Annala's derived algebraic cobordism over a general base

Talk I

July 7, 2020

Bibliography

[A] Chern classes in precobordism theories, T. Annala (2019)

[AY] Bivariant algebraic cobordism with bundles, T. Annala, S. Yokura (2019)

Contents

Algebraic cobordism (Voevodsky)

The bigraded theory $MGL^{*,*}$ on Sm_k was defined as the theory represented by the algebraic Thom spectrum $MGL \in \mathcal{SH}(k)$ in the stable motivic homotopy category and was used in order to prove the Milnor conjecture.

Algebraic cobordism over a field k of characteristic 0 (Levine-Morel)

$\Omega^* : Sm_k^{op} \to GrRing$ is the universal oriented cohomology theory.

$\Omega_* : Sch'_k \to GrAb$ is the universal oriented Borel-Moore homology theory.

A theorem of Levine establishes $MGL^{2*,*}(X) \cong \Omega^*(X)$.

The graded group $\Omega_n(X)$ can be described in terms of cobordism cycles of the form $[Y \xrightarrow{f} X, L_1, \ldots, L_r]$ with Y irreducible, f projective and $L_1, \ldots, L_r$ line bundles on Y such that $n = \dim(Y) - r$.

Double point cobordism over a field k of characteristic 0 (Levine-Pandharipande)

For $X \in Sch_k$, the group $\omega_n(X)$ is the free abelian group on cobordism cycles of the form $[Y \xrightarrow{f} X]$ with $Y \in Sm_k$ irreducible of dimension n and f projective modulo the double-point relations.

A theorem of Levine-Pandharipande establishes $\omega_*(X) \cong \Omega_*(X)$.

Algebraic cobordism of bundles over a field k of characteristic 0 (Lee-Pandharipande)

For $X \in Sch_k$, the group $\omega_{n,r}(X)$ is the free abelian group on cobordism cycles of the form $[Y \xrightarrow{f} X, E]$ with $Y \in Sm_k$ irreducible of dimension n, E a vector bundle of rank r on Y and f projective modulo analogously defined double-point relations.

In particular, one has $\omega_{*,0}(X) \cong \omega_*(X)$.

Derived algebraic cobordism over a field k of characteristic 0 (Lowrey-Schürg)

Lowrey and Schürg define algebraic cobordims groups $d\Omega_*(X)$ for quasi-projective derived schemes.

For a quasi-projective derived scheme X, one has an isomorphism $d\Omega_*(X) \cong \Omega_*(tX)$, where tX denotes the truncation of X.

Bivariant derived algebraic cobordism over a field k of characteristic 0 (Annala-Yokura)

Derived algebraic cobordism as a bivariant theory over the homotopy category of quasi-projective derived k-schemes with groups $\Omega^*(X \xrightarrow{f} Y)$ for any morphism f between quasi-projective derived schemes over k.

Associated cohomology theory: $\Omega^*(X) := \Omega^*(X \xrightarrow{id} X)$.
Associated homology theory: $\Omega_*(X) := \Omega^{-*}(X \to pt)$.

The associated homology theory coincides with the derived algebraic cobordism as defined by Lowrey-Schürg.

Next step: Bivariant derived algebraic cobordism of bundles over a field of characteristic 0!

Setting

k: field of characteristic 0

$dSch_k$: homotopy category of quasi-projective derived k-schemes with

- proper morphisms as confined morphisms,
- all homotopy Cartesian squares as independent squares,
- quasi-smooth morphisms as specialized morphisms,
- smooth morphisms as specialized projections.

Step I: Construction of $\mathcal{M}_{\mathbb{L}}^{*,*}$

Let $\mathcal{M}_{\mathbb{L}}^{i,r}(X \xrightarrow{f} Y)$ be the quotient of the free $\mathbb{L}$-module on the set of cycles of the form $[V \xrightarrow{h} X, E]$, where

- $h : V \to X$ is proper,
- the composite $f \circ h$ is quasi-smooth of virtual relative dimension $-i$,
- E is a vector bundle on V of rank r,

by the relation that taking the disjoint union on the sources corresponds to summation.

Pushforward: For $f : X \to X'$ proper and $g : X' \to Y$, one defines the pushforward map

$$f_* : \mathcal{M}_{\mathbb{L}}^{i,r}(X \xrightarrow{g \circ f} Y) \to \mathcal{M}_{\mathbb{L}}^{i,r}(X' \xrightarrow{g} Y)$$

by $f_*([V \xrightarrow{h} X, E]) = [V \xrightarrow{f \circ h} X', E]$.

