log canonical pairs is proven.

Theorem 2.3.5 (Steenbrink-type vanishing for log canonical pairs, Thm. 14.1)**.** *Let (X, D) be a log canonical pair of dimension $n \ge 2$. If $\pi : \widetilde{X} \to X$ is a log resolution of (X, D) with π-exceptional set E and $\widetilde{D} := \operatorname{supp}\big(E + \pi^{-1}\lfloor D \rfloor\big)$, then*

$$R^{n-1}\pi_*\big(\Omega^p_{\widetilde{X}}(\log \widetilde{D}) \otimes \mathscr{O}_{\widetilde{X}}(-\widetilde{D})\big) = 0 \quad \textit{for all } 0 \le p \le n.$$

This result generalises a vanishing theorem of Steenbrink [Ste85]. It follows from the statement in the special case $p = 0$, which in turn is a consequence of a recent theorem of Kollár and Kovács [KK10] stating that log canonical singularities are Du Bois.

In the second main step, in a delicate induction on the dimension of X and the dimension of X_{sing} the recent advances in the minimal model program and the theory of reflexive differentials on dlt pairs developed in the paper are applied to exclude the appearance of logarithmic poles over klt points of X. Hereby, we use the results established in the earlier paper [GKK10] to anchor the induction.

2.4 Torsion and cotorsion in the sheaf of Kähler differentials on some mild singularities

[GR11] **Daniel Greb** and Sönke Rollenske, *Torsion and cotorsion in the sheaf of Kähler differentials on some mild singularities*, Mathematical Research Letters **18** (2011), no. 6, 1-11.

As we have seen in Sections 2.2 and 2.3 above, in many respects reflexive differential forms on spaces with singularities as they appear in the Minimal Model Program play a role analogous to regular differential forms on smooth varieties.

On the other hand, given a variety Z with arbitrary singularities, the sheaf of Kähler differentials Ω^1_Z and its higher exterior powers play an important role in many contexts, most prominently deformation theory and (if Z is sufficiently nice) duality theory. For an example involving local properties one could mention Berger's conjecture that a curve is smooth if and only if its sheaf of differentials is torsionfree (see e.g. [Ber63, Gre82, Poh91]) or the Zariski–Lipman conjecture: if $(\Omega^1_Z)^*$ is locally free then Z is regular (see e.g. [Pla88, BLLS02]).

Consequently, it is natural to investigate under what kind of additional assumptions the sheaf of Kähler differentials Ω^1_X coincides with its double dual $\Omega^{[1]}_X$, the sheaf of reflexive differentials. In other words, one might ask the following.

Question 2.4.1. *If Z has* mild *singularities, is Ω^1_Z reflexive or at least torsionfree?*

Phrasing this slightly differently we ask if the natural map $\phi\colon \Omega^1_Z \to \left(\Omega^1_Z\right)^{**}$, whose kernel is the torsion submodule $\mathrm{Tors}(\Omega^1_Z)$, is bijective or at least injective. In the terminology of [Rei87, (1.6)] we say that Ω^1_Z has *cotorsion* if ϕ is not surjective.

If one translates *mild singularity* as being a local complete intersection (lci), then indeed a complete answer to Question 2.4.1 is known.

Theorem 2.4.2 ([Kun86])**.** *Let Z be a local complete intersection. Then Ω^1_Z satisfies Serre's condition S_d if and only if Z is regular in codimension d. In particular, if Z is a normal local complete intersection, then Ω^1_Z is torsionfree, and it is reflexive if and only if Z is is non-singular* *in codimension 2.*

This follows from the Auslander-Buchsbaum Formula, see Proposition 9.7 and Corollary 9.8 in [Kun86] for details.

As discussed in Section 2.1, in the context of modern birational geometry one usually measures the singularities of a normal variety in terms of discrepancies $a(X, E)$, which give rise to the definition of terminal, canonical, log terminal and other singularities [KM98, Section 2.3]. However, even terminal singularities, the mildest class considered, are in general not complete intersections, so Theorem 2.4.2 does not apply.

Somewhat contrary to our expectations, in the paper under discussion we show that the answer to Question 2.4.1 is essentially negative if one interprets "mild" in the sense of birational geometry: as soon as one leaves the world of local complete intersections the sheaf of Kähler differentials can have both torsion and cotorsion.

