

# BOUNDED $K$ -THEORY OF EXACT CATEGORIES

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ABSTRACT. This paper extends the notion of geometric control in algebraic  $K$ -theory from additive categories with split exact sequences to other exact structures. We construct boundedly controlled exact categories over a general proper metric space and recover facts familiar from bounded  $K$ -theory of free geometric modules, including controlled excision. The framework and results are used in our forthcoming computation of the algebraic  $K$ -theory of geometric groups.

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## 1. INTRODUCTION

The main goal of this paper is to establish the boundedly controlled  $K$ -theory of objects from a specific but common kind of exact category filtered by subsets of a proper metric space, similar to the bounded  $K$ -theory of geometric modules. We recover familiar features of controlled theories, such as nonconnective bounded excision, modelling the exposition on the foundational papers of E.K. Pedersen and C. Weibel [21, 22].

One important application of bounded  $K$ -theory of geometric modules is to the study of the assembly map from the homology  $H_*(\Gamma, K(R))$  of a discrete group  $\Gamma$  with coefficients in the  $K$ -theory of a ring  $R$  to the  $K$ -theory  $K_*(R[\Gamma])$  of the group ring. The integral Novikov conjecture is the statement that this assembly is a split injection in all dimensions when  $\Gamma$  has no torsion. This conjecture has been verified for various classes of groups using various methods. Some of the recent work has used bounded  $K$ -theory starting with [4] and in subsequent developments [8, 9, 11, 12, 13]. This paper is the first in a series that studies the splitting of the assembly map when it is available from the work cited above. When the splitting itself is shown to be split injective, the assembly map is an equivalence, and one obtains a proof of the Borel isomorphism conjecture.

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The bounded  $K$ -theory of exact categories introduced in this paper is the enlarged setup required for this work. Among the exact categories amenable to the construction is the most important for the application category of finitely generated modules over a noetherian ring. We should recall that the isomorphism conjecture fails without additional algebraic restrictions on the group ring  $R[\Gamma]$  or the ring  $R$  in particular. The condition that appeared in the work of Waldhausen is regular coherence of the group ring. In [6] we showed that a weaker and much more common condition *weak coherence* suffices to identify the  $K$ -theory of  $\Gamma$ -fixed filtered objects introduced here and the usual  $K$ -theory of finitely generated free  $R[\Gamma]$ -modules. This condition then provides a bridge between the theory in this paper and the classical bounded  $K$ -theory.

The rest of this section will give an outline of the paper. For standard notions of category theory we refer to Mac Lane [19].

**1.1. Definition** (Quillen exact categories). Let  $\mathbf{C}$  be an additive category. Suppose  $\mathbf{C}$  has two classes of morphisms  $\mathbf{m}(\mathbf{C})$ , called *admissible monomorphisms*, and  $\mathbf{e}(\mathbf{C})$ , called *admissible epimorphisms*, and a class  $\mathcal{E}$  of *exact sequences*, or *extensions*, of the form

$$C' : C' \xrightarrow{i} C \xrightarrow{j} C''$$

with  $i \in \mathbf{m}(\mathbf{C})$  and  $j \in \mathbf{e}(\mathbf{C})$  which satisfy the three axioms:

- a) any sequence in  $\mathbf{C}$  isomorphic to a sequence in  $\mathcal{E}$  is in  $\mathcal{E}$ ; the canonical sequence

$$C' \xrightarrow{\text{incl}_1} C' \oplus C'' \xrightarrow{\text{proj}_2} C''$$

is in  $\mathcal{E}$ ; for any sequence  $C' \xrightarrow{i} C \xrightarrow{j} C''$ ,  $i$  is a kernel of  $j$  and  $j$  is a cokernel of  $i$  in  $\mathbf{C}$ ,

- b) both classes  $\mathbf{m}(\mathbf{C})$  and  $\mathbf{e}(\mathbf{C})$  are subcategories of  $\mathbf{C}$ ;  $\mathbf{e}(\mathbf{C})$  is closed under base-changes along arbitrary morphisms in  $\mathbf{C}$  in the sense that for every exact sequence  $C' \rightarrow C \rightarrow C''$  and any morphism  $f: D'' \rightarrow C''$  in  $\mathbf{C}$ , there is a pullback commutative diagram

$$\begin{array}{ccccc} C' & \longrightarrow & D & \xrightarrow{j'} & D'' \\ = \downarrow & & \downarrow f' & & \downarrow f \\ C' & \longrightarrow & C & \xrightarrow{j} & C'' \end{array}$$

where  $j': D \rightarrow D''$  is an admissible epimorphism;  $\mathbf{m}(\mathbf{C})$  is closed under cobase-changes along arbitrary morphisms in  $\mathbf{C}$  in the (dual) sense that for every exact sequence  $C' \rightarrow C \rightarrow C''$  and any morphism  $g: C' \rightarrow D'$  in  $\mathbf{C}$ , there is a pushout diagram

$$\begin{array}{ccccc} C' & \xrightarrow{i} & C & \longrightarrow & C'' \\ g \downarrow & & g' \downarrow & & \downarrow = \\ D' & \xrightarrow{i'} & D & \longrightarrow & C'' \end{array}$$

where  $i': D' \rightarrow D$  is an admissible monomorphism,

- c) if  $f: C \rightarrow C''$  is a morphism with a kernel in  $\mathbf{C}$ , and there is a morphism  $D \rightarrow C$  so that the composition  $D \rightarrow C \rightarrow C''$  is an admissible epimorphism, then  $f$  is an admissible epimorphism; dually for admissible monomorphisms.

According to Keller [17], axiom (c) follows from the other two. We still list it in order to refer to later. We will use the standard notation  $\rightarrow$  for admissible monomorphisms and  $\twoheadrightarrow$  for admissible epimorphisms.

To each small exact category  $\mathbf{E}$ , one associates a sequence of groups  $K_i(\mathbf{E})$ ,  $i \geq 0$ , as in Quillen [24] or a connective spectrum  $K(\mathbf{E})$  whose stable homotopy groups are  $K_i(\mathbf{E})$ .

Our construction is motivated by the delooping of algebraic  $K$ -theory of a small additive category in [21] and, in particular, the introduction of bounded control in a cocomplete additive category  $\mathbf{A}$  which we briefly recall.

**1.2. Definition** (Pedersen–Weibel). Given a proper metric space  $X$ , consider an assignment of an object  $F(x)$  in  $\mathbf{A}$  to each point  $x$  satisfying the local finiteness condition: the subset  $\{x \in S \mid F(x) \neq 0\}$  should be finite for every bounded  $S \subset X$ . Such assignments define  $X$ -graded objects  $\bigoplus_{x \in X} F(x)$  in  $\mathbf{A}$  and form objects of a new category  $\mathcal{B}(X, \mathbf{A})$ . Morphisms are collections of  $\mathbf{A}$ -morphisms  $f(x, y): F(x) \rightarrow F(y)$  with the property that there is a number  $D > 0$  such that  $f(x, y) = 0$  if  $\text{dist}(x, y) > D$ . If  $\mathbf{B}$  is a subcategory of  $\mathbf{A}$  closed under the direct sum, one obtains the additive bounded category  $\mathcal{B}(X, \mathbf{B})$  as the full subcategory of  $\mathcal{B}(X, \mathbf{A})$  on objects  $F$  with  $F(x) \in \mathbf{B}$  for all  $x \in X$ . Notice that  $\mathbf{B}$  does not need to be cocomplete. The bounded algebraic  $K$ -theory  $K_i(X, \mathbf{B})$  is the  $K$ -theory spectrum associated to  $\mathcal{B}(X, \mathbf{B})$ .

To generalize this construction from additive to more general exact categories  $\mathbf{E}$ , first notice that, given an object  $F$  in  $\mathcal{B}(X, \mathbf{B})$ , to every subset  $S \subset X$  there is associated a direct sum  $F(S)$  generated by those  $F(x)$  with  $x \in S$ . The restriction to bounded homomorphisms can be described entirely in terms of these subobjects.

Recall that an *abelian category* is an additive category with kernels and cokernels such that every morphism  $f$  is balanced, that is, the canonical map from the coimage  $\text{coim}(f) = \text{coker}(\ker f)$  to the image  $\text{im}(f) = \ker(\text{coker } f)$  is an isomorphism. An abelian category has the canonical exact structure where all kernels and cokernels are respectively admissible monomorphisms and admissible epimorphisms.

**1.3. Definition.** A full subcategory  $\mathbf{H}$  of a small exact category  $\mathbf{C}$  is said to be *closed under extensions* in  $\mathbf{C}$  if  $\mathbf{H}$  contains a zero object and for any exact sequence  $C' \rightarrow C \rightarrow C''$  in  $\mathbf{C}$ , if  $C'$  and  $C''$  are isomorphic to objects from  $\mathbf{H}$  then so is  $C$ . A *thick subcategory* of an exact category is a subcategory which is closed under isomorphisms, exact extensions, admissible subobjects, and admissible quotients.

It is known from [24] that a subcategory closed under extensions inherits the exact structure from  $\mathbf{C}$ .

Recall that a category is *cocomplete* if it contains colimits of arbitrary small diagrams, cf. Mac Lane [19], chapter 5.

Let  $\mathbf{E}$  be a thick subcategory of a cocomplete abelian category  $\mathbf{F}$ , which is part of the data, and let  $\epsilon: \mathbf{E} \rightarrow \mathbf{F}$  be the embedding. We will demonstrate a specific construction of a nonconnective spectrum  $K_\epsilon^{-\infty}(\mathbf{E})$  whose stable homotopy groups are the Quillen  $K$ -groups of  $\mathbf{E}$  in positive dimensions and give expected answers in other dimensions. We will address dependence of the construction

on the embedding of  $\mathbf{E}$  in  $\mathbf{F}$  but in general avoid such questions as uniqueness and naturality. The flexibility in the choice of the embedding allows one to use natural and convenient noncanonical embeddings. For example, even the simple case of the abelian category of finitely generated modules over a noetherian ring  $R$  embedded in the cocomplete abelian category of all  $R$ -modules is meaningful and satisfies the conditions. It is used in [5, 7].

The first known delooping of the  $K$ -theory of an exact category with these properties is due to M. Schlichting [25], so the result is not new. However the construction is different and is required for the controlled excision here and the equivariant theory in [5].