Step I: Construction of $\mathcal{M}_{\mathbb{L}}^{*,*}$

Pullback: For $f : X \to X'$, $g : Y' \to Y$ and $X' = Y' \times_Y^{\mathbb{R}} X$, one defines the pullback map

$$g^* : \mathcal{M}_{\mathbb{L}}^{i,r}(X \xrightarrow{f} Y) \to \mathcal{M}_{\mathbb{L}}^{i,r}(X' \xrightarrow{f'} Y')$$

by $g^*([V \xrightarrow{h} X, E]) = [V' \xrightarrow{h'} X', E']$, where $V' = Y' \times_Y^{\mathbb{R}} V$ and $E' = (g'')^*(E)$ is the pullback of E along the projection $g'' : V' \to V$.

$$\begin{array}{ccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{h} & X & \xrightarrow{f} & Y \end{array}$$

Step I: Construction of $\mathcal{M}_{\mathbb{L}}^{*,*}$

Bivariant products: We define two bivariant products

$$\bullet_{\oplus} : \mathcal{M}_{\mathbb{L}}^{i,r}(X \xrightarrow{f} Y) \times \mathcal{M}_{\mathbb{L}}^{j,s}(Y \xrightarrow{g} Z) \to \mathcal{M}_{\mathbb{L}}^{i+j,r+s}(X \xrightarrow{g\circ f} Z)$$

$$\bullet_{\otimes} : \mathcal{M}_{\mathbb{L}}^{i,r}(X \xrightarrow{f} Y) \times \mathcal{M}_{\mathbb{L}}^{j,s}(Y \xrightarrow{g} Z) \to \mathcal{M}_{\mathbb{L}}^{i+j,rs}(X \xrightarrow{g\circ f} Z)$$

as follows: For cycles $[V \xrightarrow{h} X, E]$ and $[W \xrightarrow{k} Y, F]$, we form the following homotopy Cartesian diagram

$$\begin{array}{ccccccc}
V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\
\downarrow{\scriptstyle k''} & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle k} & & \\
V & \xrightarrow{h} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}$$

We let $E' = (k'')^*(E)$ and $F' = (f' \circ h')^*(F)$ and define

$$[V \xrightarrow{h} X, E] \bullet_{\oplus} [W \xrightarrow{k} Y, F] = [V' \xrightarrow{h\circ k''} X, E' \oplus F'],$$

$$[V \xrightarrow{h} X, E] \bullet_{\otimes} [W \xrightarrow{k} Y, F] = [V' \xrightarrow{h\circ k''} X, E' \otimes F'].$$

Step I: Construction of $\mathcal{M}_{\mathbb{L}}^{*,*}$

Proposition [AY, Proposition 5.3]

$\mathcal{M}_{\mathbb{L}}^{*,*}$ is a commutative bivariant theory with respect to both products $\bullet_{\oplus}$ and $\bullet_{\otimes}$.

We may also choose natural orientations along quasi-smooth morphisms: If $f : X \to Y$ is quasi-smooth of relative virtual dimension $-i$, then we may define the two orientations

$$\theta_{\oplus}(f) := [X \xrightarrow{id_X} X, 0] \in \mathcal{M}_{\mathbb{L}}^{i,0}(X \xrightarrow{f} Y)$$
$$\theta_{\otimes}(f) := [X \xrightarrow{id_X} X, \mathcal{O}_X] \in \mathcal{M}_{\mathbb{L}}^{i,1}(X \xrightarrow{f} Y)$$

for the two bivariant theories $(\mathcal{M}_{\mathbb{L}}^{*,*}, \bullet_{\oplus})$ and $(\mathcal{M}_{\mathbb{L}}^{*,*}, \bullet_{\otimes})$ respectively.

Step II: Relations

Recall that Annala defined his bivariant derived algebraic cobordism Ω^* as the quotient bivariant theory

$$\Omega^* := (\mathcal{M}^*_{\mathbb{L}}, \bullet_\oplus)/\langle \mathcal{R}^{LS} \rangle_{(\mathcal{M}^*_{\mathbb{L}}, \bullet_\oplus)}.$$

But $(\mathcal{M}^*_{\mathbb{L}}, \bullet_\oplus)$ can be identified with $(\mathcal{M}^{*,0}_{\mathbb{L}}, \bullet_\oplus) \subseteq (\mathcal{M}^{*,*}_{\mathbb{L}}, \bullet_\oplus)$. The bivariant ideal $\langle \mathcal{R}^{LS} \rangle_{(\mathcal{M}^*_{\mathbb{L}}, \bullet_\oplus)}$ is generated by a bivariant subset $\mathcal{R}^{LS}$.