There are two simple methods to produce non-lci singularities: quotients of finite groups and affine cones over projective varieties. In the second case injectivity of ϕ is related to global properties of the embedding $X \hookrightarrow \mathbb{P}^n$. Exploring these connections we give a criterion for the existence of torsion differentials on cones. In order to formulate our result, recall that if $X \subset \mathbb{P}^n$ is a projective variety or scheme given by an ideal sheaf $\mathscr{I}$ then X is called *projectively normal* if $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \twoheadrightarrow H^0(X, \mathscr{O}_X(d))$ for all $d \geq 0$ or, equivalently, $H^1(\mathbb{P}^n, \mathscr{I}(d)) = 0$ for all $d \geq 0$.

Theorem 2.4.3 (Thm. 1.3). *Let $X \subset \mathbb{P}^n_k$ be a smooth projective variety over an algebraically closed field k of characteristic zero and let $\mathscr{I}$ be the sheaf of ideals defining X. Let $C_X \subset \mathbb{A}^{n+1}$ be the affine cone over X.*

a) If $H^1(\mathbb{P}^n, \mathscr{I}^2(d)) = 0$ for all $d \geq 0$ then $\Omega^1_{C_X}$ is torsionfree.

b) If in addition X is projectively normal, then $\Omega^1_{C_X}$ is torsionfree if and only if also the first infinitesimal neighbourhood of X in $\mathbb{P}^n_k$ is projectively normal.[1]

In the cone situation considered so far the study of cotorsion is more problematic. However, results by Knighten and Steenbrink lead to an easily formulated sufficient criterion for cotorsion on finite quotient singularities, cf. Section 3 of the paper.

As our main class of examples, we study the affine cone over the d-th Veronese embedding of $\mathbb{P}^r$, which we denote by $X_{r,d}$. The variety $X_{r,d}$ also has a description as a cyclic quotient singularity. Consequently, using the results discussed above, we are able to control both torsion and cotorsion in the corresponding sheaves of Kähler differentials. We show that such cones have torsion differentials if and only if $d \geq 3$ and cotorsion as soon as $d \geq 2$; some significant cases are collected in Table 2.4 below[2].

In the surface case our results are optimal in the following sense: let Z be a surface singularity. If Z is terminal then it is smooth and thus Ω^1_Z is locally free. If Z is canonical but not terminal then it is one of the well-known ADE singularities, thus a hypersurface

[1] After the publication of the first version of this preprint J. Wahl brought to our attention that he had already observed this in the projectively normal case [Wah97, Proposition 1.4].

[2] The sheaf of Kähler differentials for $X_{1,2}$ and $X_{1,3}$ was also computed in [Kni73].

Table 2.1: Some Veronese cones with torsion or cotorsion in $\Omega^1_{X_{d,r}}$.

singularity	dim	type	Gorenstein	torsion	cotorsion
$X_{1,2}$ (A1)	2	canonical	yes	no	yes
$X_{1,3}$	2	log terminal	no	yes	yes
$X_{2,2}$	3	terminal	no	no	yes
$X_{2,3}$	3	canonical	yes	yes	yes
$X_{3,2}$	4	terminal	yes	no	yes
$X_{3,3}$	4	terminal	no	yes	yes
$X_{5,3}$	6	terminal	yes	yes	yes

singularity; by Theorem 2.4.2 the sheaf of Kähler differentials Ω^1_Z is torsionfree but not reflexive in this case. The most simple log terminal point is the cone over the twisted cubic $X_{1,3}$ and in this case Ω^1_Z has both torsion and cotorsion.

Furthermore, Gorenstein terminal 3-fold singularities are hypersurface singularities by a result of Reid [Rei87, (3.2) Theorem]; hence Theorem 2.4.2 shows that the sheaf of Kähler differentials is reflexive in this case.

2.5 Singular spaces with trivial canonical class

[GKP11] **Daniel Greb**, Stefan Kebekus, and Thomas Peternell, *Singular spaces with trivial canonical class*, `arXiv:1110.5250`, 2011.

In this research project we use the results concerning differential forms on singular spaces that were laid out in the previous sections to investigate singular spaces with trivial canonical class.

The minimal model program aims to reduce the birational study of projective manifolds with Kodaira dimension zero to the study of associated *minimal models*, that is, normal varieties X with terminal singularities whose canonical divisor is numerically trivial. The ideal case, where the minimal variety X is smooth, is described in the fundamental Decomposition Theorem of Beauville and Bogomolov.