Recall that a *subobject* of a fixed object  $F$  is a monic  $m: F' \rightarrow F$ . The collection of all subobjects of  $F$  forms a category where morphisms are morphisms  $j: F' \rightarrow F''$  between two subobjects of  $F$  such that  $m''j = m'$ . Notice that such  $j$  are also monic. If the category is exact, there is the subcategory of *admissible subobjects* of  $F$  represented by admissible monomorphisms. If both  $m'$  and  $m''$  are admissible, it follows from exactness axiom 3 that  $j$  is also an admissible monomorphism.

**1.4. Definition.** Given a cocomplete abelian category  $\mathbf{F}$ , the objects of the new category  $\mathbf{U}^b(X, \mathbf{F})$  are the *X-filtered* objects of  $\mathbf{F}$  which consist of an object  $F$  in  $\mathbf{F}$  and a functor from the power set  $\mathcal{P}(X)$  of  $X$  ordered by inclusion to the category of admissible subobjects of  $F$  such that  $X$  is mapped to  $F$ . The morphisms are the *boundedly controlled* morphisms  $f: F_1 \rightarrow F_2$  in  $\mathbf{F}$  such that the image  $f(F_1(S))$  factors through the subobject  $F_2(S[D])$  for a fixed number  $D \geq 0$  and all subsets  $S \subset X$ . Here  $S[D]$  stands for the metric  $D$ -enlargement  $\{x \in X \mid \text{dist}(x, S) \leq D\}$ .

It will be shown in Proposition 2.5 that  $\mathbf{U}^b(X, \mathbf{F})$  is an exact category with kernels and cokernels. To describe the exact structure in  $\mathbf{U}^b(X, \mathbf{F})$  we need to define an additional property a boundedly controlled morphism  $f: F_1 \rightarrow F_2$  may or may not have.

Given two subobjects  $m': F' \rightarrow F$ ,  $m'': F'' \rightarrow F$ , the *intersection*  $F' \cap F''$ , which is the pullback of  $m'$  along  $m''$ , is a subobject of  $F$  and can be written as the kernel of a morphism. If  $F'$  and  $F''$  are admissible subobjects then the intersection  $F' \cap F''$  is an admissible subobject.

**1.5. Definition.** A morphism  $f$  in  $\mathbf{U}^b(X, \mathbf{F})$  is called *boundedly bicontrolled* if in addition to factorizations of subobjects  $f(F_1(S)) \rightarrow F_2(S[D])$  as above, there are factorizations  $f(F_1) \cap F_2(S) \rightarrow fF_1(S[D])$  for all subsets  $S$  of  $X$ .

**1.6. Definition.** The *admissible monomorphisms* in  $\mathbf{U}^b(X, \mathbf{F})$  consist of boundedly bicontrolled morphisms  $m: F_1 \rightarrow F_2$  such that  $m: F_1(X) \rightarrow F_2(X)$  is an admissible monomorphism, that is a monic, in  $\mathbf{F}$ . The *admissible epimorphisms* are the boundedly bicontrolled morphisms  $e: F_1 \rightarrow F_2$  such that  $e: F_1(X) \rightarrow F_2(X)$  is an admissible epimorphism, that is an epi, in  $\mathbf{F}$ .

In Theorem 2.8 we show that  $\mathbf{U}^b(X, \mathbf{F})$  with these choices of admissible morphisms is an exact category.

Now suppose  $\mathbf{E}$  is a thick subcategory in  $\mathbf{F}$ . Let  $\mathcal{B}_D(X)$  stand for the collection of all subsets of  $X$  of diameter bounded by the fixed nonnegative number  $D$  and let  $\mathcal{B}(X)$  stand for the union of all  $\mathcal{B}_D(X)$ .

**1.7. Definition.** The *boundedly controlled category*  $\mathbf{B}(X, \mathbf{E})$  is the full subcategory of  $\mathbf{U}^b(X, \mathbf{F})$  on filtered objects  $F$  such that

- (1) if  $S \in \mathcal{B}(X)$  then  $F(S)$  is an object of  $\mathbf{E}$ , and
- (2) there is a number  $d \geq 0$  such that for any subset  $S$  of  $X$ ,

$$F(S) \subset \sum_{x \in S} F(B_d(x)).$$

This category is clearly additive but now it can be given a larger class of exact sequences with boundedly bicontrolled monics with cokernels in  $\mathbf{B}(X, \mathbf{E})$  and boundedly bicontrolled epis with kernels in  $\mathbf{B}(X, \mathbf{E})$  as respectively admissible monomorphisms and admissible epimorphisms.

We should point out that the embedding of the exact category  $\mathbf{E}$  in a pseudoabelian category is chosen for convenience when defining the control conditions. It is not necessary. For example, saying that the image of  $f|_{F_1(S)}$  factors through  $F_2(S[D])$  is equivalent to saying that the restriction  $f|_{F_1(S)}$  can be factored as  $F_1(S) \rightarrow F_2(S[D]) \rightarrow F_2(X)$ . The map  $f$  is bicontrolled if whenever the restriction  $f|_{F_1(S[D])}$  can be factored through an admissible subobject  $J$  as  $F_1(S[D]) \rightarrow J \rightarrow F_2(X)$ , the admissible subobject  $F_2(S)$  also factors  $F_2(S) \rightarrow J \rightarrow F_2(X)$ . Since naturality and uniqueness issues don't come up in our applications, we prefer the less intrinsic but simpler framework.

The excision results and the definition of nonconnective deloopings  $K_{\epsilon}^{-\infty}(\mathbf{E})$  will require a construction of exact quotient categories of  $\mathbf{B}(X, \mathbf{E})$  using localization. Notice that even if  $\mathbf{F}$  is an abelian category,  $\mathbf{U}^b(X, \mathbf{F})$  is not abelian in general, so we are not in the familiar domain of algebraic  $K$ -theory of abelian categories. The role of Serre subcategories is played by certain full exact subcategories  $\mathbf{B}(X, \mathbf{E})_{<Z}$  of  $\mathbf{B}(X, \mathbf{E})$  for every subspace  $Z$  of  $X$ .

**1.8. Definition.** Let  $\mathbf{B}(X, \mathbf{E})_{<Z}$  be the full subcategory of  $\mathbf{B}(X, \mathbf{E})$  on those objects  $F$  that satisfy  $F(X) = \sum F(S)$  over all  $S \in \mathcal{B}_D(X)$  with  $S \cap Z[d] \neq \emptyset$  for some pair of numbers  $d \geq 0$  and  $D \geq 0$ .

In Proposition 3.2 we show that  $\mathbf{B}(X, \mathbf{E})_{<Z}$  is indeed a thick subcategory of  $\mathbf{B}(X, \mathbf{E})$ , so it is an exact subcategory.

At this point we need to assume that  $\mathbf{E}$  is idempotent complete which can always be achieved by idempotent completion without affecting the positive  $K$ -theory of  $\mathbf{E}$ . This makes  $\mathbf{B}(X, \mathbf{E})$  idempotent complete and allows us to use localization theorems from [25] to produce an exact quotient category  $\mathbf{B}/\mathbf{Z}$ . Identification of  $K(X, \mathbf{E})_{<Z}$  with  $K(Z, \mathbf{E})$  gives the following result.

**1.9. Theorem (Localization).** *The quotient sequence of exact categories*

$$\mathbf{B}(X, \mathbf{E})_{<Z} \rightarrow \mathbf{B}(X, \mathbf{E}) \rightarrow \mathbf{B}/\mathbf{Z}$$

*induces a homotopy fibration*

$$K(Z, \mathbf{E}) \rightarrow K(X, \mathbf{E}) \rightarrow K(\mathbf{B}/\mathbf{Z}).$$

Applying this theorem to the inclusion  $X \subset \mathbb{Z}^{\geq 0} \times X$  and using the fact that  $K(X \times \mathbb{Z}^{\geq 0}, \mathbf{E})$  is contractible for any metric space  $X$ , we obtain a map  $K(X, \mathbf{E}) \rightarrow \Omega K(X \times \mathbb{Z}, \mathbf{E})$  which is a weak equivalence in positive dimensions. If  $S^k \mathbf{E}$  stands for the boundedly controlled category  $\mathbf{B}(\mathbb{Z}^k, \mathbf{E})$ , iterations of this construction give weak equivalences  $K(S^k \mathbf{E}) \rightarrow \Omega^{k+1} K(S^{k+1} \mathbf{E})$ .

For a thick subcategory  $\mathbf{E}$  of  $\mathbf{F}$ , the nonconnective delooping of  $K(\mathbf{E})$  relative to the embedding  $\epsilon: \mathbf{E} \rightarrow \mathbf{F}$  is defined as

$$K_\epsilon^{-\infty}(\mathbf{E}) \stackrel{\text{def}}{=} \overline{\text{hocolim}}_{k>0} \Omega^k K(S^k \mathbf{E}^\wedge),$$

where  $\mathbf{E}^\wedge$  is the idempotent completion of  $\mathbf{E}$ . The positive homotopy groups of  $K^{-\infty}(\mathbf{E})$  coincide with those of  $K(\mathbf{E})$  as desired.

Using the same idea one gets a nonconnective delooping of the controlled  $K$ -theory. If  $\mathbf{E}$  is a subcategory closed under extensions in a cocomplete abelian category and  $X$  is a proper metric space, there is a nonconnective spectrum

$$K^{-\infty}(X, \mathbf{E}) \stackrel{\text{def}}{=} \overline{\text{hocolim}}_{k>0} \Omega^k K\mathbf{B}(X \times \mathbb{Z}^k, \mathbf{E}^\wedge)$$

whose positive stable homotopy groups are the Quillen  $K$ -groups of  $\mathbf{B}(X, \mathbf{E})$ .

If a proper metric space  $X$  is the union of subspaces  $X_1$  and  $X_2$ , let  $\mathbf{B}(X_1, X_2; \mathbf{E})$  stand for the intersection of the thick subcategories  $\mathbf{B}(X, \mathbf{E})_{<X_1}$  and  $\mathbf{B}(X, \mathbf{E})_{<X_2}$  in  $\mathbf{B}(X, \mathbf{E})$ . The following is the main result of the paper.

**1.10. Theorem** (Nonconnective excision). *The commutative diagram*

$$\begin{array}{ccc} K^{-\infty}(X_1, X_2; \mathbf{E}) & \longrightarrow & K^{-\infty}(X_1, \mathbf{E}) \\ \downarrow & & \downarrow \\ K^{-\infty}(X_2, \mathbf{E}) & \longrightarrow & K^{-\infty}(X, \mathbf{E}) \end{array}$$

*is a homotopy pushout.*

An important technical feature in this paper is the passage to derived categories of exact categories which allows to use a package of theorems from Waldhausen  $K$ -theory. For example, the derived category of bounded chain complexes in  $\mathbf{B}(X, \mathbf{E})$  has a useful Waldhausen structure with degree-wise admissible monomorphisms as cofibrations and chain maps with mapping cones homotopy equivalent to an acyclic complex as weak equivalences. Its  $K$ -theory is weakly equivalent to that of  $\mathbf{B}(X, \mathbf{E})$ , so the excision theorem which is first proven in derived  $K$ -theory can be restated as Theorem 1.10.

