Differential geometry of complex vector bundles — Exercises

Problem Set 3

**Problem 1  (Sheaves)**

Let $X$ be a topological space. A sheaf of a abelian groups $\mathcal{F}$ on $X$ consists of the data

* for every open subset $U \subset X$ an abelian group $\mathcal{F}(U)$, and
* for every inclusion $V \subset U$ of open sets, a morphism of abelian groups

$$r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V),$$

satisfying the following conditions

1. $\mathcal{F}(\emptyset) = 0$.
2. $r_{UU} = id_{\mathcal{F}(U)}$.
3. If $W \subset V \subset U$ are three open subsets, then $r_{UV} = r_{VW} \circ r_{UV}$.
4. If $U$ is an open subset, $\{V_i\}$ an open covering of $U$, and $s \in \mathcal{F}(U)$ such that $r_{UV_i}(s) = 0$ for all $i$, then $s = 0$.
5. If $U$ is an open subset, $\{V_i\}$ an open covering of $U$, and $s_i \in \mathcal{F}(V_i)$ are such that

$$r_{V_i(V_i \cap V_j)}(s_i) = r_{V_j(V_i \cap V_j)}(s_j)$$

for all $i, j$, then there exists a unique $s \in \mathcal{F}(U)$ such that $r_{UV_i}(s) = s_i$.

An element $s \in \mathcal{F}(U)$ is called a section, and $r_{UV}$ is called the restriction from $U$ to $V$.

Let now $X$ be a complex manifold, and $E$ a holomorphic vector bundle over $X$. For every open set $U \subset X$, let $\Gamma(U, E)$ be the vector space of sections of $E|_U$. Furthermore, if $V \subset U$ is another open set, we let $r_{UV} : \Gamma(U, E) \to \Gamma(V, E)$ be given by restriction of sections of $E|_U$ to the open subset $V$. Show that this defines a sheaf of abelian groups on $X$, called the sheaf of sections of $E$, and denoted by $\mathcal{O}_X(E)$. In particular, from the trivial line bundle $X \times \mathbb{C}$ we obtain the sheaf $\mathcal{O}_X$ of holomorphic functions on $X$.

Note that the same statement and proof holds for the spaces of differentiable sections.
**Problem 2**  *(Holomorphic line bundles and invertible sheaves)*

Let $\mathcal{F}, \mathcal{G}$ be two sheaves on the topological space $X$. A sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ is a family of group homomorphisms $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$, compatible with restriction maps. If all the $\alpha_U$ are isomorphisms, we call $\alpha$ an isomorphism, and we write $\mathcal{F} \cong \mathcal{G}$.

Let $X$ be a complex manifold. We say a sheaf of abelian groups $\mathcal{F}$ on $X$ is invertible if there exists an open covering $(U_i)_{i \in I}$ of $X$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}$. Show that there exists a bijection between invertible sheaves and holomorphic line bundles on $X$.

**Problem 3**  *(Images and kernels of bundle homomorphisms)*

Let $X$ be a complex manifold, and let $\pi : E \to X$ and $\psi : F \to X$ be holomorphic vector bundles on $X$. A morphism of vector bundles of rank $k$ is a holomorphic map $\phi : E \to F$ such that $\pi = \psi \circ \phi$, and such that for every $x \in X$, the induced map of vector spaces

$$\phi_x : E_x \to F_x$$

is $C$-linear of rank $k$. Show that the image of $E$ is a holomorphic subbundle of rank $k$ of $F$. We set

$$\ker \phi := \{e \in E \mid \phi_{\pi(e)}(e) = 0 \in F_{\pi(e)}\}.$$

Show that $\ker \phi$ is a holomorphic subbundle of rank $\text{rk}E - k$ of $E$.

**Problem 4**  *(Pullback of forms)*

Let $f : X \to Y$ be a holomorphic map between complex manifolds. Show that the pull-back

$$f^* : C^\infty(Y, \Omega^k_Y) \to C^\infty(X, \Omega^k_X)$$

respects the decomposition into forms of type $(p, q)$, i.e., if $\alpha$ has type $(p, q)$, then $f^*(\alpha)$ also has type $(p, q)$.

**Due:** Friday, November 28th, 2014, at the end of the lecture.