Theorem 2.5.1 (Beauville-Bogomolov Decomposition, [Bea83] and references therein)**.** *Let X be a compact Kähler manifold whose canonical divisor is numerically trivial. Then there exists a finite étale cover, $X' \to X$ such that X' decomposes as a product*

$$X' = T \times \prod_{\nu} X_{\nu} \tag{2.2}$$

where T is a compact complex torus, and where the X_{ν} are irreducible and simply-connected Calabi-Yau– or holomorphic-symplectic manifolds.

The decomposition (2.2) induces a decomposition of the tangent bundle $T_{X'}$ into a direct sum whose summands have vanishing Chern class, and are integrable in the sense of Frobenius' theorem. Those summands that correspond to the X_{ν} are stable with respect to any polarisation.

In view of recent progress in minimal model theory, an analogue of Theorem 2.5.1 for minimal models would be a substantial step towards a complete structure theory for varieties of Kodaira dimension zero. However, since the only known proof of Theorem 2.5.1 heavily uses Kähler-Einstein metrics and the solution of the Calabi conjecture, an immediate generalisation of the Beauville-Bogomolov Decomposition Theorem 2.5.1 to the singular setting is not possible.

The main result of our paper is a Decomposition Theorem for the tangent sheaf of minimal varieties with vanishing first Chern class: Presenting the tangent sheaf as a direct sum of integrable subsheaves which are stable with respect to any polarisation, it can be seen as an infinitesimal or linear analogue of the Beauville-Bogomolov Decomposition 2.5.1 in the singular setting.

Theorem 2.5.2 (Decomposition of the tangent sheaf, Thm. 1.3)**.** *Let X be a normal projective variety with at worst canonical singularities defined over the complex numbers. Assume that the canonical divisor of X is numerically trivial: $K_X \equiv 0$. Then there exists an Abelian variety A as well as a projective variety $\widetilde{X}$ with at worst canonical singularities, a finite cover $f : A \times \widetilde{X} \to X$,* *étale in codimension one, and a decomposition*

$$\mathscr{T}_{\widetilde{X}} \cong \bigoplus \mathscr{E}_i \tag{2.3}$$

such that the following holds.

a) *The $\mathscr{E}_i$ are integrable saturated subsheaves of $\mathscr{T}_{\widetilde{X}}$, with trivial determinants.*

Further, if $g : \widehat{X} \to \widetilde{X}$ is any finite cover, étale in codimension one, then the following properties hold in addition.

b) *The sheaves $(g^*\mathscr{E}_i)^{**}$ are stable with respect to any ample polarisation on $\widehat{X}$.*

c) *The irregularity of $\widehat{X}$ is zero, i.e., $h^1(\widehat{X}, \mathscr{O}_{\widehat{X}}) = 0$.*

In particular, taking g to be the identity, we see that the irregularity of $\widetilde{X}$ is zero, and that the summands $\mathscr{E}_i$ are stable with respect to any polarisation. Additionally, the decomposition found in Theorem 2.5.2 satisfies a certain uniqueness property, see Remark 7.5 of the paper.

In the course of the proof, we show the following two additional results, pertaining to stability of the tangent bundle and to wedge products of differentials forms that are defined on the smooth locus of a minimal model.

Proposition 2.5.3 (Stability of $\mathscr{T}_X$ does not depend on polarisation, Prop. 5.7)**.** *Let X be a normal projective variety having at worst canonical singularities. Assume that K_X is numerically equivalent to zero. If the tangent sheaf $\mathscr{T}_X$ is stable with respect to one polarisation, then it is also stable with respect to any other polarisation.*

Proposition 2.5.4 (Non-degeneracy of the wedge product, Prop. 6.1)**.** *Let X be a normal n-dimensional projective variety X having at worst canonical singularities. Denote the smooth locus of X by X_{reg}. Suppose that the canonical divisor is trivial. If $0 \le p \le n$ is any number, then the natural pairing given by the wedge product on X_{reg},*

$$\bigwedge : H^0\big(X_{\mathrm{reg}}, \Omega^p_{X_{\mathrm{reg}}}\big) \times H^0\big(X_{\mathrm{reg}}, \Omega^{n-p}_{X_{\mathrm{reg}}}\big) \longrightarrow H^0\big(X_{\mathrm{reg}}, \omega_{X_{\mathrm{reg}}}\big) \cong H^0\big(X, \omega_X\big) \cong \mathbb{C},$$

is non-degenerate.