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## 2. CONTROLLED CATEGORIES OF FILTERED OBJECTS

Let  $X$  be a proper metric space, in the sense that all closed metric balls in  $X$  are compact, and let  $\mathbf{F}$  be a cocomplete abelian category. In general, a category is cocomplete if it has colimits of all small diagrams. The power set  $\mathcal{P}(X)$  is partially ordered by inclusion which makes it into the category with subsets of  $X$  as objects and unique morphisms  $(S, T)$  when  $S \subset T$ . A *presheaf* of  $\mathbf{F}$ -objects on  $X$  is a functor  $F: \mathcal{P}(X) \rightarrow \mathbf{F}$ . This corresponds to the usual notion of presheaf of  $\mathbf{F}$ -objects on the discrete topological space  $X^\delta$  if the chosen Grothendieck topology on  $\mathcal{P}(X)$  is the partial order given by inclusion, cf. section II.1 of [16]. We will use standard in sheaf theory terms such as *structure maps* when referring to the morphisms  $F(S, T)$ .

A presheaf  $F$  is a  $X$ -filtered object if all structure maps of  $F$  are admissible monomorphisms. For each presheaf  $F$  there is an associated  $X$ -filtered object given by  $F_X(S) = \text{im} F(S, X)$ .

**2.1. Definition.** The objects of the *uncontrolled* category  $\mathbf{U}(X, \mathbf{F})$  are the  $X$ -filtrations of objects  $F$  in  $\mathbf{F}$ . The morphisms  $F \rightarrow G$  are the morphisms  $F(X) \rightarrow G(X)$  in  $\mathbf{F}$ .

Let  $S[D]$  denote the subset  $\{x \in X \mid \text{dist}(x, S) \leq D\}$ . A morphism  $f: F \rightarrow G$  in  $\mathbf{U}(X, \mathbf{F})$  is *boundedly controlled* if there is a number  $D \geq 0$  such that the image of  $\phi$  restricted to  $F(S)$  is a subobject of  $G(S[D])$  for every subset  $S \subset X$ . The *bounded category*  $\mathbf{U}^b(X, \mathbf{F})$  is the subcategory of  $\mathbf{U}(X, \mathbf{F})$  with the same objects and the boundedly controlled morphisms.

If  $f$  in addition has the property that for all subsets  $S \subset X$  the pullback  $\text{im}(f) \cap G(S)$  factors through  $fF(S[D])$  as a subobject of the image of  $f$ , then it is called *boundedly bicontrolled*. In this case we say that  $f$  has filtration degree  $D$  and write  $\text{fil}(f) \leq D$ .

**2.2. Remark.** If  $\mathbf{F}$  is an abelian category then a morphism in  $\mathbf{U}^b(X, \mathbf{F})$  is boundedly bicontrolled if and only if it is balanced.

**2.3. Lemma.** Let  $f_1: F \rightarrow G$ ,  $f_2: G \rightarrow H$  be in  $\mathbf{U}(X, \mathbf{F})$  and  $f_3 = f_2 f_1$ .

- (1) If  $f_1, f_2$  are boundedly bicontrolled morphisms and either  $f_1: F(X) \rightarrow G(X)$  is an epi or  $f_2: G(X) \rightarrow H(X)$  is a monic, then  $f_3$  is also boundedly bicontrolled.
- (2) If  $f_1, f_3$  are boundedly bicontrolled and  $f_1$  is epic then  $f_2$  is also boundedly bicontrolled; if  $f_3$  is only boundedly controlled then  $f_2$  is also boundedly controlled.
- (3) If  $f_2, f_3$  are boundedly bicontrolled and  $f_2$  is monic then  $f_1$  is also boundedly bicontrolled; if  $f_3$  is only boundedly controlled then  $f_1$  is also boundedly controlled.

*Proof.* Suppose  $\text{fil}(f_i) \leq D$  and  $\text{fil}(f_j) \leq D'$  for  $\{i, j\} \subset \{1, 2, 3\}$ , then in fact  $\text{fil}(f_{6-i-j}) \leq D + D'$  in each of the three cases. For example, there are factorizations

$$\begin{aligned} f_2 G(S) &\subset f_2 f_1 F(S[D]) = f_3 F(S[D]) \subset H(S[D + D']) \\ f_2 G(X) \cap H(S) &\subset f_3 F(S[D']) = f_2 f_1 F(S[D']) \subset f_2 G(S[D + D']) \end{aligned}$$

which verify part 2 with  $i = 1, j = 3$ .  $\nabla$

**2.4. Lemma.** In any additive category, given a morphism  $h$ ,  $\ker(h) = 0$  if and only if  $h$  is monic. Similarly,  $\text{coker}(h) = 0$  if and only if  $h$  is epic.

*Proof.* Left to the reader.  $\nabla$

**2.5. Proposition.**  $\mathbf{U}^b(X, \mathbf{F})$  is an additive category with kernels and cokernels.

*Proof.* Additive properties are inherited from  $\mathbf{F}$ . In particular, the biproduct is given by the filtration-wise operation  $(F \oplus G)(S) = F(S) \oplus G(S)$  in  $\mathbf{F}$ . For any boundedly controlled morphism  $f: F \rightarrow G$ , the kernel of  $f$  in  $\mathbf{F}$  has the

$X$ -filtration  $K$  where  $K(S) = \ker(f) \cap F(S)$  is the image of the pullback

$$\begin{array}{ccc} P_\kappa & \longrightarrow & F(S) \\ \downarrow & & \downarrow \\ \ker(f) & \xrightarrow{\kappa} & F(X) \end{array}$$

which gives the kernel of  $f$  in  $\mathbf{U}^b(X, \mathbf{F})$ . The canonical monic  $\kappa: K \rightarrow F$  has filtration degree 0. It follows from part 3 of Lemma 2.3 that  $K$  has the universal properties of the kernel in  $\mathbf{U}^b(X, \mathbf{F})$ . Similarly, let  $I$  be the  $X$ -filtration of the image of  $f$  in  $\mathbf{F}$  by the images of the pullbacks

$$\begin{array}{ccc} P_l & \longrightarrow & G(S) \\ \downarrow & & \downarrow \\ \text{im}(f) & \xrightarrow{l} & G(X) \end{array}$$

Then there is a bipresheaf  $C$  over  $X$  with  $C(S) = G(S)/I(S)$  for  $S \subset X$ . Of course  $C(X)$  is the cokernel of  $f$  in  $\mathbf{F}$ . Recall that there is an  $X$ -filtered object  $C_X$  associated to  $C$  given by  $C_X(S) = \text{im } C(S, X)$ . The canonical morphism  $\sigma: G(X) \rightarrow C(X)$  gives a filtration 0 morphism  $\sigma: G \rightarrow C_X$  since

$$\text{im}(\sigma G(S, X)) = \text{im } C(S, X) = C_X(S).$$

This in conjunction with part 2 of Lemma 2.3 also verifies the universal properties of  $C_X$  and  $\sigma$  in  $\mathbf{U}^b(X, \mathbf{F})$ .  $\nabla$

**2.6. Remark.** For an explicit description of a boundedly controlled morphism in  $\mathbf{U}(\mathbb{Z}, \mathbf{Mod}(R))$  which is an isomorphism of left  $R$ -modules but whose inverse is not boundedly controlled, we refer to Example 1.5 of [21]. This indicates that in general  $\mathbf{U}^b(X, \mathbf{F})$  is not an abelian category and that under any embedding of such  $\mathbf{U}^b(X, \mathbf{F})$  in an abelian category the kernels and/or cokernels of some morphisms will be different from those in  $\mathbf{U}^b(X, \mathbf{F})$ .

**2.7. Definition.** The *admissible monomorphisms*  $\mathbf{mU}^b(X, \mathbf{F})$  in  $\mathbf{U}^b(X, \mathbf{F})$  consist of boundedly bicontrolled morphisms  $m: F_1 \rightarrow F_2$  such that  $m: F_1(X) \rightarrow F_2(X)$  is a monic in  $\mathbf{F}$ . The *admissible epimorphisms*  $\mathbf{eU}^b(X, \mathbf{F})$  are the boundedly bicontrolled morphisms  $e: F_1 \rightarrow F_2$  such that  $m: F_1(X) \rightarrow F_2(X)$  is an epi in  $\mathbf{F}$ . The class  $\mathcal{E}$  of exact sequences consists of the sequences

$$E^\cdot: \quad E' \xrightarrow{i} E \xrightarrow{j} E''$$

with  $i \in \mathbf{mU}^b(X, \mathbf{F})$  and  $j \in \mathbf{eU}^b(X, \mathbf{F})$  which are exact at  $E$  in the sense that  $\text{im}(i)$  and  $\ker(j)$  represent the same subobject of  $E$ .

**2.8. Theorem.**  $\mathbf{U}^b(X, \mathbf{F})$  with the choices above is an exact category.

*Proof.* (a) It follows from Lemma 2.3 that any short exact sequence  $F^\cdot$  isomorphic to some  $E^\cdot \in \mathcal{E}$  is also in  $\mathcal{E}$ , that

$$F' \xrightarrow{[\text{id}, 0]} F' \oplus F'' \xrightarrow{[0, \text{id}]^T} F''$$

is in  $\mathcal{E}$ , and that  $i = \ker(j)$ ,  $j = \text{coker}(i)$  in any  $E^\cdot \in \mathcal{E}$ .