Definition [AY, Definition 5.7]

Bivariant algebraic cobordism with vector bundles is defined as the quotient bivariant theory $(\Omega^{*,*}, \bullet_\oplus) := (\mathcal{M}^{*,*}_{\mathbb{L}}, \bullet_\oplus)/\langle \mathcal{R}^{LS} \rangle_{(\mathcal{M}^{*,*}_{\mathbb{L}}, \bullet_\oplus)}$, where we regard $\mathcal{R}^{LS}$ as a bivariant subset of $(\mathcal{M}^{*,*}_{\mathbb{L}}, \bullet_\oplus)$ via the canonical identification $(\mathcal{M}^*_{\mathbb{L}}, \bullet_\oplus) = (\mathcal{M}^{*,0}_{\mathbb{L}}, \bullet_\oplus)$.

Basic facts about $\Omega^{*,*}$

The bivariant theory $(\Omega^{*,*}, \bullet_{\oplus})$ is obviously commutative and inherits natural orientations

$$\theta_{\oplus}(f) := [X \xrightarrow{id_X} X, 0] \in \Omega_{+}^{i,0}(X \xrightarrow{f} Y)$$

along quasi-smooth morphisms $f : X \to Y$ of relative virtual dimension $-i$.

Associated cohomology theory: $\Omega^{*,*}(X) := \Omega^{*,*}(X \xrightarrow{id} X)$.

Associated homology theory: $\Omega_{*,*}(X) := \Omega^{-*,*}(X \to pt)$.

Basic facts about $\Omega^{*,*}$

There are two canonical Grothendieck transformations

$$\mathcal{Z} : (\Omega^*, \bullet_\oplus) \to (\Omega^{*,0}, \bullet_\oplus), [V \xrightarrow{h} X] \mapsto [V \xrightarrow{h} X, 0],$$
$$\mathcal{F} : (\Omega^{*,*}, \bullet_\oplus) \to (\Omega^*, \bullet_\oplus), [V \xrightarrow{h} X, E] \mapsto [V \xrightarrow{h} X],$$

which give a canonical isomorphism $\Omega^* \cong \Omega^{*,0}$.

The theory $\Omega^{*,*}$ also becomes bivariant with respect to the bivariant product $\bullet_\otimes$ on $\mathcal{M}^{*,*}_{\mathbb{L}}$. Then the subtheory $(\Omega^{*,1}, \bullet_\otimes)$ of $(\Omega^{*,*}, \bullet_\otimes)$ automatically becomes a bivariant theory equipped with an inclusion

$$(\Omega^*, \bullet_\oplus) \hookrightarrow (\Omega^{*,1}, \bullet_\otimes), [V \xrightarrow{h} X] \mapsto [V \xrightarrow{h} X, \mathcal{O}_V].$$

The subtheory $(\Omega^{*,1}, \bullet_\otimes)$ is called *bivariant algebraic cobordism with line bundles*.

Comparison theorem

Theorem [AY, Theorem 5.28]

For any quasi-projective derived scheme X, there is a natural isomorphism $\Omega_{*,*}(X) \cong \omega_{*,*}(tX)$, where tX denotes the truncation of X.

One considers a natural cross product map

$$\omega_{*,*}(pt) \otimes_{\mathbb{L}} \Omega_*(X) \to \Omega_{*,*}(X)$$

defined on cycles by

$$[V \to pt, E] \otimes_{\mathbb{L}} [W \to X] \mapsto [V \times W \to X, pr_V^*(E)]$$

and then identifies $\omega_{*,*}(pt) \otimes_{\mathbb{L}} \Omega_*(X) \cong \omega_{*,*}(pt) \otimes_{\mathbb{L}} \omega_*(tX)$ with $\omega_{*,*}(tX)$.

Setting

Question: What is the "correct" definition of bivariant algebraic cobordism over a general base scheme?

A: Noetherian ring of finite Krull dimension

$dSch_A$: homotopy category of quasi-projective derived A-schemes with

- proper morphisms as confined morphisms,
- all homotopy Cartesian squares as independent squares,
- quasi-smooth morphisms as specialized morphisms,
- smooth morphisms as specialized projections.

Construction of $\mathcal{M}_+^{*,*}$

Let $\mathcal{M}_+^{i,r}(X \xrightarrow{f} Y)$ be the quotient of the free abelian group on the set of cycles of the form $[V \xrightarrow{h} X, E]$, where

- $h : V \to X$ is proper,
- the composite $f \circ h$ is quasi-smooth of virtual relative dimension $-i$,
- E is a vector bundle on V of rank r,

by the relation that taking the disjoint union on the sources corresponds to summation.