It is the proof of this second result that crucially relies on the extension theorems discussed in Sections 2.2 and 2.3.

The proof of the main result proceeds by taking a maximally destabilising subsheaf $\mathscr{F}$ of the tangent sheaf $\mathscr{T}_X$ (if necessary on some finite cover which is étale in codimension one) and then proving that, possibly after further finite covers, the determinant of $\mathscr{F}$ is trivial. Using this triviality and the non-degeneracy of the wedge product as established in Proposition 2.5.4, the existence of a complementary subsheaf $\mathscr{G}$ with $\mathscr{T}_X = \mathscr{F} \oplus \mathscr{G}$ is established. I.e., $\mathscr{F}$ can be split off, and the procedure can be repeated. Finally, integrability of the individual summands of the decomposition (2.3) follows from a result of Demailly [Dem02].

Based on the Decomposition Theorem 2.5.2, we argue in Chapter 8 of the paper that the natural building blocks for a structure theory of projective manifolds with Kodaira dimension zero are canonical varieties with *strongly stable* tangent sheaf. Strong stability is a formalisation of condition (2.5.2.b) appearing in the Decomposition Theorem 2.5.2.

In dimension no more than five, we show that canonical varieties with strongly stable tangent sheaf fall into two classes, which naturally generalise the notions of irreducible Calabi-Yau– and irreducible holomorphic-symplectic manifolds, respectively. There is ample evidence suggesting that this dichotomy should hold in arbitrary dimension.

Chapter 3

Lagrangian fibrations on hyperkähler manifolds

3.1 Introduction

As we have already seen in Section 2.5, the classical decomposition theorem of Beauville–Bogomolov, Theorem 2.5.1, asserts that every compact Kähler manifold with vanishing first Chern class admits a finite cover which decomposes as a product of complex tori, irreducible Calabi–Yau manifolds, and hyperkähler manifolds. While complex tori are quite well-understood, a classification of Calabi–Yau and hyperkähler manifolds is still far out of reach. Only in dimension two, where Calabi–Yau and hyperkähler manifolds coincide, the theory of K3–surfaces provides a fairly complete picture.

The research surveyed in this chapter is concerned with *hyperkähler manifolds*, that is, compact, simply–connected Kähler manifolds X such that $H^0(X, \Omega^2_X)$ is spanned by a holomorphic symplectic form σ. From a differential-geometric point of view hyperkähler manifolds are Riemannian manifolds whose holonomy is equal to the full unitary–symplectic group $\mathrm{Sp}(n)$.

In order to understand the structure of any (compact) complex manifold X, it is important to study fibrations $f\colon X \to B$ of X over complex spaces B of smaller dimension. For the special class of hyperkähler manifolds it is known by work of Matsushita [Mat99, Mat00, Mat01, Mat03] that in case such a fibration f exists, it is a *Lagrangian fibration*: I.e., $\dim X = 2 \dim B$, and the holomorphic symplectic form σ restricts to zero on the general fibre of f. As a consequence, it follows from a holomorphic version of the classical Arnold–Liouville theorem that the general fibre of f is a smooth (Lagrangian) torus.

On the other hand, in accordance with the case of K3–surfaces (and also motivated by string theory and the idea of mirror symmetry), a simple version of the so–called *Hyperkähler SYZ–conjecture* asks if every hyperkähler manifold can be deformed to a hyperkähler manifold admitting a Lagrangian fibration, cf. [SYZ96][1]. With this as a starting

[1] We refer the reader to [Ver10] for a historical discussion and for a description of known cases.

point, an approach to a rough classification of hyperkähler manifolds has been proposed, see e.g. [Saw03]. Hence, is an important question to determine conditions under which a given hyperkähler manifold admits a (Lagrangian) fibration.

3.2 Lagrangian fibrations on hyperkähler manifolds - On a question of Beauville

[GLR11a] **Daniel Greb**, Christian Lehn, and Sönke Rollenske, *Lagrangian fibrations on hyperkähler manifolds - On a question of Beauville*, `arXiv:1105.3410`, 2011.

By the aforementioned results of Matsushita the general fibre of a Lagrangian fibration is a Lagrangian submanifold biholomorphic to a complex torus. Based on this observation, it is a natural question to ask whether the converse holds. This was formulated by Beauville in [Bea11, Sect. 1.6].