(b) The collections of morphisms  $\mathbf{mU}^b(X, \mathbf{F})$  and  $\mathbf{eU}^b(X, \mathbf{F})$  are closed under composition by part 1 of Lemma 2.3. Given  $E' \in \mathcal{T}$  and any  $f: A \rightarrow E'' \in \mathbf{U}^b(X, \mathbf{F})$ , there is a base change diagram

$$\begin{array}{ccccc} E' & \longrightarrow & E \times_f A & \xrightarrow{j'} & A \\ = \downarrow & & \downarrow f' & & \downarrow f \\ E' & \longrightarrow & E & \xrightarrow{j} & E'' \end{array}$$

where  $m: E \times_f A \rightarrow E \oplus A$  is the kernel of the epi  $j \operatorname{pr}_1 - f \operatorname{pr}_2: E \oplus A \rightarrow E''$  and  $f' = \operatorname{pr}_1 m$ ,  $j' = \operatorname{pr}_2 m$ . The  $X$ -filtration is given by

$$(E \times_f A)(S) = E \times_f A \cap (E(S) \times A(S))$$

defined as the image of the usual pullback, so that  $j'$  is boundedly controlled and has the same kernel as  $j$ . In fact,

$$\operatorname{im}(j') \cap A(S) \subset (E \times_f A)(S[D(f) + D(j)])$$

since  $fA(S) \subset E''(S[D(f)])$ , so  $j'$  is boundedly bicontrolled of filtration degree  $D(f) + D(j)$ . Now given an admissible subobject  $F \subset E \times A$ , the restriction  $j'|_F$  is the pullback of the admissible epimorphism  $f(F) \rightarrow f j'(F)$ . This shows that the class of admissible epimorphisms is closed under base change by arbitrary morphisms in  $\mathbf{U}^b(X, \mathbf{F})$ . Cobase changes by admissible monomorphisms are similar.

(c) Suppose  $f_1: B \rightarrow A \in \mathbf{U}^b(X, \mathbf{F})$  and  $f_2: A \rightarrow A'' \in \mathbf{U}^b(X, \mathbf{F})$  such that  $\ker(f_2) \in \mathbf{U}^b(X, \mathbf{F})$ . As an addendum to Lemma 2.3, it is easy to see that if the composition  $f_2 f_1$  is a boundedly bicontrolled epi then so is  $f_2$ . So  $f_2 f_1 \in \mathbf{eU}^b(X, \mathbf{F})$  implies  $f_2 \in \mathbf{eU}^b(X, \mathbf{F})$ . The case of admissible monomorphisms is easier.  $\nabla$

Now let  $\mathbf{E}$  be a thick subcategory of  $\mathbf{F}$ .

**2.9. Definition.** The *boundedly controlled* category  $\mathbf{B}(X, \mathbf{E})$  is the full subcategory of  $\mathbf{U}^b(X, \mathbf{F})$  on objects  $F$  such that

- (1) if  $T \in \mathcal{B}(X)$  then  $F(T)$  is an object of  $\mathbf{E}$ , and
- (2) there is a number  $d = d(F) \geq 0$  such that for any subset  $S$  of  $X$ ,

$$F(S) \subset \sum_{x \in S} F(B_d(x)).$$

The terminology adopted here is convenient and should not suggest relations to boundedly controlled spaces and maps introduced earlier by Anderson and Munkholm [1].

**2.10. Remark.** The exact subcategory  $\mathbf{E}$  is not assumed to be cocomplete. In fact, the construction is most interesting when it is not. Notice also that the notation  $\mathbf{B}(X, \mathbf{E})$  does not suggest that the objects  $F$  have the terminal piece  $F(X)$  in  $\mathbf{E}$ , unlike the notation for  $\mathbf{U}^b(X, \mathbf{F})$  where  $F(X)$  are in  $\mathbf{F}$ . The object  $F(X)$  is contained in the cocompletion of  $\mathbf{E}$  in  $\mathbf{F}$ .

**2.11. Theorem.**  $\mathbf{B}(X, \mathbf{E})$  is closed under isomorphisms and exact extensions in  $\mathbf{U}^b(X, \mathbf{F})$ .

*Proof.* Since isomorphisms in  $\mathbf{U}^b(X, R)$  are boundedly bicontrolled, the closure under isomorphisms is clear. Let

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in  $\mathbf{U}^b(X, \mathbf{E})$  and  $D \geq 0$  be a number that serves as one characteristic bound for both  $f$  and  $g$  as boundedly bicontrolled maps, and also for both  $E'$  and  $E''$  as boundedly controlled filtered modules in  $\mathbf{B}(X, \mathbf{E})$ . We are going to check the two properties of boundedly controlled  $X$ -filtered modules for  $E$ .

Here  $B_D(x) = \{x\}[D]$  is the metric ball in  $X$  centered in  $x$  with radius  $D$ . Consider  $E(S)$ , then  $gE(S) \subset E''(S[D])$  and so

$$gE(S) \subset \sum_{x \in S[D]} E''(B_D(x)).$$

For each  $x \in S[D]$ ,

$$E''(B_D(x)) \subset gE(B_{2D}(x)),$$

so

$$E(S) \subset \sum_{x \in S[D]} E(B_{2D}(x)) + \sum_{x \in S[D]} fE'(B_{3D}(x)).$$

Therefore

$$E(S) \subset \sum_{x \in S[D]} E(B_{4D}(x)) \subset \sum_{x \in S} E(B_{5D}(x)).$$

In order to see that each subobject  $E(S)$  for bounded  $S$  is in  $\mathbf{E}$ , we use the fact that  $\mathbf{E}$  is thick in  $\mathbf{F}$ . So  $g|: E(S) \rightarrow gE(S)$  is an admissible epimorphism onto an admissible subobject of  $E''(S[D])$ , which is in  $\mathbf{E}$ . The kernel of  $g|E(S)$  is the admissible subobject  $\ker(g) \cap E(S)$  of  $E(S)$ , which is also in  $\mathbf{E}$ . Now  $E(S)$  is in  $\mathbf{E}$  by closure under exact extensions in  $\mathbf{F}$ .  $\nabla$

**2.12. Corollary.**  $\mathbf{B}(X, \mathbf{E})$  is an exact category in the sense of Quillen. The additive category  $\mathcal{B}(X, \mathbf{E})$  of geometric objects with the induced split exact structure is an exact subcategory of  $\mathbf{B}(X, \mathbf{E})$ .

*Proof.* The  $X$ -filtration of the geometric objects in  $\mathcal{B}(X, \mathbf{E})$  is the obvious one with  $F(S) = \bigoplus_{x \in S} F(x)$ , and the structure maps are the inclusions and projections onto direct summands.  $\nabla$

Recall that a morphism  $e: F \rightarrow F$  is an idempotent if  $e^2 = e$ . Categories in which every idempotent is the projection onto a direct summand of  $F$  are called *idempotent complete*.

**2.13. Corollary.** If the thick subcategory  $\mathbf{E}$  of  $\mathbf{F}$  is idempotent complete then  $\mathbf{B}(X, \mathbf{E})$  is idempotent complete.

*Proof.* Since the restriction of an idempotent  $e$  to the image of  $e$  is the identity, every idempotent is boundedly bicontrolled of filtration 0. It follows easily that the splitting of  $e$  in  $\mathbf{F}$  is in fact a splitting in  $\mathbf{B}(X, \mathbf{E})$ .  $\nabla$

For additional flexibility, one may want to impose weaker requirements on objects in  $\mathbf{U}^b(X, \mathbf{F})$ . Restricting as in Definition 2.9 to objects  $F$  with a locally finite covering  $\mathcal{U}$  by bounded subsets  $U \in \mathcal{B}(X)$  such that  $F(X) = \sum_{U \in \mathcal{U}} F(U)$  gives another exact category. In this case, one may also relax the bounded control conditions on the maps to those of Lipschitz type. Similar modifications

have become useful in recent work of Hambleton–Pedersen [15] and Pedersen–Weibel [23] in controlled  $K$ -theory.

### 3. LOCALIZATION IN CONTROLLED CATEGORIES

In the next two sections we assume that the exact category  $\mathbf{E}$  is idempotent complete.

**3.1. Definition.** A subobject  $E$  of  $F \in \mathbf{U}^b(X, \mathbf{F})$  is *supported on* a subset  $S \subset X$  if  $E$  is an admissible subobject of  $F(S)$ . For a subset  $T \subset X$  and a number  $D > 0$ , let  $F(T, D)$  denote the subobject of  $F$  generated as the sum  $\sum F(S)$  over all subsets  $S \subset X$  with  $\text{diam}(S) \leq D$  and  $S \cap T \neq \emptyset$ . Similarly, if  $Z$  is a metric subspace of  $X$  and  $d > 0$ , there is a subobject  $F(Z, d, D)$  of  $F$  generated as  $\sum F(S)$  over all  $S \in \mathcal{B}_D(X)$  with  $S \cap Z[d] \neq \emptyset$ . In other words,  $F(Z, d, D) = F(Z[d], D)$ .

Let  $F$  be an object of  $\mathbf{B}(X, \mathbf{E})$ . We say  $F$  is *supported near*  $Z$  if there are two numbers  $d \geq 0$  and  $D \geq 0$  so that  $F(X) = F(Z, d, D)$ . Let  $\mathbf{B}(X, \mathbf{E})_{<Z}$  be the full subcategory of  $\mathbf{B}(X, \mathbf{E})$  on objects supported near  $Z$ . Notice that if  $\mathbf{B}_{d,D}(X, \mathbf{E})_{<Z}$  denotes the full subcategory of  $\mathbf{B}(X, \mathbf{E})$  with objects  $F$  as above then

$$\mathbf{B}(X, \mathbf{E})_{<Z} = \text{colim}_{d,D} \mathbf{B}_{d,D}(X, \mathbf{E})_{<Z}.$$

**3.2. Proposition.**  $\mathbf{B}(X, \mathbf{E})_{<Z}$  is a thick subcategory of  $\mathbf{B}(X, \mathbf{E})$ .

*Proof.* Let

$$F' \xrightarrow{\phi_1} F \xrightarrow{\phi_2} F''$$

be an exact sequence. Let  $D_i$  be filtration degrees of  $\phi_i$  and  $F'$  and  $F''$  be supported near  $Z$  so that  $F' = F'(Z, d', D')$  and  $F'' = F''(Z, d'', D'')$ . Since  $F(X) = K_2(X) + M$ , where  $K_2 = \ker(\phi_2)$  and  $M$  is any subobject  $M \subset F(X)$  with  $\phi_2(M) = F''(X)$ , it suffices to show that for some  $d \geq 0$  and  $D \geq 0$

$$K_2(X) = I_1(X) = I_1(Z, d, D) \rightarrow F(Z, d, D)$$

where  $I_1 = \text{im}(\phi_1)$ , and that  $M$  can be chosen to be a subobject of  $F(Z, d, D)$ . Indeed,

$$\begin{aligned} I_1(X) &= \phi_1 F'(X) = \phi_1 F'(Z, d', D') \rightarrow F(Z, d' + D_1, D' + D_1), \\ F''(X) &= \phi_2 F(X) \cap F''(Z, d'', D'') \rightarrow \phi_2 F(Z, d'' + D_2, D'' + D_2). \end{aligned}$$

Let  $M = F(Z, d'' + D_2, D'' + D_2)$ . If  $d$  is chosen as  $\max\{d' + D_1, d'' + D_2\}$  and  $D$  as  $\max\{D' + D_1, D'' + D_2\}$  then  $F$  is in  $\mathbf{B}(X, \mathbf{E})_{<Z}$ .