Pushforward: For $f : X \to X'$ proper and $g : X' \to Y$, one defines the pushforward map

$$f_* : \mathcal{M}_+^{i,r}(X \xrightarrow{g \circ f} Y) \to \mathcal{M}_+^{i,r}(X' \xrightarrow{g} Y)$$

by $f_*([V \xrightarrow{h} X, E]) = [V \xrightarrow{f \circ h} X', E]$.

Construction of $\mathcal{M}_+^{*,*}$

Pullback: For $f : X \to X'$, $g : Y' \to Y$ and $X' = Y' \times_Y^{\mathbb{R}} X$, one defines the pullback map

$$g^* : \mathcal{M}_+^{i,r}(X \xrightarrow{f} Y) \to \mathcal{M}_+^{i,r}(X' \xrightarrow{f'} Y')$$

by $g^*([V \xrightarrow{h} X, E]) = [V' \xrightarrow{h'} X', E']$, where $V' = Y' \times_Y^{\mathbb{R}} V$ and $E' = (g'')^*(E)$ is the pullback of E along the projection $g'' : V' \to V$.

$$\begin{array}{ccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{h} & X & \xrightarrow{f} & Y \end{array}$$

Construction of $\mathcal{M}_+^{*,*}$

Bivariant products: We define two bivariant products

$$\bullet_{\oplus} : \mathcal{M}_+^{i,r}(X \xrightarrow{f} Y) \times \mathcal{M}_+^{j,s}(Y \xrightarrow{g} Z) \to \mathcal{M}_+^{i+j,r+s}(X \xrightarrow{g \circ f} Z)$$
$$\bullet_{\otimes} : \mathcal{M}_+^{i,r}(X \xrightarrow{f} Y) \times \mathcal{M}_+^{j,s}(Y \xrightarrow{g} Z) \to \mathcal{M}_+^{i+j,rs}(X \xrightarrow{g \circ f} Z)$$

as follows: For cycles $[V \xrightarrow{h} X, E]$ and $[W \xrightarrow{k} Y, F]$, we form the following homotopy Cartesian diagram

$$\begin{array}{ccccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\ \downarrow{\scriptstyle k''} & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle k} & & \\ V & \xrightarrow{h} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

We let $E' = (k'')^*(E)$ and $F' = (f' \circ h')^*(F)$ and define

$$[V \xrightarrow{h} X, E] \bullet_{\oplus} [W \xrightarrow{k} Y, F] = [V' \xrightarrow{h \circ k''} X, E' \oplus F'],$$
$$[V \xrightarrow{h} X, E] \bullet_{\otimes} [W \xrightarrow{k} Y, F] = [V' \xrightarrow{h \circ k''} X, E' \otimes F'].$$

Construction of $\mathcal{M}_+^{*,*}$

Proposition

$\mathcal{M}_+^{*,*}$ is a commutative bivariant theory with respect to both products $\bullet_\oplus$ and $\bullet_\otimes$.

If $f : X \to Y$ is quasi-smooth of relative virtual dimension $-i$, then we may define the two orientations along f

$$\theta_\oplus(f) := [X \xrightarrow{id_X} X, 0] \in \mathcal{M}_+^{i,0}(X \xrightarrow{f} Y)$$
$$\theta_\otimes(f) := [X \xrightarrow{id_X} X, \mathcal{O}_X] \in \mathcal{M}_+^{i,1}(X \xrightarrow{f} Y)$$

for the two bivariant theories $(\mathcal{M}_+^{*,*}, \bullet_\oplus)$ and $(\mathcal{M}_+^{*,*}, \bullet_\otimes)$ respectively (and, in particular, for the subtheories $(\mathcal{M}_+^{*,0}, \bullet_\oplus)$ and $(\mathcal{M}_+^{*,1}, \bullet_\otimes)$).

Euler classes

Associated cohomology theory: $\mathcal{M}_+^{*,*}(X) := \mathcal{M}_+^{*,*}(X \xrightarrow{id} X)$.

Associated homology theory: $\mathcal{M}^+_{*,*}(X) := \mathcal{M}_+^{-*,*}(X \to pt)$.

If $f : X \to Y$ is proper and quasi-smooth of relative virtual dimension $-i$, then the orientation $\theta_\oplus(f)$ along f induces a *Gysin pushforward homomorphism*

$$f_! = f_*(- \bullet_\oplus \theta_\oplus(f)) : \mathcal{M}_+^{k,r}(X) \to \mathcal{M}_+^{k+i,r}(Y).$$

Definition

Let E be a vector bundle of rank r on a quasi-projective derived A-scheme X with zero section $s : X \to E$. Then we define its *Euler class* by $e(E) = s^*(s_!(1_X)) = [V(s) \to X, 0] \in \mathcal{M}_+^{r,0}(X)$.

