$$MGL_{2*,*} \xleftarrow{\vartheta} \Omega_*$$

Goal: Prove that ϑ is an isomorphism

① Define MGL

Look at classifying spaces BGL_n; this admits a universal rank n bundle $U_n \to BGL_n$

zero section $BGL_n \to U_n$

Define $MGL_n := Th(U_n) = U_n / U_n \setminus 0$

These assemble into a T-spectrum:

$$MGL = (MGL_0, MGL_1, \ldots)$$

$MGL_n \wedge \mathbb{P}^1 \longrightarrow MGL_{n+1}$ come from

$$\begin{array}{c} Th(U_n \oplus \mathcal{O}) \longrightarrow Th(U_{n+1}) \\ \wr\wr \\ Th(U_n) \wedge \mathbb{P}^1 \end{array}$$

The associated cohomology theory is $MGL^{*,*}(X) = [\Sigma_T^\infty X, \Sigma^{*,*} MGL]_{SH(k)}$

Review of six operations for SH:

For any scheme B, $SH(B)$ is a closed symmetric monoidal stable ∞-category

$\underline{\mathrm{Hom}}$ $\quad -\wedge-, \mathbb{1}_B$

- For any morphism $f: Y \to X$, we have an adjunction

$$f^* : SH(X) \rightleftarrows SH(Y) : f_* \quad \text{w/ } f^* \text{ symmetric monoidal}$$

For $f: Y \to X$ separated, of finite type, we have an adjunction

$$f_! : SH(Y) \rightleftarrows SH(X) : f^!$$

If f is proper, then $f_! \to f_*$ is an isomorphism.

If f is smooth, then f^* has a further left adjoint $f_\#$

- Various exchange transformations
- Associated with $i: Z \hookrightarrow X$, open complement $j: U \hookrightarrow X$, there are cofiber sequences

$$j_! j^! \to \mathrm{id} \to i_* i^*$$

$$i_! i^! \to \mathrm{id} \to j_* j^*$$

• Purity: If f is smooth, there are equivalences

$$f_! \simeq f_\# \circ \underbrace{\Sigma^{-\Omega_f}}_{Th(-\mathbb{V}(\Omega_f)) \wedge -} \qquad f^! \simeq \Sigma^{\Omega_f} \circ f^*$$

On to the definition $MGL^{BM}_{*,*}(X)$

$$MGL^{BM}_{*,*}(X) := MGL^{-*,-*}(\pi_{X!}(\mathbb{1}_X)) \quad \text{, where } \pi_X : X \longrightarrow \operatorname{Spec} k$$

$$\pi_{X!} : SH(X) \longrightarrow SH(k)$$

$$\mathbb{1}_X \longmapsto X_{BM/k}$$

Slightly trickier to describe functoriality:

for $f: X \to Y$ proj. of rel. dim. d, we have

$$f_* : MGL^{*,*}(X) \longrightarrow MGL^{*-2d,*-d}(Y)$$

Upshot: $MGL^{BM}_{2*,*}$ defines an OBMHT in the sense of the last talk

Only thing here: Chern classes for line bundles.

Suppose $L \to X$ line bundle w/ X smooth projective$/k$

Jouanolou device for X [an affine space bundle over X, that is affine]

$\rightsquigarrow$ we can assume that L is generated by global sections, i.e.

there is a morphism $f: X \to \mathbb{P}^N$ s.t. $L \cong f^* \mathcal{O}(1)$

$$\begin{array}{c} \mathbb{P}^N \\ \downarrow \\ \mathbb{P}^\infty = BGL_1 \end{array}$$

Recall $MGL_1 = Th(\mathcal{O}_{\mathbb{P}^\infty}(1))$, the structure maps in MGL give maps

$$MGL_1 \wedge (\mathbb{P}^1)^{\wedge n} \longrightarrow MGL_{n+1}, \text{ i.e.}$$

$$\Sigma^\infty_{\mathbb{P}^1} MGL_1 \longrightarrow \mathbb{P}^1 \wedge MGL \simeq S^{2,1} \wedge MGL \rightsquigarrow [\iota] \in MGL^{2,1}(MGL_1)$$

To define $c_1(\mathcal{O}_{\mathbb{P}^\infty}(1)) \in MGL^{2,1}(\mathbb{P}^\infty)$, look $\pi: \mathbb{P}^\infty \xrightarrow{0} \mathcal{O}_{\mathbb{P}^\infty}(1) \longrightarrow MGL_1$:

$$c_1(\mathcal{O}_{\mathbb{P}^\infty}(1)) := \pi^*([\iota]) \in MGL^{2,1}(\mathbb{P}^\infty) \xrightarrow{f^*} MGL^{2,1}(X)$$

$$\longmapsto c_1(L).$$

<u>Fact:</u> This makes $MGL^{BM}_{*,*}$ into an OBMHT

Universality of Ω_* gives a map <u>$\vartheta: \Omega_* \longrightarrow MGL_{2*,*}$</u>.