If  $\phi: F \rightarrow G$  is an admissible subobject, that is a boundedly bicontrolled monic with control bound  $\ell$ , then for  $G(X) = G(Z, d, D)$  one has  $F(X) = F(Z, d, D + \ell)$ . Similarly, if  $\phi$  is an admissible quotient with control bound  $\ell$ , and  $F(X) = F(Z, d, D)$  then  $G(X) = G(Z, d, D + \ell)$ . Since  $\mathbf{B}(X, \mathbf{E})_{<Z}$  is clearly closed under isomorphisms, this proves the assertion.  $\nabla$

**3.3. Definition.** A class of morphisms  $\Sigma$  in an additive category  $\mathbf{C}$  admits a *calculus of left fractions* if

- (1) the identity of each object is in  $\Sigma$ ,
- (2)  $\Sigma$  is closed under composition,

- (3) each diagram  $F' \xleftarrow{s} F \xrightarrow{f} G$  with  $s \in \Sigma$  can be completed to a commutative square

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow s & & \downarrow t \\ F' & \xrightarrow{f'} & G' \end{array}$$

with  $t \in \Sigma$ , and

- (4) if  $f$  is a morphism in  $\mathbf{C}$  and  $s \in \Sigma$  such that  $fs = 0$  then there exists  $t \in \Sigma$  such that  $tf = 0$ .

In this case there is a construction of the *localization*  $\mathbf{C}[\Sigma^{-1}]$  which has the same objects as  $\mathbf{C}$ . The morphism sets  $\text{Hom}(F, G)$  in  $\mathbf{C}[\Sigma^{-1}]$  consist of equivalence classes of diagrams

$$(f, s): F \xrightarrow{f} G' \xleftarrow{s} G$$

with the equivalence relation generated by  $(f_1, s_1) \sim (f_2, s_2)$  if there is a map  $h: G'_1 \rightarrow G'_2$  so that  $hf_1 = f_2$  and  $hs_1 = s_2$ . Let  $(f|s)$  denote the equivalence class of  $(f, s)$ . The composition of morphisms in  $\mathbf{C}[\Sigma^{-1}]$  is defined by

$$(f|s) \circ (g|t) = (g'f|s't)$$

where  $g'$  and  $s'$  fit in the commutative square

$$\begin{array}{ccc} G & \xrightarrow{g} & H' \\ \downarrow s & & \downarrow s' \\ F' & \xrightarrow{g'} & H'' \end{array}$$

from axiom 3.

**3.4. Proposition.** *The localization  $\mathbf{C}[\Sigma^{-1}]$  is a category. The morphisms of the form  $(s|\text{id})$  where  $s \in \Sigma$  are isomorphisms in  $\mathbf{C}[\Sigma^{-1}]$ . The rule  $P_\Sigma(f) = (f|\text{id})$  gives a functor  $P_\Sigma: \mathbf{C} \rightarrow \mathbf{C}[\Sigma^{-1}]$  which is universal among the functors making the morphisms  $\Sigma$  invertible. Also  $P_\Sigma$  preserves finite pushouts.*

*Proof.* The proofs of these facts can be found in Chapter I of [10]. The inverse of  $(s|\text{id})$  is  $(\text{id}|s)$ . The last statement is Proposition I.3.1, loc. cit.  $\nabla$

Suppose  $\mathbf{E}$  is a thick subcategory of an exact pseudoabelian category  $\mathbf{F}$ , and let  $\mathbf{Z}$  be the subcategory  $\mathbf{B}(X, \mathbf{E})_{<Z}$  of  $\mathbf{B} = \mathbf{B}(X, \mathbf{E})$  for a fixed choice of  $Z \subset X$ . Let the class of *weak equivalences*  $\Sigma$  consist of all finite compositions of admissible monomorphisms with cokernels in  $\mathbf{Z}$  and admissible epimorphisms with kernels in  $\mathbf{Z}$ .

**3.5. Lemma.** *If  $F$  is an object of  $\mathbf{Z}$  then every morphism  $f: F \rightarrow G$  in  $\mathbf{B}$  can be factored as a composition of a morphism  $g: F \rightarrow G'$  with  $G'$  in  $\mathbf{Z}$  and an admissible monomorphism  $i: G' \rightarrow G$ .*

*Proof.* Suppose  $F = F(Z, d, D)$ , in the sense that  $F$  is generated by all  $F(S)$  with  $\text{diam}(S) \leq D$  and  $S \cap Z[d] \neq \emptyset$ . If  $f$  is bounded by  $l$  then choose  $G' = G(Z, d, D + l)$ .  $\nabla$

**3.6. Proposition.** *The class  $\Sigma$  admits a calculus of left fractions.*

*Proof.* The properties 1 and 2 are clear. To see property 3 use induction on the length of  $s$  to reduce to two cases. When the given weak equivalence  $s$  is an admissible monomorphism, the pushout is an object of  $\mathbf{B}$  since  $\mathbf{B}$  is closed under cobase-changes. Given a diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow s & & \\ F' & & \end{array}$$

where  $s$  is an admissible epimorphism with kernel  $i: Z \rightarrow F$  in  $\mathbf{Z}$ , use Lemma 3.5 to factor the composition  $fi$  as  $Z \rightarrow Z' \rightarrow G$ , where  $Z'$  is in  $\mathbf{Z}$  and the map  $j: Z' \rightarrow G$  is an admissible monomorphism. Take  $G'$  to be the cokernel of  $j$ , then the corresponding admissible epimorphism  $t: G \rightarrow G'$  and the induced map of quotients  $f': F' \rightarrow G'$  form the required commutative square

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow s & & \downarrow t \\ F' & \xrightarrow{f'} & G' \end{array}$$

Property 4 is proved by similar reduction. If  $s$  is an admissible epimorphism then  $f$  is the zero map. If  $s: E \rightarrow F$  is an admissible monomorphism with cokernel in  $\mathbf{Z}$  and  $f: F \rightarrow G$  is a morphism with  $fs = 0$ , let  $g: Z \rightarrow G$  be the induced universal map. By Lemma 3.5,  $g$  factors as a composition of  $g': Z \rightarrow G'$  with  $G'$  in  $\mathbf{Z}$  and an admissible monomorphism  $i: G' \rightarrow G$ . Taking  $t$  to be the cokernel of  $i$  gives  $tf = 0$ .  $\nabla$

**3.7. Definition.** The *quotient category*  $\mathbf{B}/\mathbf{Z}$  is the localization  $\mathbf{B}[\Sigma^{-1}]$ .

It is clear that the quotient  $\mathbf{B}/\mathbf{Z}$  is an additive category, and  $P_{\Sigma}$  is an additive functor. In fact, we have the following.

**3.8. Theorem.** *The short sequences in  $\mathbf{B}/\mathbf{Z}$  which are isomorphic to images of exact sequences from  $\mathbf{B}$  form a Quillen exact structure.*

*Proof.* This follows from Proposition 1.16 of Schlichting [26] as soon as we check that the subcategory  $\mathbf{Z}$  is left  $s$ -filtering in  $\mathbf{B}$ . We need to see that if  $f: F \rightarrow G$  is an admissible epimorphism with  $G$  in  $\mathbf{Z}$  then there exist  $E$  in  $\mathbf{Z}$  and an admissible monomorphism  $i: E \rightarrow F$  such that  $fi$  is an admissible epimorphism. Let  $G = G(Z, d, D)$  in terms of Definition 3.1. If  $\text{fil}(f) \leq l$  then the subobject  $F(Z, d, D+l)$  is in  $\mathbf{Z}$  and the inclusion  $i: F(Z, d, D+l) \rightarrow F$  is the desired admissible monomorphism.  $\nabla$

The main tool in proving controlled excision in the boundedly controlled  $K$ -theory will be the following localization theorem.

**3.9. Theorem** (Theorem 2.1 of Schlichting [26]). *Let  $\mathbf{Z}$  be an idempotent complete left  $s$ -filtering subcategory of an exact category  $\mathbf{B}$ . Then the sequence of exact categories  $\mathbf{Z} \rightarrow \mathbf{B} \rightarrow \mathbf{B}/\mathbf{Z}$  induces a homotopy fibration of  $K$ -theory spectra*

$$K(\mathbf{Z}) \rightarrow K(\mathbf{B}) \rightarrow K(\mathbf{B}/\mathbf{Z}).$$

## 4. NONCONNECTIVE BOUNDED EXCISION

The proof of controlled excision in the boundedly controlled  $K$ -theory requires the context of Waldhausen  $K$ -theory of derived categories.

**4.1. Definition** (Waldhausen categories). A *Waldhausen category* is a category  $\mathbf{D}$  with a zero object  $0$  together with two chosen subcategories of *cofibrations*  $\text{co}(\mathbf{D})$  and *weak equivalences*  $\mathbf{w}(\mathbf{D})$  satisfying the four axioms:

- (1) every isomorphism in  $\mathbf{D}$  is in both  $\text{co}(\mathbf{D})$  and  $\mathbf{w}(\mathbf{D})$ ,
- (2) every map  $0 \rightarrow D$  in  $\mathbf{D}$  is in  $\text{co}(\mathbf{D})$ ,
- (3) if  $A \rightarrow B \in \text{co}(\mathbf{D})$  and  $A \rightarrow C \in \mathbf{D}$  then the pushout  $B \cup_A C$  exists in  $\mathbf{D}$ , and the canonical map  $C \rightarrow B \cup_A C$  is in  $\text{co}(\mathbf{D})$ ,
- (4) (“gluing lemma”) given a commutative diagram

$$\begin{array}{ccccc} B & \xleftarrow{a} & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \xleftarrow{a'} & A' & \longrightarrow & C' \end{array}$$

in  $\mathbf{D}$ , where the morphisms  $a$  and  $a'$  are in  $\text{co}(\mathbf{D})$  and the vertical maps are in  $\mathbf{w}(\mathbf{D})$ , the induced map  $B \cup_A C \rightarrow B' \cup_{A'} C'$  is also in  $\mathbf{w}(\mathbf{D})$ .

A Waldhausen category  $\mathbf{D}$  with weak equivalences  $\mathbf{w}(\mathbf{D})$  is often denoted by  $\mathbf{wD}$  as a reminder of the choice. A functor between Waldhausen categories is exact if it preserves cofibrations and weak equivalences.