Definition of naive cobordism theories

We now consider the bivariant theory $(\mathcal{M}_+^*, \bullet_\oplus) := (\mathcal{M}_+^{*,0}, \bullet_\oplus)$.

Definition [AY, Definition 6.1(1)]

Let $(\mathbb{B}^*, \bullet_\oplus)$ be a quotient bivariant theory of $\mathcal{M}_+^*$. We say that $(\mathbb{B}^*, \bullet_\oplus)$ is a *naive cobordism theory* if, given a morphism $f : X \to Y$ and a projective morphism $W \to \mathbb{P}^1 \times X$ such that $W \to \mathbb{P}^1 \times X \xrightarrow{id_{\mathbb{P}^1} \times f} \mathbb{P}^1 \times Y$ is quasi-smooth of relative virtual dimension d, then

$$[W_0 \to X] = [W_\infty \to X] \in \mathbb{B}^{-d}(X \xrightarrow{f} Y),$$

where W_0 and W_∞ are the homotopy fibres of $W \to \mathbb{P}^1 \times X$ over $\{0\} \times X$ and $\{\infty\} \times X$ respectively.

Definition of precobordism theories

Definition [AY, Definition 6.1(2)]

Let $(\mathbb{B}^*, \bullet_\oplus)$ be a naive cobordism theory. We say that $(\mathbb{B}^*, \bullet_\oplus)$ is a *precobordism theory* if, given line bundles L_1 and L_2 on a quasi-projective derived scheme X, we have

$$\begin{aligned} e(L_1 \otimes L_2) = e(L_1) + e(L_2) \\ - e(L_1) \bullet e(L_2) \bullet [\mathbb{P}_1 \to X] \\ - e(L_1) \bullet e(L_2) \bullet e(L_1 \otimes L_2) \bullet ([\mathbb{P}_2 \to X] - [\mathbb{P}_3 \to X]) \end{aligned}$$

in $\mathbb{B}^1(X \xrightarrow{id_X} X)$, where $\bullet = \bullet_\oplus$ and

$$\mathbb{P}_1 := \mathbb{P}_X(L_1 \oplus \mathcal{O}),$$
$$\mathbb{P}_2 := \mathbb{P}_X(L_1 \oplus (L_1 \otimes L_2) \oplus \mathcal{O}),$$
$$\mathbb{P}_3 := \mathbb{P}_{\mathbb{P}_X(L_1 \oplus (L_1 \otimes L_2))}(\mathcal{O}(-1) \oplus \mathcal{O}).$$

Equivalent definition of precobordism theories

Proposition [AY, Proposition 6.3]

A naive cobordism theory $(\mathbb{B}^*, \bullet_\oplus)$ is a precobordism theory if and only if, given a quasi-smooth morphism $W \to \mathbb{P}^1 \times X$ with homotopy fibres W_0 and W_∞ of $W \to \mathbb{P}^1 \times X$ over $\{0\} \times X$ and $\{\infty\} \times X$ respectively such that W_∞ is equivalent to the sum of divisors $A \to W$ and $B \to W$, the double point cobordism formula

$$[W_0 \to X] = [A \to X] + [B \to X] - [\mathbb{P} \to X] \in \mathbb{B}^1(X),$$

holds, where $\mathbb{P} := \mathbb{P}_Z(\mathcal{O}(A) \oplus \mathcal{O})$ holds with Z the derived intersection of A and B in W.

Associated precobordism with line bundles

Note that we have an inclusion $(\mathcal{M}_+^*, \bullet_\oplus) \hookrightarrow (\mathcal{M}_+^{*,1}, \bullet_\otimes)$ given by

$$[V \xrightarrow{h} X, 0] \mapsto [V \xrightarrow{h} X, \mathcal{O}_X].$$

Definition [AY, Definition 6.6]

For a precobordism theory $(\mathbb{B}^*, \bullet_\oplus) = (\mathcal{M}_+^*, \bullet_\oplus)/\mathbb{I}$, its *associated precobordism with line bundles* is defined as the bivariant theory $(\mathbb{B}^{*,1}, \bullet_\otimes) := (\mathcal{M}_+^{*,1}, \bullet_\otimes)/\langle \mathbb{I} \rangle_{(\mathcal{M}_+^{*,1}, \bullet_\otimes)}$.