Question B. *Let X be a hyperkähler manifold and $L \subset X$ a Lagrangian submanifold biholomorphic to a complex torus. Is L a fibre of a (meromorphic) Lagrangian fibration $f\colon X \to B$?*

Building on work of Campana, Oguiso, and Peternell [COP10] the paper under discussion gives a positive answer in case X is not projective.

Theorem 3.2.1 (Thm. 4.1)**.** *Let X be a non–projective hyperkähler manifold of dimension $2n$ containing a Lagrangian subtorus L. Then the algebraic dimension of X is n, and there exists an algebraic reduction $f\colon X \to B$ of X that is a holomorphic Lagrangian fibration with fibre L.*

Let us give a short outline of the proof: first, we study the deformation theory of L in X, and, using results of Ran [Ran92, Thm. 2.2], we show that deformations of L cover X. In a second step, we consider the field $\mathcal{M}_X(X)$ of meromorphic functions on X. As X is non-projective and Kähler, the transcendence degree of $\mathcal{M}_X(X)$ cannot be maximal. In other words, the algebraic reduction of X is either trivial or a proper fibration. Structure results about (fibres of) algebraic reductions obtained by Campana, Oguiso and Peternell [COP10] imply that in case the algebraic reduction is not a Lagrangian fibration, its fibres are *isotypically semisimple*, i.e., up to a finite correspondence, the fibres are of the form $S \times \cdots \times S$, where S is a simple compact Kähler manifold, i.e., not covered by proper non-trivial subvarieties. The existence of the covering family of tori obtained from L is then used to derive a contradiction.

Since non-projective hyperkähler manifolds are dense in the universal deformation space of a given (projective) hyperkähler manifold, a natural approach to the projective case of Beauville's question is to use the deformation theory of the pair (X, L): either the pair (X, L) can be deformed to a pair (X', L') that is not projective, or (X, L) is what we call *stably projective*. In the first case, looking at the deformation theory of the Lagrangian fibration on X' guaranteed by Theorem 3.2.1, we are able to carry over the result from the non-projective to the projective case. More precisely, we prove the following result.

Theorem 3.2.2 (Thm. 5.8)**.** *Let X be a projective hyperkähler manifold and $L \subset X$ a Lagrangian subtorus. Then, the pair (X, L) is not stably projective if and only if X admits an almost holomorphic Lagrangian fibration with strong fibre L.*

Here, *strong fibre* means that f is holomorphic near L, and L is a fibre of the corresponding holomorphic map.

The local deformation theory of hyperkähler manifolds (and pairs such as (X, L)) is quite well-understood in terms of the hyperkähler period domain and its (linear) subvarieties; e.g., see [Huy03a, Chap. 25] as well as [Voi92]. Using this rather explicit special geometry we are able to give a completely intrinsic characterisation of stably projective pairs. This in turn leads to the following precise criterion for the existence of a Lagrangian fibration on a projective hyperkähler manifold.[2]

Proposition 3.2.3 (Cor. 5.6)**.** *Let X be a projective hyperkähler manifold and L a Lagrangian subtorus of X. If (X, L) is stably projective, then no effective divisor D in X restricts to zero on L; that is, $c_1(\mathscr{O}_L(D|_L)) \in H^{1,1}(L, \mathbb{R})$ is non–zero for every effective divisor D on X. Consequently, if there exists an effective divisor D on X which restricts to zero on L, then there exists an almost holomorphic Lagrangian fibration $f\colon X \to B$ with strong fibre L.*

If (X, L) is not stably projective and if $f\colon X \dashrightarrow B$ is an almost holomorphic Lagrangian fibration (whose existence is guaranteed by Theorem 3.2.2), then it is natural to search for a holomorphic model of f in the same birational equivalence class. Using the fact that the Minimal Model Program terminates under certain additional assumptions ([BCHM10, CDH11]), the following result is proven in our paper.

Theorem 3.2.4 (Thm. 6.2)**.** *Let X be a projective hyperkähler manifold with an almost holomorphic Lagrangian fibration $f\colon X \dashrightarrow B$. Then there exists a holomorphic model for f on a birational hyperkähler manifold X'. In other words, there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, dashed, "f"'] & X' \arrow[d, "f'"] \\ B \arrow[r, dashed] & B' \end{tikzcd}$$

where f' is a holomorphic Lagrangian fibration on X' and the horizontal maps are birational.