A Waldhausen category may or may not satisfy the following additional axioms.

**4.2. Saturation axiom.** Given two morphisms  $\phi: F \rightarrow G$  and  $\psi: G \rightarrow H$  in  $\mathbf{D}$ , if any two of  $\phi$ ,  $\psi$ , or  $\psi\phi$ , are in  $\mathbf{w}(\mathbf{D})$  then so is the third.

**4.3. Extension axiom.** Given a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & G & \longrightarrow & H \\ \downarrow \phi & & \downarrow \psi & & \downarrow \mu \\ F' & \longrightarrow & G' & \longrightarrow & H' \end{array}$$

with exact rows, if both  $\phi$  and  $\mu$  are in  $\mathbf{w}(\mathbf{D})$  then so is  $\psi$ .

A *cylinder functor* on  $\mathbf{D}$  is a functor  $C$  from the category of morphisms  $f: F \rightarrow G$  in  $\mathbf{D}$  to  $\mathbf{D}$  together with three natural transformations  $j_1: F \rightarrow C(f)$ ,  $j_2: G \rightarrow C(f)$ , and  $p: C(f) \rightarrow G$  such that  $pj_2 = \text{id}_G$  and  $pj_1 = f$  for all  $f$ , and which has a number of properties listed in point 1.3.1 of [29] which will be rather automatic for the functors we construct later.

**4.4. Cylinder axiom.** A cylinder functor  $C$  satisfies this axiom if for all morphisms  $f: F \rightarrow G$  the required map  $p$  is in  $\mathbf{w}(\mathbf{D})$ .

**4.5. Definition.** In any additive category, a sequence of morphisms

$$E^\cdot: 0 \longrightarrow E^1 \xrightarrow{d_1} E^2 \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} E^n \longrightarrow 0$$

is called a (*bounded*) *chain complex* if the compositions  $d_{i+1}d_i$  are the zero maps for all  $i = 1, \dots, n-1$ . A *chain map*  $f: F^\cdot \rightarrow E^\cdot$  is a collection of morphisms  $f^i: F^i \rightarrow E^i$  such that  $f^i d_i = d_i f^i$ . A chain map  $f$  is *null-homotopic* if there are

morphisms  $s_n: F^{i+1} \rightarrow E^i$  such that  $f = ds + sd$ . Two chain maps  $f, g: F \rightarrow E$  are *chain homotopic* if  $f - g$  is null-homotopic. Now  $f$  is a *chain homotopy equivalence* if there is a chain map  $h: E \rightarrow F$  such that the compositions  $fh$  and  $hf$  are chain homotopic to the respective identity maps.

The Waldhausen structures on categories of bounded chain complexes are based on homotopy equivalence as a weakening of the notion of isomorphism of chain complexes.

**4.6. Definition.** A sequence of maps in an exact category is called *acyclic* if it is assembled out of short exact sequences in the sense that each map factors as the composition of the cokernel of the preceding map and the kernel of the succeeding map.

It is known that the class of acyclic complexes in an exact category is closed under isomorphisms in the homotopy category if and only if the category is idempotent complete, which is also equivalent to the property that each contractible chain complex is acyclic, cf. [18, sec. 11].

**4.7. Definition.** Given an exact category  $\mathbf{E}$ , there is a standard choice for the Waldhausen structure on the derived category  $\mathbf{E}'$  of bounded chain complexes in  $\mathbf{E}$  where the degree-wise admissible monomorphisms are the cofibrations and the chain maps whose mapping cones are homotopy equivalent to acyclic complexes are the weak equivalences  $\mathbf{v}(\mathbf{E}')$ .

**4.8. Proposition.** *The category  $\mathbf{vE}'$  is a Waldhausen category satisfying the extension and saturation axioms and has cylinder functors satisfying the cylinder axiom.*

*Proof.* The pushouts along cofibrations in  $\mathbf{E}'$  are the complexes of pushouts in each degree. All standard Waldhausen axioms including the gluing lemma are clearly satisfied. The saturation and the extension axioms are also clear. The cylinder functor  $C$  for  $\mathbf{vE}'$  is defined using the canonical homotopy pushout as in point 1.1.2 in Thomason–Trobeaugh [29]. Given a chain map  $f: F \rightarrow G$ ,  $C(f)$  is the canonical homotopy pushout of  $f$  and the identity  $\text{id}: F \rightarrow F$ . With this construction, the map  $p: C(f) \rightarrow G$  is a chain homotopy equivalence, so the cylinder axiom is also satisfied.  $\nabla$

**4.9. Definition.** There are two choices for the Waldhausen structure on the bounded derived category  $\mathbf{B}' = \mathbf{B}'(X, \mathbf{E})$ . One is  $\mathbf{vB}'$  as in Definition 4.7. Given a metric subspace  $Z$  in  $X$ , the other choice for the weak equivalences  $\mathbf{w}(\mathbf{B}')$  is the chain maps whose mapping cones are homotopy equivalent to acyclic complexes in the quotient  $\mathbf{B}/Z$ .

**4.10. Corollary.** *The categories  $\mathbf{vB}'$  and  $\mathbf{wB}'$  are Waldhausen categories satisfying the extension and saturation axioms and have cylinder functors satisfying the cylinder axiom.*

*Proof.* All axioms and constructions, including the cylinder functor, for  $\mathbf{wB}'$  are inherited from  $\mathbf{vB}'$ .  $\nabla$

The  $K$ -theory functor from the category of small Waldhausen categories  $\mathbf{D}$  and exact functors to connective spectra is defined in terms of  $S$ -construction as in Waldhausen [31]. It extends to simplicial categories  $\mathbf{D}$  with cofibrations

and weak equivalences and inductively delivers the connective spectrum  $n \mapsto |\mathbf{w}S^{(n)} \mathbf{D}|$ . We obtain the functor assigning to  $\mathbf{D}$  the connective  $\Omega$ -spectrum

$$K(\mathbf{D}) = \Omega^\infty |\mathbf{w}S^{(\infty)} \mathbf{D}| = \varinjlim_{n \geq 1} \Omega^n |\mathbf{w}S^{(n)} \mathbf{D}|$$

representing the Waldhausen algebraic  $K$ -theory of  $\mathbf{D}$ . For example, if  $\mathbf{D}$  is the additive category of free finitely generated  $R$ -modules with the canonical Waldhausen structure, then the stable homotopy groups of  $K(\mathbf{D})$  are the usual  $K$ -groups of the ring  $R$ . In fact, there is a general identification of the two theories. Recall that for any exact category  $\mathbf{E}$ , the derived category  $\mathbf{E}'$  has the Waldhausen structure  $\mathbf{v}\mathbf{E}'$  as in Definition 4.7.

**4.11. Theorem.** *The Quillen  $K$ -theory of an exact category  $\mathbf{E}$  is equivalent to the Waldhausen  $K$ -theory of  $\mathbf{v}\mathbf{E}'$ .*

*Proof.* The proof is based on repeated applications of the additivity theorem, cf. Thomason's Theorem 1.11.7 [29]. Thomason's proof of his Theorem 1.11.7 can be repeated verbatim here. It is in fact simpler in this case since condition 1.11.3.1 is not required.  $\nabla$

Let  $\mathbf{E}$  be a thick subcategory of a cocomplete abelian category  $\mathbf{F}$  and let  $X$  be a proper metric space with subspace  $Z$ . We will use the notation  $\mathbf{B} = \mathbf{B}(X, \mathbf{E})$  and  $\mathbf{Z} = \mathbf{B}(X, \mathbf{E})_{<Z}$ .

**4.12. Theorem (Localization).** *If  $\mathbf{E}$  is idempotent complete, there is a homotopy fibration*

$$K(Z, \mathbf{E}) \longrightarrow K(X, \mathbf{E}) \longrightarrow K(\mathbf{B}/\mathbf{Z}).$$

This should be a direct application of Theorem 3.9 as soon as we identify  $K(Z, \mathbf{E})$  with  $\mathbf{Z}$ . The proof requires the use of an approximation theorem due to Waldhausen [31], sec. 1.6; the necessary improvements are due to Thomason [29].

Let  $\mathbf{D}$  be a small Waldhausen category with respect to two categories of weak equivalences  $\mathbf{v}(\mathbf{D}) \subset \mathbf{w}(\mathbf{D})$  with a cylinder functor  $T$  both for  $\mathbf{v}\mathbf{D}$  and for  $\mathbf{w}\mathbf{D}$  satisfying the cylinder axiom for  $\mathbf{w}\mathbf{D}$ . Suppose also that  $\mathbf{w}(\mathbf{D})$  satisfies the extension and saturation axioms. Define  $\mathbf{v}\mathbf{D}^{\mathbf{w}}$  to be the full subcategory of  $\mathbf{v}\mathbf{D}$  whose objects are  $F$  such that  $0 \rightarrow F \in \mathbf{w}(\mathbf{D})$ . Then  $\mathbf{v}\mathbf{D}^{\mathbf{w}}$  is an additive Waldhausen category with cofibrations  $\text{co}(\mathbf{D}^{\mathbf{w}}) = \text{co}(\mathbf{D}) \cap \mathbf{D}^{\mathbf{w}}$  and weak equivalences  $\mathbf{v}(\mathbf{D}^{\mathbf{w}}) = \mathbf{v}(\mathbf{D}) \cap \mathbf{D}^{\mathbf{w}}$ . The cylinder functor  $T$  for  $\mathbf{v}\mathbf{D}$  induces a cylinder functor for  $\mathbf{v}\mathbf{D}^{\mathbf{w}}$ . If  $T$  satisfies the cylinder axiom then the induced functor does so too.

**4.13. Theorem (Approximation theorem).** *Let  $E: \mathbf{D}_1 \rightarrow \mathbf{D}_2$  be an exact functor between two small saturated Waldhausen categories. It induces a map of  $K$ -theory spectra*

$$K(E): K(\mathbf{D}_1) \longrightarrow K(\mathbf{D}_2).$$

*Assume that  $\mathbf{D}_1$  has a cylinder functor satisfying the cylinder axiom. If  $E$  satisfies two conditions:*

- (1) *a morphism  $f \in \mathbf{D}_1$  is in  $\mathbf{w}(\mathbf{D}_1)$  if and only if  $E(f) \in \mathbf{D}_2$  is in  $\mathbf{w}(\mathbf{D}_2)$ ,*
- (2) *for any object  $D_1 \in \mathbf{D}_1$  and any morphism  $g: E(D_1) \rightarrow D_2$  in  $\mathbf{D}_2$ , there is an object  $D'_1 \in \mathbf{D}_1$ , a morphism  $f: D_1 \rightarrow D'_1$  in  $\mathbf{D}_1$ , and a weak equivalence  $g': E(D'_1) \rightarrow D_2 \in \mathbf{w}(\mathbf{D}_2)$  such that  $g = g'E(f)$ ,*

*then  $K(E)$  is a homotopy equivalence.*



*Proof.* This is Theorem 1.6.7 of [31]. The presence of the cylinder functor with the cylinder axiom allows to make condition 2 weaker than that of Waldhausen, see point 1.9.1 in [29].  $\nabla$

4.14. *Notation.*  $K'(X, \mathbf{E}) \stackrel{\text{def}}{=} K(\mathbf{vB}(X, \mathbf{E})')$ .