We obtain an induced inclusion

$$(\mathbb{B}^*, \bullet_\oplus) \hookrightarrow (\mathbb{B}^{*,1}, \bullet_\otimes)$$

and an induced forgetful Grothendieck transformation

$$\mathcal{F} : (\mathbb{B}^{*,1}, \bullet_\otimes) \to (\mathbb{B}^*, \bullet_\oplus).$$

Associated precobordism with line bundles

Theorem [AY, Theorem 6.12]

Let $\mathbb{B}^*$ be a precobordism theory. Then the group $\mathbb{B}^{*,1}(pt)$ has an $\mathbb{B}^*(pt)$-linear basis given by $([\mathbb{P}^i \to pt, \mathcal{O}(1)])_{i \geq 0}$.

Theorem [AY, Theorem 6.13]

Let $\mathbb{B}^*$ be a precobordism theory. Then the natural cross product map

$$\mathbb{B}^{*,1}(pt) \otimes_{\mathbb{B}^*(pt)} \mathbb{B}^*(X \to Y) \to \mathbb{B}^{*,1}(X \to Y),$$

$$([V \to pt, L], [W \to X]) \mapsto [V \times W \to X, \pi_V^*(L)],$$

is an isomorphism, where $\pi_V : V \times W \to V$ is the canonical projection.

Examples

Example 1 [AY, Example 6.4]

If $A = k$ is a field of characteristic 0, then Annala's bivariant derived algebraic cobordism Ω^* is a precobordism theory. The associated precobordism with line bundles is just Annala's bivariant algebraic cobordism $\Omega^{*,1}$ with line bundles.

Example 2 [AY, Construction 6.5]

One easily obtains the *universal precobordism theory* $\underline{\Omega}^*$ as a quotient bivariant theory from $\mathcal{M}^*_+$ by enforcing the relations from the definition of precobordism theories. Any precobordism theory $\mathbb{B}^*$ is a quotient of $\underline{\Omega}^*$.

Weak projective bundle formula

Theorem [AY, Theorem 6.22]

Let $\mathbb{B}^*$ be a bivariant precobordism theory and $X \to Y$ a morphism between derived schemes over a Noetherian ring A of finite Krull dimension. Then the assignments $\alpha \mapsto e(\mathcal{O}(1))^i \bullet \theta(\pi) \bullet \alpha$ induce an isomorphism

$$\bigoplus_{i=0}^{n} \mathbb{B}^{*-i+n}(X \to Y) \cong \mathbb{B}^*(\mathbb{P}^n \times X \to Y)$$

of $\mathbb{B}^*(pt)$-modules, where π is the projection $\mathbb{P}^n \times X \to X$.

Weak projective bundle formula

Corollary [AY, Corollary 6.24]

There is a natural isomorphism $\mathbb{B}^*(\mathbb{P}^n \times X) \cong \mathbb{B}^*(X)[t]/\langle t^{n+1}\rangle$ of rings, where $t = e(\mathcal{O}(1))$.

Proof.

Since $\mathbb{B}^*$ is a commutative bivariant theory, the homotopy Cartesian square

$$\begin{array}{ccc} \mathbb{P}^n \times X & \xrightarrow{id} & \mathbb{P}^n \times X \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{id} & X \end{array}$$

implies that $\theta(\pi) \bullet_\oplus \alpha = \pi^*(\alpha) \bullet_\oplus \theta(\pi)$ for all $\alpha \in \mathbb{B}^*(X)$. □

Weak projective bundle formula

Proof (continued).

Since the orientations along quasi-smooth morphisms are all nice (i.e. stable under pullbacks), the homomorphisms

$$- \bullet_\oplus \theta(\pi) : \mathbb{B}^{*+n}(\mathbb{P}^n \times X) \to \mathbb{B}^*(\mathbb{P}^n \times X \to X)$$

are isomorphisms. So the previous theorem implies that the homomorphisms

$$t^i \bullet_\oplus \pi^* : \mathbb{B}^{*-i}(X) \to \mathbb{B}^*(\mathbb{P}^n \times X)$$

induce an isomorphism

$$\bigoplus_{i=0}^n \mathbb{B}^{*-i}(X) \to \mathbb{B}^*(\mathbb{P}^n \times X).$$

Since $t^i = [\mathbb{P}^{n-i} \times X \to \mathbb{P}^n \times X, 0]$, it follows that $t^{n+1} = 0$. □

Formal group law

Now let $n, m \geq 1$ and consider the line bundle

$$\mathcal{O}_{n,m}(1,1) := \pi_1^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^m}(1))$$

on $\mathbb{P}^n \times \mathbb{P}^m$. Using the previous result, one obtains an isomorphism

$$\mathbb{B}^*(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{B}^*(pt)[x,y]/\langle x^{n+1}, y^{m+1}\rangle.$$

In particular, there are unique coefficients $a_{ij}^{n,m} \in \mathbb{B}^*(pt)$ such that

$$e(\mathcal{O}_{n,m}(1,1)) = \sum_{i,j} a_{ij}^{n,m} x^i y^j \in \mathbb{B}^*(pt)[x,y]/\langle x^{n+1}, y^{m+1}\rangle.$$

These coefficients do not depend on n, m as long as $i \leq n$ and $j \leq m$.

