Geometric Algebraic Cobordism

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Overview

Theorem (Levine-Morel)

Let k be a field of characteristic zero.

1. *There is a universal OCT on* $\mathbf{Sm}_k$, $\Omega^* : \mathbf{Sm}_k^{\mathrm{op}} \to \mathrm{Gr}\mathbf{Ring}$
2. *There is a universal OBMHT on* $\mathbf{Sch}_k$, $\Omega_* : \mathbf{Sch}'_k \to \mathrm{Gr}\mathbf{Ab}$
3. *For* $X \in \mathbf{Sm}_k$ *of dimension* d, $\Omega^n(X) = \Omega_{d-n}(X)$
4. *For* $X \in \mathbf{Sch}_k$ *with closed immersion* $i : Z \to X$ *and open complement* $j : U \to X$, *the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

is exact

$\Omega_n(X)$ has generators as $\mathbb{L}$-module the *cobordism cycles*

$$(f : Y \to X, L_1, \ldots, L_r)$$

- f projective, $Y \in \mathbf{Sm}_k$ irreducible
- $L_1, \ldots, L_r$ line bundles on Y
- $n = \dim_k Y - r$
- modulo isomorphism over X and reordering the L_i:

For $g : (Y, f) \xrightarrow{\sim} (Y', f')$, $\sigma \in S_r$ and $L_i \cong g^* L'_{\sigma(i)}$

$$(f : Y \to X, L_1, \ldots, L_r) = (f' : Y' \to X, L'_1, \ldots, L'_r)$$

For $g : X \to X'$ proper,

$$g_*((f : Y \to X, L_1, \ldots, L_r)) = (g \circ f : Y \to X', L_1, \ldots, L_r)$$

For $g : X' \to X$ smooth,

$$g^*((f : Y \to X, L_1, \ldots, L_r)) = (p_{X'} : Y \times_X X' \to X', g^* L_1, \ldots, g^* L_r)$$

For $L \in \mathrm{Pic}(X)$, the operator $\tilde{c}_1(L)$

$$\tilde{c}_1(L)((f : Y \to X, L_1, \ldots, L_r)) = (f : Y \to X, L_1, \ldots, L_r, f^* L)$$

The following relation hold:

1. (Dim) For $\dim_k Z - r < 0$, and $g : Y \to Z$ smooth morphism in $\mathbf{Sm}_k$, $L_1, \ldots, L_r$ line bundles on g.
$(f : Y \to X, g^* L_1, \ldots, g^* L_r) = 0$
2. (Sect) Let $i : D \to Y$ be a codimension one closed immersion in $\mathbf{Sm}_k$ (Cartier divisor) and let $L = O_Y(D)$. Then

$$\tilde{c}_1(L)((f : Y \to X, L_1, \ldots, L_r)) = (f \circ i : D \to X, i^* L_1, \ldots, i^* L_r))$$

3. (FGL) Let $F_{\mathbb{L}}(u, v) \in \mathbb{L}[[u, v]]$ be the universal FGL. For L, M line bundles on X

$$\tilde{c}_1(L \otimes M) = F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))$$

There is another presentation of $\Omega_*(X)$:

Definition (Levine-Pandharipande)

A pair $(p : Y \to \mathbb{P}^1, f : Y \to X)$ is a *double-point cobordism* if

- p is dominant, $Y \in \mathbf{Sm}_k$
- f is projective
- $p^{-1}(\infty)$ is smooth (possibly empty)
- $p^{-1}(0) = A \cup B$ with A, B and $C := A \cap B$ smooth and $C \subset A$ of codimension 1 (any of A, B, C could be empty).

Let $N_{C \subset A}$ = the normal bundle of C in A.