4.15. **Lemma.** *For any pair of proper metric spaces  $Z \subset X$ , there is a weak equivalence  $K'(Z, \mathbf{E}) \simeq K'(X, \mathbf{E})_{<Z}$ .*

*Proof.* There is a fully faithful embedding  $E: \mathbf{B}(Z, \mathbf{E}) \rightarrow \mathbf{B}(X, \mathbf{E})_{<Z}$  defined by associating to each filtered object  $F \in \mathbf{B}(Z, \mathbf{E})$  the extension  $E(F) \in \mathbf{B}(X, \mathbf{E})_{<Z}$  given by  $E(F)(S) = F(S \cap Z)$ . We will apply the approximation theorem to the induced embedding of derived categories. First, observe that for any object  $F$  in  $\mathbf{B}(X, \mathbf{E})_{<Z}$  such that  $F(Z, d, D) = F(X)$ ,  $F$  is supported on  $Z[D + d]$ , so there is a bounded function  $r: Z[D + d] \rightarrow Z$  which generates an object  $R(F)$  of  $\mathbf{B}(Z, \mathbf{E})$  by the rule  $R(F)(S) = F(r^{-1}(S))$ . If  $l$  is a bound for  $r$ ,  $R(F)$  is generated by all subobjects  $R(F)(T)$ ,  $T \subset Z$ , with  $\text{diam}(T) \leq D + l$ . The identity map  $R(F) \rightarrow F$  is boundedly controlled. Now to check condition 2 in the approximation theorem, take a chain complex

$$E': \quad 0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots \rightarrow E^n \rightarrow 0$$

in  $\mathbf{B}(Z, \mathbf{E})$ , a chain complex

$$F': \quad 0 \rightarrow F^1 \xrightarrow{\phi_1} F^2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} F^n \rightarrow 0$$

and a morphism  $g: E' \rightarrow F'$  in  $\mathbf{B}(X, \mathbf{E})'_{<Z}$ . Using a sufficiently large constant  $D$  and the corresponding choice of the function  $r: Z[D + d] \rightarrow Z$ , perform the construction above in all degrees to obtain  $\hat{F}' = R(F')$  with the chain map  $f: E' \rightarrow \hat{F}'$  induced from  $g$ . The identity  $R(F') = F'$  induces a weak equivalence, as required.  $\nabla$

Now Theorem 4.12 follows from the localization fibration.

The best computational tools in bounded  $K$ -theory, the controlled excision theorems [3, 21, 22], can now be adapted to  $\mathbf{B}(X, \mathbf{E})$ . We will obtain a direct analogue, which is one of the main results of this paper.

Let  $\mathbf{E}$  be a thick subcategory of a cocomplete pseudoabelian category  $\mathbf{F}$  and let  $X$  be a proper metric space. Suppose  $X_1$  and  $X_2$  are subspaces in a proper metric space  $X$ , and  $X = X_1 \cup X_2$ . We use the notation  $\mathbf{B} = \mathbf{B}(X, \mathbf{E})$ ,  $\mathbf{B}_i = \mathbf{B}(X, \mathbf{E})_{<X_i}$  for  $i = 1$  or  $2$ , and  $\mathbf{B}_{12}$  for the intersection  $\mathbf{B}_1 \cap \mathbf{B}_2$ .

Now there is a commutative diagram

$$(+) \quad \begin{array}{ccccc} K(\mathbf{B}_{12}) & \longrightarrow & K(\mathbf{B}_1) & \longrightarrow & K(\mathbf{B}_1/\mathbf{B}_{12}) \\ & & \downarrow & & \downarrow_{K(I)} \\ K(\mathbf{B}_2) & \longrightarrow & K(\mathbf{B}) & \longrightarrow & K(\mathbf{B}/\mathbf{B}_2) \end{array}$$

where the rows are homotopy fibrations. We should not expect the map induced by the rightmost exact inclusion  $I: \mathbf{wB}'_1 \rightarrow \mathbf{wB}'$  to be an equivalence of categories as in similar applications in [3] and [27], but we claim that  $K(I)$  is almost a weak equivalence.

Let  $Z$  be a subset of  $X$ , so  $\mathbf{Z} = \mathbf{B}(X, \mathbf{E})_{<Z}$  is a thick subcategory of  $\mathbf{B}$ , and recall that  $\hat{\mathbf{C}}$  is the idempotent completion of an exact category  $\mathbf{C}$ .

**4.16. Lemma.** *If  $f^\cdot : F^\cdot \rightarrow G^\cdot$  is either a degreewise admissible monomorphism with cokernels in  $\mathbf{Z}$  or a degreewise admissible epimorphism with kernels in  $\mathbf{Z}$  then  $f^\cdot$  is a weak equivalence in  $\mathbf{v}((\mathbf{B}/\mathbf{Z})^\wedge)$ .*

*Proof.* We need to see that the mapping cone  $Cf^\cdot$  is the zero complex in the bounded homotopy category of  $\mathbf{B}/\mathbf{Z}$ . In the first case,  $Cf^\cdot$  is weakly equivalent to the cokernel of  $f^\cdot$ , by Lemma 11.6 of [18], which is zero in  $\mathbf{B}/\mathbf{Z}$ . In the second case,  $F^\cdot$  is weakly equivalent to the mapping cone of  $\ker(f^\cdot)$  in  $\mathbf{B}/\mathbf{Z}$ , which is again weakly equivalent to  $G^\cdot$ .  $\nabla$

An exact subcategory  $\mathbf{C}$  of an exact category  $\mathbf{E}$  is *cofinal* if it is closed under extensions and for every  $E$  in  $\mathbf{E}$  there is  $E'$  so that  $E \oplus E'$  is isomorphic to an object from the subcategory  $\mathbf{C}$ .

**4.17. Theorem** (Cofinality theorem, Staffeldt [27]). *If  $\mathbf{C}$  is cofinal in  $\mathbf{E}$  then the Waldhausen  $K$ -theory sequence  $K(\mathbf{v}\mathbf{C}') \rightarrow K(\mathbf{v}\mathbf{E}') \rightarrow BG$ , where  $G = K_0(\mathbf{E})/K_0(\mathbf{C})$ , is a fibration.*

**4.18. Lemma.**  *$K(I): K(\mathbf{w}\mathbf{B}'_1) \rightarrow K(\mathbf{w}\mathbf{B}')$  is a weak equivalence of spectra in positive dimensions.*

*Proof.* Applying the Cofinality theorem to the inclusion  $I: \mathbf{B}_1/\mathbf{B}_{12} \rightarrow (\mathbf{B}/\mathbf{B}_2)^\wedge$ , for any  $E$  in  $(\mathbf{B}/\mathbf{B}_2)^\wedge$  choose  $E'$  so that  $E \oplus E'$  is isomorphic to an object

$$F^\cdot: 0 \rightarrow F^1 \xrightarrow{\phi_1} F^2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} F^n \rightarrow 0$$

in  $\mathbf{B}/\mathbf{B}_2$ . Recall the notation  $F(T, d, D)$  for the subobject of  $F$  generated by all subobjects  $F(S)$  such that  $\text{diam}(S) \leq D$  and  $S \cap T[d] \neq \emptyset$ . Fixing nonnegative numbers  $D, d$ , and some  $l$  that serves as a control bound for all  $\phi_i$ , we define  $E'^i = F^i(X_1, d, D + il)$  and  $\phi'_i$  to be the restrictions of  $\phi_i$  to  $E'^i$ . This gives a chain subcomplex  $E'^\cdot$  of  $F^\cdot$  in  $\mathbf{B}'_1$  so that the mapping cone of the inclusion is homotopy equivalent to the cokernel and so is contractible. Therefore, it is acyclic in the idempotent complete category  $(\mathbf{B}/\mathbf{B}_2)^\wedge$ .  $\nabla$

Let  $\mathbb{Z}, \mathbb{Z}^{\geq 0}$ , and  $\mathbb{Z}^{\leq 0}$  denote the metric spaces of integers, nonnegative integers, and nonpositive integers with the restriction of the usual metric on the real line  $\mathbb{R}$ . Let  $\mathbf{E}$  be an idempotent complete thick category of a pseudoabelian category  $\mathbf{F}$ . Then for any proper metric space  $X$ , we have the following instance of commutative diagram ( $\dagger$ )

$$\begin{array}{ccccc} K(X, \mathbf{E}) & \longrightarrow & K(X \times \mathbb{Z}^{\geq 0}, \mathbf{E}) & \longrightarrow & K(\mathbf{B}_1/\mathbf{B}_{12}) \\ \downarrow & & \downarrow & & \downarrow K(I) \\ K(X \times \mathbb{Z}^{\leq 0}, \mathbf{E}) & \longrightarrow & K(X \times \mathbb{Z}, \mathbf{E}) & \longrightarrow & K(\mathbf{B}/\mathbf{B}_2) \end{array}$$

**4.19. Lemma.** *The spectra  $K(\mathbb{Z}^{\geq 0} \times X, \mathbf{E})$  and  $K(\mathbb{Z}^{\leq 0} \times X, \mathbf{E})$  are contractible.*

*Proof.* This follows from the fact that these controlled categories are flasque, that is, the usual shift functor  $T$  in the positive (respectively negative) direction along  $\mathbb{Z}^{\geq 0}$  (respectively  $\mathbb{Z}^{\leq 0}$ ) interpreted in the obvious way is an exact endofunctor, and there is a natural equivalence  $1 \oplus \pm T \cong \pm T$ . Contractibility follows from the additivity theorem, cf. Pedersen–Weibel [21].  $\nabla$

In view of Lemma 4.18, we obtain a map  $K(X, \mathbf{E}) \rightarrow \Omega K(X \times \mathbb{Z}, \mathbf{E})$  which induces isomorphisms of  $K$ -groups in positive dimensions. Iterations of this construction give weak equivalences

$$\Omega^k K(X \times \mathbb{Z}^k, \mathbf{E}) \rightarrow \Omega^{k+1} K(X \times \mathbb{Z}^{k+1}, \mathbf{E})$$

for  $k \geq 2$ .