Theorem 3.2.4 proves a special version of the Hyperkähler SYZ–conjecture. Related results were obtained by Amerik and Campana [AC08, Thm. 3.6] in dimension four. Note furthermore that birational hyperkähler manifolds are deformation–equivalent by work of Huybrechts [Huy99, Thm. 4.6], so Theorem 3.2.4 might also lead to a new approach to the general case of the Hyperkähler SYZ–conjecture.

The connection to this circle of ideas is also manifest in the following generalisation of Matsushita's results mentioned in Section 3.1, which we obtain as a corollary of Theorem 3.2.4.

[2]Proposition 3.2.3 has in fact been used by Jun-Muk Hwang and Richard Weiss in their recent preprint [HW12] to give a positive answer to Beauville's question in the remaining projective case.

Theorem 3.2.5 (Thm. 6.7)**.** *Let X be a projective hyperkähler manifold and $f\colon X \dashrightarrow B$ an almost holomorphic map with connected fibres onto a normal projective variety B. If $0 < \dim B < \dim X$, then $\dim B = \frac{1}{2}\dim X$, and f is an almost holomorphic Lagrangian fibration.*

To answer Beauville's question completely it remains to exclude the case of a projective hyperkähler manifold X containing a Lagrangian subtorus L such that (X, L) is stably projective, i.e., does not deform to a non-projective pair. In this case, some deformations of L intersect L in unexpected ways. This suggests a local geometric approach to the problem, which my coauthors and myself explored in [GLR11b]. The results we obtained are summarised in Section 3.3 below.

3.3 Lagrangian fibrations on hyperkähler manifolds: Intersections of Lagrangians, L-reduction, and fourfolds

[GLR11b] **Daniel Greb**, Christian Lehn, and Sönke Rollenske, *Lagrangian fibrations on hyperkähler manifolds: intersections of Lagrangians, L-reduction, and fourfolds*, `arXiv:1110.2680`, 2011.

This paper picks up the study of Lagrangian tori in hyperkähler manifolds described in the previous section. Recall our setup: we are given a hyperkähler manifold X together with a Lagrangian torus $L \subset X$, and we aim to answer Beauville's question if in this situation X admits a (meromorphic) Lagrangian fibration with fibre L.

Our approach in the paper under discussion rests on a more detailed study of the deformation theory of L in X. For this, consider the component $\mathfrak{B}$ of the Barlet space that contains the point $[L]$ corresponding to L, together with its universal family $\mathfrak{U}$ and the evaluation map to X:

$$\begin{tikzcd} \mathfrak{U} \arrow[r, "\epsilon"] \arrow[d, "\pi"'] & X \\ \mathfrak{B}. \end{tikzcd}$$

It was shown in our previously discussed article that ϵ is surjective and generically finite, and that X admits an almost holomorphic Lagrangian fibration if and only if $\deg(\epsilon) = 1$, see [GLR11a, Lemma 3.1].

If the degree of ϵ is strictly greater than one, some deformations of L intersect L in unexpected ways. In order to deal with this, we introduce the notion of *L-reduction*: for each projective hyperkähler manifold containing a Lagrangian torus L we establish the existence of a projective variety $\mathfrak{T}$ and a rational map $\phi_L\colon X \dashrightarrow \mathfrak{T}$, uniquely defined up to birational equivalence, whose fibre through a general point x coincides with the connected component of the intersection of all deformations of L going through x.

In this situation, we say that X is *L-separable* if ϕ_L is birational, and prove the following result:

Theorem 3.3.1 (Thm. 3.5)**.** *Let X be a projective hyperkähler manifold and $L \subset X$ a Lagrangian subtorus. Then X admits an almost holomorphic fibration with strong fibre L if and only if X is not L-separable .*

If X is a hyperkähler fourfold, then we can exclude the case that X is L-separable by an elementary argument in symplectic linear algebra, hence proving that any hyperkähler fourfold containing a Lagrangian torus admits an almost holomorphic Lagrangian fibration. Furthermore, based upon the rather explicit knowledge concerning the birational geometry of hyperkähler fourfolds [WW03] we even obtain a positive answer to the strongest form of Beauville's question:

Theorem 3.3.2 (Thm. 5.1)**.** *Let X be a four-dimensional hyperkähler manifold containing a Lagrangian torus L. Then X admits a holomorphic Lagrangian fibration with fibre L.*

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