Theorem (Levine-Pandharipande)

Let k be a field of characteristic zero and let $X \in \mathbf{Sch}_k$. Then $\Omega_*(X)$ is the free abelian group on cobordism cycles of the form $(f : Y \to X)$ modulo the double-point relations*:*

Let $(p : Y \to \mathbb{P}^1, f : Y \to X)$ be a double point cobordism with $f_\infty : Y_\infty \to X$ the fiber of p over ∞ and $f_A : A \to X$, $f_B : B \to X$, $f_C : C \to X$ the maps induced by $f : p^{-1}(0) = A \cup B \to X$. Then

$$[(f_\infty : Y_\infty \to X)] = [f_A : A \to X] + [f_B : B \to X] - [p_C : \mathbb{P}_C(O_C \oplus N_{C\subset A}) \to X]$$

where p_C is $\mathbb{P}_C(O_C \oplus N_{C\subset A}) \to C \xrightarrow{f_C} X$.

Data for an OBMHT A on $\mathbf{Sch}_k$:
$\mathbf{Sch}_k \supset \mathbf{Sch}'_k$ = the subcategory of projective morphisms in $\mathbf{Sch}_k$.

D1. A functor $A_* : \mathbf{Sch}'_k \to \mathrm{Gr}\mathbf{Ab}$
D2. For $g : X \to X'$ lci of relative dimension d,
$g^* : A_*(X') \to A_{*+d}(X)$
D3. An element $1 \in A_0(k)$ and external products

$$\times : A_*(X) \otimes A_*(Y) \to A_*(X \times_k Y)$$

associative, commutative, with unit 1.

(D1)+(D2)$\rightsquigarrow$ 1st Chern class operators: For $L \to Y$ a line bundle on Y with 0-section $s : Y \to L$ define $\tilde{c}_1 : A_*(Y) \to A_{*-1}(Y)$ by

$$\tilde{c}_1 := s^* \circ s_*$$

Oriented Borel-Moore homology theories

Relations

These satisfy:

BM1. The lci pull-backs are functorial
BM2. Given a transverse (Tor-independent) cartesian square in $\mathbf{Sch}_k$

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

with g lci and f projective, $g^* f_* = f'_* g'^*$
BM3. $(f \times g)_*(u \times v) = f_*(u) \times g_*(v)$ for f, g projective and $(f \times g)^*(u \times v) = f^*(u) \times g^*(v)$ for f, g lci.

PB. Let $E \to X$ be a rank $n+1$ vector bundle on X giving $q : \mathbb{P}(E) \to X$ with quotient line bundle $q^*E \to O_E(1)$. Let

$$\xi^{(i)} := \tilde{c}_1(O_E(1))^i \circ q^* : A_{*+i-n}(X) \to A_*(\mathbb{P}(E))$$

then

$$\sum_{i=0}^{n} \xi^{(i)} : \oplus_{i=0}^{n} A_{*+i-n}(X) \to A_*(\mathbb{P}(E))$$

is an isomorphism.

EH. Let $p : V \to X$ be an affine space bundle (torsor for a vector bundle) of rank r. Then $p^* : A_*(X) \xrightarrow{\sim} A_{*+r}(V)$.

CD. Let $W = (\mathbb{P}^N)^r$, let $p_i : W \to \mathbb{P}^N$ be the ith projection, choose $n_i \geq 0$, and let $i : E \to W$ be the Cartier divisor defined by $\prod_{i=1}^{r} p_i^*(X_0^{n_i}) = 0$. Then $i_* : A_*(E) \to A_*(W)$ is injective.

The projective bundle formula gives a formal group law for $\tilde{c}_1$:

- On $\mathbb{P}^n \times \mathbb{P}^m$ we have $O_{n,m}(1,1) := p_1^* O_{\mathbb{P}^n}(1) \otimes p_2^* O_{\mathbb{P}^m}(1)$ and

$$c_1(O_{n,m}(1,1)) := \tilde{c}_1(O_{n,m}(1,1))(1_{\mathbb{P}^n \times \mathbb{P}^m})$$
$$= \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij}^{n,m} \cdot \tilde{c}_1(O_{\mathbb{P}^n}(1))^i(1_{\mathbb{P}^n}) \times \tilde{c}_1(O_{\mathbb{P}^m}(1))^j(1_{\mathbb{P}^m});$$
$$a_{ij}^{n,m} \in A_*(k).$$

$\rightsquigarrow F_A(u,v) = \sum_{i,j} a_{ij} u^i v^j \in A_*(k)[[u,v]]$ ($a_{ij}^{n,m}$ stable in n, m).