Given a small exact category  $\mathbf{E}$ , let  $\widehat{\mathbf{E}}$  denote its idempotent completion. If  $\mathbf{E}$  is thick in a pseudoabelian category  $\mathbf{F}$  then  $\widehat{\mathbf{E}}$  can also be identified with a thick subcategory of  $\mathbf{F}$ .

**4.20. Definition.** The *nonconnective controlled  $K$ -theory* of  $\mathbf{E}$ , relative to the embedding  $\epsilon: \mathbf{E} \rightarrow \mathbf{F}$ , over a proper metric space  $X$  is the spectrum

$$K_\epsilon^{-\infty}(X, \mathbf{E}) \stackrel{\text{def}}{=} \varinjlim_k \Omega^k K(X \times \mathbb{Z}^k, \widehat{\mathbf{E}}).$$

Since  $\mathbf{B}(X, \mathbf{E})$  can be identified with  $\mathbf{E}$  for a bounded metric space  $X$ , this definition gives the *nonconnective  $K$ -theory* of  $\mathbf{E}$

$$K_\epsilon^{-\infty}(\mathbf{E}) \stackrel{\text{def}}{=} \varinjlim_{k>0} \Omega^k K(\mathbb{Z}^k, \widehat{\mathbf{E}}).$$

As  $K_\epsilon^{-\infty}(\mathbf{E})$  is an  $\Omega$ -spectrum in positive dimensions, the positive homotopy groups of  $K_\epsilon^{-\infty}(\mathbf{E})$  coincide with those of  $K(\mathbf{E})$  as desired. The class group  $K_{\epsilon,0}(\mathbf{E})$  is the class group of the idempotent completion  $K_0(\widehat{\mathbf{E}})$ .

**4.21. Example.** If  $\mathbf{E}$  is an arbitrary small exact category, there is the full Gabriel-Quillen embedding of  $\mathbf{E}$  in the cocomplete abelian category  $\mathbf{F}$  of left exact functors  $\mathbf{E}^{\text{op}} \rightarrow \mathbf{Mod}(\mathbb{Z})$  with the standard exact structure. The embedding is always closed under extensions in  $\mathbf{F}$ . It is usually not thick, for example when  $\mathbf{E}$  is not balanced. But if  $\mathbf{E}$  is abelian, this gives a canonical delooping of  $K(\mathbf{E})$ .

On the other hand, it is often convenient to use noncanonical embeddings in pseudoabelian categories that closely reflect algebraic and geometric properties of  $\mathbf{E}$  as illustrated in the following examples.

**4.22. Example.** One may start with the cocomplete abelian category  $\mathbf{Mod}(R)$  of modules over a ring  $R$  with the standard abelian exact structure where the admissible monomorphisms and epimorphisms are respectively all monics and epis. If  $R$  is a noetherian ring, the subcategory  $\mathbf{E}$  may be taken to be the non-cocomplete abelian category of finitely generated  $R$ -modules  $\mathbf{Modf}(R)$ . Now  $K_\epsilon^{-\infty}(\mathbf{E})$  gives the algebraic  $G$ -theory of  $R$ .

**4.23. Example.** The negative  $K$ -theory of a regular ring  $R$  is trivial in the sense that  $K_i(\mathbf{Modf}(R)) = 0$  for all  $i < 0$ . This is well-known in Bass' theory [2]. A proof that the negative  $K$ -theory is trivial for general abelian categories can be given using the same strategy as in chapter 9 of [25].

Of course, when the exact category  $\mathbf{E}$  is itself cocomplete, its  $K$ -theory is contractible because of the Eilenberg swindle type argument as in the proof of Lemma 4.19.

**4.24. Example.** Let  $R$  be a noetherian ring and let  $\Gamma$  be a finitely generated group which is *weakly coherent* in the sense of [6]. Fixing a finite generating set makes  $\Gamma$  into a proper metric space with the associated word metric. Let  $\mathbf{L}(\Gamma, R)$  be

the full subcategory of *lean*  $\Gamma$ -filtered modules as defined in loc. cit. It can be shown that  $\mathbf{L}(\Gamma, R)$  is closed under extensions in  $\mathbf{U}^b(\Gamma, R)$  and so is an exact subcategory. It can be further shown that  $\mathbf{L}(\Gamma, R)$  is closed under cokernels.

Consider the left translation action by  $\Gamma$  on itself. This makes  $\mathbf{L}(\Gamma, R)$  a  $\Gamma$ -equivariant category. The embedding of fixed objects  $\mathbf{L}(\Gamma, R)^\Gamma$  in the exact pseudoabelian category  $\mathbf{U}^b(\Gamma, R)^\Gamma$  inherits the properties above. In addition, the subcategory  $\mathbf{L}(\Gamma, R)^\Gamma$  is closed under admissible subobjects as a consequence of the weak coherence property of  $\Gamma$ , so the embedding  $\epsilon: \mathbf{L}(\Gamma, R)^\Gamma \rightarrow \mathbf{U}^b(\Gamma, R)^\Gamma$  can be used to form the delooping  $K_\epsilon^{-\infty}(\Gamma, R)^\Gamma$ . Notice also that if  $Y$  is another proper metric space with a left  $\Gamma$ -action, one obtains inductively a plethora of useful exact categories such as  $\mathbf{B}(Y, \mathbf{L}(\Gamma, R)^\Gamma)$ ,  $\mathbf{L}(Y, \mathbf{L}(\Gamma, R)^\Gamma)$ , etc.

**4.25. Definition.** Let  $R$  be a noetherian ring and make the choice of  $\mathbf{E}$  and  $\mathbf{F}$  as in Example 4.22. We use notation  $\mathbf{B}(X, R)$  for  $\mathbf{B}(X, \mathbf{E})$ . Now  $\mathbf{B}(X, R)$  is an exact subcategory of the cocomplete pseudoabelian category  $\mathbf{U}^b(X, R)$  closed under extensions, so we have the *nonconnective controlled  $K$ -theory* of  $\mathbf{E}$ , relative to the embedding  $\epsilon: \mathbf{E} \rightarrow \mathbf{F}$ , over a proper metric space  $X$

$$K_\epsilon^{-\infty}(X, \mathbf{E}) \stackrel{\text{def}}{=} \operatorname{hocolim}_k \Omega^k K(X \times \mathbb{Z}^k, \mathbf{E}^\wedge).$$

**4.26. Theorem** (Nonconnective excision). *There is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc} K^{-\infty}(\mathbf{B}_{12}) & \longrightarrow & K^{-\infty}(\mathbf{B}_1) \\ \downarrow & & \downarrow \\ K^{-\infty}(\mathbf{B}_2) & \longrightarrow & K^{-\infty}(\mathbf{B}) \end{array}$$

where the maps of spectra are induced from the exact inclusions.

*Proof.* Let us write  $S^k \mathbf{B}$  for  $\mathbf{B}(X \times \mathbb{Z}^k, \mathbf{E})$  whenever  $\mathbf{B}$  is the boundedly controlled category for a general metric space  $X$ . If  $Z$  is a subset of  $X$ , consider the fibration

$$K(Z, \mathbf{E}) \rightarrow K(X, \mathbf{E}) \rightarrow K(\mathbf{B}/Z)$$

from Theorem 4.12. Notice that there is a map  $K(\mathbf{B}/Z) \rightarrow \Omega K(S\mathbf{B}/S Z)$  which is a weak equivalence in positive dimensions by the Five Lemma. If one defines

$$K^{-\infty}(\mathbf{B}/Z) = \operatorname{hocolim}_k \Omega^k K(S^k \mathbf{B}/S^k Z),$$

there is an induced fibration

$$K^{-\infty}(Z, \mathbf{E}) \rightarrow K^{-\infty}(X, \mathbf{E}) \rightarrow K^{-\infty}(\mathbf{B}/Z)$$

The theorem follows from the commutative diagram

$$\begin{array}{ccccc} K^{-\infty}(\mathbf{B}_{12}) & \longrightarrow & K^{-\infty}(\mathbf{B}_1) & \longrightarrow & K^{-\infty}(\mathbf{B}_1/\mathbf{B}_{12}) \\ \downarrow & & \downarrow & & \downarrow^{K^{-\infty}(I)} \\ K^{-\infty}(\mathbf{B}_2) & \longrightarrow & K^{-\infty}(\mathbf{B}) & \longrightarrow & K^{-\infty}(\mathbf{B}/\mathbf{B}_2) \end{array}$$

and the fact that now  $K^{-\infty}(I): K^{-\infty}(\mathbf{B}_1/\mathbf{B}_{12}) \rightarrow K^{-\infty}(\mathbf{B}/\mathbf{B}_2)$  is a weak equivalence.  $\nabla$

**4.27. Remark.** As in all other versions of controlled  $K$ -theory, there is no excision theorem similar to Theorem 4.26 which employs the connective  $K$ -theory. This time the reason is that the map  $K(I)$  is not necessarily a weak equivalence. The difference is detected at the level of  $K_0$  which makes the use of cofinality theorem essential to the proof. To give an idea why condition 2 in the approximation theorem fails when applied to the inclusion of Waldhausen categories  $I: \mathbf{wB}'_1 \rightarrow \mathbf{wB}'$ , suppose  $E^\cdot$  is a chain complex in  $\mathbf{B}_1$ ,  $F^\cdot$  is a chain complex in  $\mathbf{B}$ , and  $g: E^\cdot \rightarrow F^\cdot$ . One needs to construct a subcomplex  $E'^\cdot$  of  $F^\cdot$  such that the inclusion  $h$  is a weak equivalence in  $\mathbf{wB}'$  and  $hf = g$  for some  $f: E^\cdot \rightarrow E'^\cdot$ . Of course, we can assume that all  $F^i = F^i(X_1, d, D)$  for some fixed numbers  $d$  and  $D$  and that some  $l$  serves as a control bound for all  $\phi_i$  and can indeed easily define  $E'^i$  as in the proof of Lemma 4.18. This gives a chain subcomplex of  $F^\cdot$  in  $\mathbf{B}'_1$  with cokernel in  $\mathbf{B}'_2$ . The mapping cone  $C(h)$  of the inclusion  $h$  is contractible in  $\mathbf{B}/\mathbf{Z}$ . The point is that unless  $\mathbf{B}/\mathbf{Z}$  is idempotent complete, this does not necessarily imply that  $C(h)$  is acyclic.

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