Formal group law

Therefore one obtains the following result:

Corollary [AY, Corollary 6.25]

Let $\mathbb{B}^*$ be a precobordism theory. Then there is a formal group law given by

$$F_{\mathbb{B}^*}(x,y) := \sum_{i,j} a_{ij} x^i y^j \in \mathbb{B}^*(pt)[[x,y]]$$

such that, for any two (globally generated) line bundles L_1, L_2 on a quasi-projective derived scheme X, one has

$$e(L_1 \otimes L_2) = F_{\mathbb{B}^*}(e(L_1), e(L_2)) = \sum_{i,j} a_{ij} e(L_1)^i e(L_2)^j \in \mathbb{B}^*(X).$$

The formal group law above induces a homomorphism of rings

$$\mathbb{L} \to \mathbb{B}^*(pt).$$

Construction of Chern classes

For all vector bundles E of rank r on a quasi-projective derived A-scheme X one constructs

1. a projective quasi-smooth morphism $\pi_{X,E} : \tilde{X}_E \to X$ of relative virtual dimension 0 which is natural in the sense that, for any morphism $f : Y \to X$, there is a homotopy Cartesian square

$$\begin{array}{ccc} \tilde{Y}_{f^*(E)} & \xrightarrow{f'} & \tilde{X}_E \\ \downarrow{\scriptstyle \pi_{Y,f^*(E)}} & & \downarrow{\scriptstyle \pi_{X,E}} \\ Y & \xrightarrow{f} & X \end{array}$$

2. $\pi^*_{X,E}(E)$ has a natural filtration $0 = E_0 \subseteq E_1 \subseteq \dots \subseteq E_r = \pi^*_{X,E}(E)$ with line bundles $L_1, \dots, L_r$ as the associated graded pieces;

3. a class $\eta_{X,E} \in \underline{\Omega}^0(\tilde{X}_E)$ pushing forward to $1_X \in \underline{\Omega}^0(X)$ which is natural in the sense that $f'^* \eta_{X,E} = \eta_{Y,f^*(E)}$ for f and f' as above.

Construction of Chern classes

Definition

Let E be a vector bundle of rank r on a quasi-projective derived A-scheme X. Its *ith Chern class* is defined as

$$c_i(E) := \pi_{X,E!}(s_i(e(L_1), \dots, e(L_r)) \bullet_{\oplus} \eta_{X,E}) \in \underline{\Omega}^i(X)$$

for $1 \leq i \leq r$ and its *total Chern class* is defined as

$$c(E) = 1_X + c_1(E) + \dots + c_r(E) \in \underline{\Omega}^*(X).$$

Euler classes in the universal precobordism theory

Lemma [A, Lemma 4.1]

Let X be a quasi-projective derived A-scheme and

$$0 \to E' \to E \to E'' \to 0$$

be a short exact sequence of vector bundles over X. Then

$$e(E) = e(E') \bullet_{\oplus} e(E'') \in \underline{\Omega}^*(X)$$

Euler classes in the universal precobordism theory

Lemma [A, Lemma 4.2]

Let E be a vector bundle of rank r on a quasi-projective derived A-scheme X. Then its Euler class $e(E) \in \underline{\Omega}^*(X)$ is nilpotent.

Sketch of proof.

If $r = 1$, we can write $L = L_1 \otimes L_2^\vee$ with L_1, L_2 globally generated. By the defining relation of a precobordism theory, we can hence assume that L is globally generated.
In this case, we can find global sections $s_1, ..., s_n$ which generate L. Since the derived intersection of the vanishing loci $V(s_1), ..., V(s_n)$ is empty and $e(L) = [V(s_1) \to X, 0] = ... = [V(s_n) \to X, 0]$, the claim follows.
If $r > 1$, one constructs a surjection $E^n \to L$ on a line bundle L for some $n >> 0$, so $e(E)^n = e(E^{\oplus n})$ is multiple of $e(L)$. □

Construction of Chern classes

One constructs $\pi_{X,E} : \tilde{X}_E \to X$, $L_1, \dots, L_r$ and $\eta_{X,E} \in \underline{\Omega}^0(\tilde{X}_E)$ by induction on r:

If $r = 1$, simply define $\tilde{X}_E = X$, $\pi_{X,E} = id_X$, choose the trivial filtration $0 \subseteq E$ and $\eta_{X,E} = 1_X \in \underline{\Omega}^0(X)$.

If $r > 1$, we let Z be the vanishing locus of the zero section of E. Furthermore, we let $\tilde{X} = Bl_Z(X)$ and denote by $\pi : \tilde{X} \to X$ the structure morphism of $\tilde{X}$.

Then there is a canonical exact sequence

$$0 \to \mathcal{O}(\mathcal{E}) \to \pi^*(E) \to Q \to 0$$

of vector bundles on $\tilde{X}$, where $\mathcal{E}$ is the exceptional divisor.

Construction of Chern classes

One proves that the element $\eta'_{X,E} \in \underline{\Omega}^*(\tilde{X})$ defined as

$$\sum_{i=0}^{\infty} e(E)^i \bullet [\mathbb{P}_{\tilde{X}}(\mathcal{O} \oplus \pi^*(E))]^i \bullet (1_{\tilde{X}} - e(\mathcal{O}(\mathcal{E})) \bullet [\mathbb{P}_{\tilde{X}}(\mathcal{O}(\mathcal{E}) \oplus \mathcal{O})])$$

with $\bullet = \bullet_{\oplus}$ satisfies $\pi_!(\eta'_{X,E}) = 1_X$ and defines:

$$\tilde{X}_E := \widetilde{Bl_Z(X)}_Q$$

$$\pi_{X,E} : \widetilde{Bl_Z(X)}_Q \xrightarrow{\pi_{Bl_Z(X),Q}} Bl_Z(X) \xrightarrow{\pi} X$$

$$\eta_{X,E} := \eta_{Bl_Z(X),Q} \bullet_{\oplus} \pi^*_{Bl_Z(X),Q}(\eta'_{X,E})$$

We obtain the desired filtration of $\pi^*_{X,E}(E)$ by combining the pullback of $\mathcal{O}(\mathcal{E})$ to $\tilde{X}_E$ and the pullback of the filtration of $\pi^*_{Bl_Z(X),Q}(Q)$ along the surjection $E \to Q$.

Construction of Chern classes

One then checks that all the desired properties of $\tilde{X}_E$ are satisfied, e.g.

$$\begin{aligned}\pi_{X,E!}(\eta_{X,E}) &= \pi_!(\pi_{Bl_Z(X),Q!}(\eta_{Bl_Z(X),Q} \bullet_\oplus \pi^*_{Bl_Z(X),Q}(\eta'_{X,E})))\\ &= \pi_!(\pi_{Bl_Z(X),Q!}(\eta_{Bl_Z(X),Q}) \bullet_\oplus \eta'_{X,E}))\\ &= \pi_!(1_{Bl_Z(X)} \bullet_\oplus \eta'_{X,E})\\ &= \pi_!(\eta'_{X,E})\\ &= 1_X.\end{aligned}$$

Projective bundle formula

Theorem [A, Theorem 5.16(1)]

Let $\mathbb{B}^*$ be a bivariant precobordism theory and $X \to Y$ a morphism between derived schemes over a Noetherian ring A of finite Krull dimension. Let E be a vector bundle of rank r over X. Then there is a natural isomorphism

$$\mathbb{B}^*(\mathbb{P}(E)) \cong \mathbb{B}^*(X)[t]/\langle f \rangle$$

of rings, where $t = e(\mathcal{O}(1))$ and $f = \sum_{i=0}^{r} (-1)^i c_{r-i}(E^\vee) t^i$.

Bivariant projective bundle formula

Theorem [A, Theorem 5.16(2)]

Let $\mathbb{B}^*$ be a bivariant precobordism theory and $X \to Y$ a morphism between derived schemes over a Noetherian ring A of finite Krull dimension. Let E be a vector bundle of rank r over X. Then the assignment $\alpha \otimes \beta \mapsto \alpha \bullet \theta(\pi) \bullet \beta$ induces an isomorphism

$$\mathbb{B}^*(\mathbb{P}(E)) \otimes_{\mathbb{B}^*(X)} \mathbb{B}^*(X \to Y) \cong \mathbb{B}^*(\mathbb{P}(E) \to Y)$$

of $\mathbb{B}^*(\mathbb{P}(E))$-modules, where π is the structure morphism $\mathbb{P}(E) \to X$.