- (Commutativity): Use $\tau : \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\sim} \mathbb{P}^m \times \mathbb{P}^n$, $\tau^*(O(1,1)) \cong O(1,1) \rightsquigarrow a_{ij} = a_{ji}$.
- (Associativity) Use $O_{n,m,l}(1,1,1)$ and the Segre embeddings $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^l \to \mathbb{P}^N \times \mathbb{P}^l$, $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^l \to \mathbb{P}^n \times \mathbb{P}^M$.

As Pic is "represented" by $\{\mathbb{P}^n, O_{\mathbb{P}^n}(1)\}$ get

$$\tilde{c}_1(L \otimes M)(1_Y) = F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y)$$

$\forall Y$ lci over k.
The *Lazard ring* $\mathbb{L}$ is the coefficient ring of the universal fgl $F_{\mathbb{L}}(u, v) \in \mathbb{L}[[u, v]] \rightsquigarrow$ the classifying homomorphism

$$\phi_A : \mathbb{L} \to A_*(k)$$

Oriented cohomology theories

The axioms for an OCT on $\mathbf{Sm}_k$ are a modification of the axioms for a OBMHT:

D1': $A^* : \mathbf{Sm}_k^{\mathrm{op}} \to \mathrm{Gr}\mathbf{Ring}$

D2': For $f : Y \to X$ projective, relative dimension d, have $f_* : A^*(Y) \to A^{*-d}(X)$, $A^*(X)$-linear.

1st Chern classes: For $L \to Y$ a line bundle, $Y \in \mathbf{Sm}_k$, with zero-section $s : Y \to L$, define $c_1(L) := s^* s_*(1_Y) \in A^1(Y)$.

(O1): The projective pushforward maps are functorial
(O2), (O3), (PB), (EH): the analogs of (BM2), (BM3), (PB) and (EH) are satisfied.

Oriented cohomology theories

Proposition

Let A be an OCT on $\mathbf{Sm}_k$. There is a fgl $F_A(u,v) \in A^*(k)[[u,v]]$ with

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)).$$

Theorem

If A_* is an OBMHT on $\mathbf{Sch}_k$, defining $A^n(Y) = A_{\dim Y - n}(Y)$ gives an OCT on $\mathbf{Sm}_k$ with ring structure on $A^*(Y)$ defined by

$$\alpha \cup \beta = \delta_Y^*(\alpha \times \beta).$$

For $Y \in \mathbf{Sm}_k$, $L \to Y$ a line bundle and $\alpha \in A^*(Y)$, we have

$$\tilde{c}_1(L)(\alpha) = c_1(L) \cup \alpha.$$

A_* and A^* have the same fgl.

Examples

1. The Chow groups $X \mapsto \mathrm{CH}_*(X)$ is an OBMHT on $\mathbf{Sch}_k$ with OCT on $\mathbf{Sm}_k$ the Chow ring $Y \mapsto \mathrm{CH}^*(Y)$.
2. The Grothendieck group of coherent sheaves $X \mapsto G_0(X)[\beta, \beta^{-1}]$ $(\deg \beta = 1)$ is an OBMHT on $\mathbf{Sch}_k$ with OCT on $\mathbf{Sm}_k$ the Grothendieck group of vector bundles $Y \mapsto K_0(Y)[\beta, \beta^{-1}]$ $(\deg \beta = -1)$.
3. If $\mathrm{char}\, k = 0$, algebraic cobordism $X \mapsto \Omega_*(X)$ is the universal OBMHT on $\mathbf{Sch}_k$ with OCT $Y \mapsto \Omega^*(Y)$ the universal OCT on $\mathbf{Sm}_k$. The classifying map $\phi_\Omega : \mathbb{L} \to \Omega(k)$ is an isomorphism.
4. $(\mathrm{char}\, k = 0)$ Let $\mathbb{L} \to R_*$ be a (graded) ring homomorphism $\rightsquigarrow F_R(u, v) \in R_*[[u, v]]$ fgl. Then $X \mapsto \Omega_*(X) \otimes_{\mathbb{L}} R_*$ is the universal OBMHT on $\mathbf{Sch}_k$ having fgl $= (F_R, R_*)$. Similarly for the OCT $Y \mapsto \Omega^*(Y) \otimes_{\mathbb{L}} R^*$ $(R^n = R_{-n})$

Ω and $\mathbb{L}$

Theorem

($\mathrm{char}\, k = 0$) *The classifying map* $\phi_\Omega : \mathbb{L} \to \Omega(k)$ *is an isomorphism.*

Idea of proof.

(1) Compute $\Omega_0(k) = \mathbb{Z}$ by degree using explicit "elementary cobordisms

(2) Let $\phi_a : \mathbb{L} \to \mathbb{Z}$ classify $F_a(u, v) = u_v \in \mathbb{Z}[[u, v]]$ and consider $\Omega_*^{ad} := \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$ and the induced map $\mathbb{L}_*/\mathbb{L}_*^+ \to \Omega_*^{ad}(k) = \Omega_*(k)/\mathbb{L}^+$. Prove by induction on n that this is an iso as follows:

(3) Show using a "blow-up formula" and weak factorization that Ω_*^{ad} is *birational*: If $f : Z \to X$ and $f' : Z' \to X$ are projective morphisms, $Z, Z' \in \mathbf{Sm}_k$, $g : Z \to Z'$ proper birational over X, then $[f : Z \to X] = [f' : Z' \to X] \in \Omega_*^{ad}(X)$. □

Ω and $\mathbb{L}$

Idea of proof-continued.

(4) Using (PB) and (3), show $[\mathbb{P}(E) \to Y] = 0 \in \Omega_n^{ad}(Y)$ for $E \to Y$ rank $n+1 \geq 2$,starting with $[\mathbb{P}^1 \to \mathrm{Spec}\, k] = -a_{11} \in \mathbb{L}_1$.

(5) Take a generator $[Y]$ for $\Omega_n^{ad}(k)$, $Y \in \mathbf{Sm}_k$ projective, dimension n over k.

Project to a hypersurface of degree d, $p : Y \to \bar{Y} \subset \mathbb{P}_k^{n+1}$.

Resolve singularities: $\pi : \tilde{\mathbb{P}} \to \mathbb{P}_k^{n+1}$, $\pi^{-1}(\bar{Y}) = \tilde{Y} + \sum_{i=1}^r n_i E_i$.

(6) By (3) and weak factorization $[\tilde{Y}] = [Y] \in \Omega_n^{ad}(k)$. By (4), fgl and induction $[\tilde{Y} + \sum_{i=1}^r n_i E_i] = [\tilde{Y}] \in \Omega_n^{ad}(k)$.

(7) Let $S \subset \mathbb{P}^{n+1}$ be a general degree d hypersurface, so $\tilde{S} := \pi^{-1}(S)$ is smooth. Then $[S] = [\tilde{S}] = [\tilde{Y} + \sum_{i=1}^r n_i E_i] = [\tilde{Y}]$ by (3)-(6).

(8) Degenerate S to d hyperplanes in general position. Then as for (7) $[S] = d \cdot [\mathbb{P}^n] = 0 \in \Omega_n^{ad}(k)$ (continued) □

Idea of proof-continued.

Steps (6,7,8) rely on the cobordism class of a SNC divisor:

$E \subset Y$ a SNC divisor on a smooth dim n Y, $|E| :=$ support of E, $i : |E| \to Y$. There is $[E \to |E|] \in \Omega_{n-1}(|E|)$ with $i_*[E \to |E|] = c_1(\mathcal{O}_Y(E)) \in \Omega_{n-1}(Y)$.

For E smooth, $[E \to |E|] = 1_E$. $i_*(1_E) = c_1(\mathcal{O}_Y(E))$ by (Sect)

For more general E, "decompose" (Sect) using the fgl, see below: □

Idea of proof-continued.

We describe $[E \to |E|]$ for $E = E_1 + E_2$:

$E_{12} = E_1 \cap E_2$, $i_j : E_j \to |E|$, $i_{12} : E_{12} \to |E|$
$\tilde{i}_j : E_j \to Y$, $\tilde{i}_{12} : E_{12} \to Y$. For $\alpha \in \Omega_*(E_{12})$, applying (Sect) twice $\Rightarrow i_{12*}(i_{12}^*\alpha) = c_1(\mathcal{O}_Y(E_1)) \cdot c_1(\mathcal{O}_Y(E_2)) \cdot \alpha$.

The fgl $F_\Omega(u, v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j = u + v + uv \cdot g(u, v)$.

Define $[E \to |E|] :=$
$i_{1*}(1_{E_1}) + i_{2*}(1_{E_2}) + i_{12*}(\tilde{i}_{12}^* g(c_1(\mathcal{O}_Y(E_1)), c_1(\mathcal{O}_Y(E_2)))$, so

$$
\begin{aligned}
i_*[E \to |E|] &= \tilde{i}_{1*}(1_{E_1}) + \tilde{i}_{2*}(1_{E_2}) + \tilde{i}_{12*}(\tilde{i}_{12}^* g(c_1(\mathcal{O}_Y(E_1)), c_1(\mathcal{O}_Y(E_2))) \\
&= c_1(\mathcal{O}_Y(E_1)) + c_1(\mathcal{O}_Y(E_2)) \\
&\quad + c_1(\mathcal{O}_Y(E_1)) \cdot c_1(\mathcal{O}_Y(E_2)) \cdot g(c_1(\mathcal{O}_Y(E_1)), c_1(\mathcal{O}_Y(E_2))) \\
&= F_\Omega(c_1(\mathcal{O}_Y(E_1)), c_1(\mathcal{O}_Y(E_2))) \\
&= c_1(\mathcal{O}_Y(E))
\end{aligned}
$$

Theorem

The fgl for $K_0[\beta, \beta^{-1}]$ *is* $F_K(u, v) = F_\mu(u, v) := u + v - \beta \cdot uv$ *and* $Y \mapsto K_0(Y)[\beta, \beta^{-1}$ *is the universal OCT on* $\mathbf{Sm}_k$ *with this fgl.*

Corollary

Assume $\mathrm{char} k = 0$.
Let $\phi_m : \mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$ *be the homomorphism classifying* $F(u, v) = u + v - \beta uv$. *Then the canonical map of OCTs* $\Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to K_0[\beta, \beta^{-1}]$ *is an isomorphism.*

Theorem

1. The fgl for CH_* *is* $F_{\mathrm{CH}}(u,v) = F_a(u,v) := u + v \in \mathbb{Z}[[u,v]]$.

2. ($\mathrm{char}\, k = 0$) *Let* $\phi_a : \mathbb{L} \to \mathbb{Z}$ *be the homomorphism classifying* $(F_a(u,v), \mathbb{Z})$. *Then the canonical map of OBMHTs* $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z} \to \mathrm{CH}_*$ *is an isomorphism.*

3. $X \mapsto \mathrm{CH}_*(X)$ *is the universal OBMHT on* $\mathbf{Sch}_k$ *with fgl* $(F_a(u,v), \mathbb{Z})$

4. $X \mapsto \mathrm{CH}^*(X)$ *is the universal OCT on* $\mathbf{Sm}_k$ *with fgl* $(F_a(u,v), \mathbb{Z})$.