Theorem: $\vartheta : \Omega_* \longrightarrow MGL_{2*,*}$ is an isomorphism.

Proof strategy: ① For fields $\Omega_*(F) = \mathbb{L}_* = \mathbb{L}^{-*}$ Construct an isomorphism

$$\varphi_F : \mathbb{L}^* \xrightarrow{\cong} MGL^{2*,*}(F)$$ for $F = k(X)$, X smooth irreducible/k

and a surjection $\psi_F : F^\times \otimes \mathbb{L}^* \longrightarrow MGL^{2*+1,*+1}(F)$

② Suppose $j : U \hookrightarrow X \in Sm/k$, construct $\psi : \mathcal{O}_U^\times(U) \longrightarrow MGL^{1,1}(U)$

$i : D = X \setminus U \hookrightarrow X$

$$\cdots \longrightarrow MGL^{1,1}(X) \xrightarrow{j^*} MGL^{1,1}(U) \xrightarrow{\partial_{X,D}} MGL^{2,1}_D(X) \xrightarrow{i_*} MGL^{2,1}(X) \longrightarrow \cdots$$

$$MGL^{2,1}_D(X) = MGL^{2,1}(X/X\setminus D)$$

we'll need a computation of $\underline{\partial_{X,D}(\psi(u))}$.

③ Prove the theorem by induction on the maximal dimension of irreducible components of X.

On (1): Suppose X smooth and irreducible $/k$, $F = k(X)$

We know $\Omega^*(F) \cong \mathbb{L}^*$, so take $\varphi = \vartheta : \mathbb{L}^* \cong \Omega^*(F) \longrightarrow MGL^{2*,*}(F)$

To define $\psi : \Gamma(X, \mathcal{O}_X^\times) \longrightarrow MGL^{1,1}(X)$, construct

$$\begin{tikzcd} B\mathbb{G}_m = \mathbb{P}^\infty \arrow[rr] \arrow[dr, "\mathcal{O}"'] & & MGL_1 \\ & \mathcal{O}_{\mathbb{P}^\infty}(1) \arrow[ur] & \end{tikzcd}$$

This gives $\mathbb{G}_m(X) \longrightarrow [X_+, \Omega_s B\mathbb{G}_m]_* \simeq [S^1 \wedge X_+, B\mathbb{G}_m]_* \longrightarrow \cdots$

$$\longrightarrow [S^1 \wedge X_+, MGL_1]_* \xrightarrow{?\circ\Sigma^\infty} [S^1 \wedge \Sigma^\infty_{\mathbb{P}^1} X_+, S^{2,1} \wedge MGL] \cong$$

$$\cong [\Sigma^\infty_{\mathbb{P}^1} X_+, S^{1,1} \wedge MGL] = MGL^{1,1}(X).$$

Motivic Cohomology $H\mathbb{Z}$ is oriented, so it admits $\rho : MGL \longrightarrow H\mathbb{Z}$

Lemma $\rho \circ \psi : \Gamma(X, \mathcal{O}_X^\times) \longrightarrow H^{1,1}(X)$ is an isomorphism.

Sketch: $\mathcal{O}_X^*(X) \cong \mathrm{Pic}(X \times \mathbb{A}^1, X \times \{0,1\}) \underset{\cong}{\xrightarrow{c_1}} H^{2,1}(X \times \mathbb{A}^1 / X \times \{0,1\}) = H^{1,1}(X)$

etc. //

Use Hopkins-Morel-Hoyois spectral sequence:

$$E_2^{p,q}(n) = H^{p-q, n-q}(F) \otimes \mathbb{L}^q \Longrightarrow MGL^{p+q,n}(F)$$

$H^{0,0}(F) = \mathbb{Z}$, $H^{1,1}(F) = F^\times$, $\mathbb{L}^q = 0$ if $q > 0$

Formula E_2:

$$E_2^{p,q}(n) = \begin{cases} 0 & p > n \\ \mathbb{L}^n & p = q = n \\ F^\times \otimes \mathbb{L}^{n-1} & p = n,\ q = n-1 \end{cases}$$

$\Rightarrow$ we get surjections

$$\mathbb{L}^n = E_2^{n,n}(n) \xrightarrow{\varphi} E_\infty^{n,n}(n) = MGL^{2n,n}(F)$$

$$F^\times \otimes \mathbb{L}^{n-1} = E_2^{n,n-1}(n) \xrightarrow{\psi} E_\infty^{n,n-1}(n) = MGL^{2n-1,n}(F)$$

Injectivity of φ by picking an embedding $k \hookrightarrow \mathbb{C}$ and compare MGL and MU. //

Assuming ②, the proof in ③ goes as follows: