An Introduction to derived algebraic geometry

"add derived intersection"

classical schemes \(\xrightarrow{\sim}\) derived schemes

\[
\text{algebraic} \quad 1\text{-stacks} \quad \xrightarrow{\sim} \quad \text{derived} \quad \text{algebraic} \quad 1\text{-stacks}
\]

\[
\text{algebraic} \quad (\infty,1)\text{-stacks} \quad \xrightarrow{\sim} \quad \text{derived} \quad \text{algebraic} \quad (\infty,1)\text{-stacks}
\]

"add homotopy quotients"

[derived] stack \(F\) is \(n\)-algebraic

if \(\mathcal{E}\) [derived] scheme \(X\), smooth \(\mathcal{E} \xrightarrow{\text{epi}} F\) s.t.

\[
\text{Y}(\text{derived})\text{ scheme } \text{Y} \quad \xrightarrow{\text{Y} \times X} \quad \text{Y}\}
\]

\(Y_F X\) is \((\infty,1)\)-algebraic

TODAY: \(\text{QCoh, Perf, vector bundles, vanishing locus of a section of a vector bundle}\)
Stable ∞-categories of quasi-coherent sheaves

Recollection:

Def. An ∞-category $C$ is stable if

1. there exists a zero object $0 \in C$ (i.e. $\text{Hom}_{\mathcal{C}}(0, X) \cong \{0\}$ for all $X$)
2. $\forall f : X \to Y$ there exist fibers and cofibers
   
   \[
   \begin{array}{c}
   X \to Y \\
   \downarrow \theta \downarrow \\
   0 \to \text{cofib}(f)
   \end{array}
   \begin{array}{c}
   \downarrow \\
   0
   \end{array}
   \begin{array}{c}
   \downarrow \\
   0 \to Y
   \end{array}
   \]
3. A "triangle" in $C$ is a fiber sequence
   if it is a cofiber sequence

   "triangle":

   \[
   \begin{array}{c}
   X \xrightarrow{f} Y \\
   \downarrow \theta \downarrow \\
   0 \to Z
   \end{array}
   \]

Det. If $C$ has $0$, define $X[1]$ as $\frac{C - C}{\text{ad}}$. $X[1]$ is a functor.

Prop. If $C$ is a stable ∞-category
then $\text{hC}$ is a triangulated category with

distinguished triangles $X \to Y \to Z \to X[1]$
that are lifts of $X \to Y \to 0 \to Z \to W$

Where to find stable ∞-categories?

1) $C$ - differential graded category $\to \text{Ndg}(C)$ - ∞-category
   (over a comm. ring $k$)
   
   E.g. if $A$ is an additive category,
   then $\text{Ndg}(\text{Ch}(A))$ is a stable ∞-category
   
   In fact, \{pre-triangulated differential graded categories over $k$\} $\sim$ \{idempotent-complete $k$-linear\}
   \text{Morita} \{\text{stable} \infty\text{-categories}\}

2) if $C$ is an ∞-category with $0$ and finite limits,
   then $\text{Sp}(C) : = \text{holim}_{C \to C}$
   is a stable ∞-category
   (spectrum objects of $C$
   
   Prop. Let $A$ be a simplicial commutative ring over $\mathbb{Q}$,
   then the following stable ∞-categories are equivalent:
   
   (i) dg-modules over the dg-algebra corresponding to $A$
   (ii) stabilization of simplicial modules over $A$
   (iii) stabilization of simplicial comm. algebras over $A$
   (iv) $(\text{Mod}A)$ defined last time
**Qcoh**

**Def.** If $X$ is a derived affine scheme represented by simplicial comm. ring $A$, let $\text{Qcoh}(X) := \text{Mod}_A$.

If $X$ is a derived scheme,

let $\text{Qcoh}(X) := \varinjlim_{S \to X} \text{Qcoh}(S)$

$S$ is affine open subscheme.

If $X$ is a derived stack,

let $\text{Qcoh}(X) := \varinjlim_{S \to X} \text{Qcoh}(S)$

$S$ is affine derived scheme.

**Rk.** If $X$ is a (classical) scheme, then $\text{hQcoh}(X) \cong D(X)$.

In particular, if $X$ is quasi-projective over a field, then $\text{Qcoh}(X) \cong \text{Mod}_A$ for some $A \in$ simp. comm. algebra (to construct $A$ find a 'generator' of $\text{Qcoh}(X)$ and take its endomorphisms as dg/simplicial algebra).

**Question:** Does [derived] algebraic cobordism respect Hori’s gradient?

**Probably not.**

**Properties:**

- $\text{Qcoh}(X)$ is a stable $\infty$-category with $\boxtimes_X$.
- $A \xrightarrow{f} B$ - morphism of simplicial comm. rings
- $\boxtimes_A B = f^* : \text{Mod}_A \hookrightarrow \text{Mod}_B : f^* - \text{forgetful}$
- $X \xrightarrow{f} Y$ - morphism of derived schemes/stacks
  $\text{Qcoh}(Y) \xrightarrow{f^*} \text{Qcoh}(X)$
- $-11 = f^* \circ f^* \circ f^*$

**Claim:** These definitions agree.

**Rk:** affine open subschemes of $X$ are in 1-1 correspondence with affine open subschemes of $tX$.

projection formula $f^*(F) \boxtimes G = f^*(F \boxtimes f^* G)$
Prop. Given a (derived) pullback square
\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow^g & & \downarrow^g \\
X & \xrightarrow{f} & Y
\end{array}
\]
of derived schemes, there is a natural equivalence of functors
\[
g^* f^* \simeq g'^* f'^* g'^*
\]

"Proof": enough to consider \(d\text{Aff}\)

- \(\text{Spec } B \xrightarrow{g'} \text{Spec } C\)
- \(\text{Spec } B \xrightarrow{f} \text{Spec } A\)

\[
M \in \text{Qcoh}(\text{Spec } B), \quad g^* f^* (M) = M \otimes_A C
\]
\[
g'^* f'^* (M) = M \otimes_B (B \otimes_C A)
\]

Rk. This "tautological" result uses the derived pullback and does not hold for classical schemes. In fact, let \(X, Y, Y' \in \text{Sch}_{/X} \), \(x' = x \times_X Y'\) then the following are equivalent:

1) \(Lg^* Rf^* = Rf'^* Lg'^* \quad \text{D}(X \to Y')\)

2) the derived pullback is isomorphic to the classical pullback

3) \(f, g\) are Tor-independent (aka transversal)

Under relevant "finiteness" assumptions 1) yields base change formula for \(K_0\) algebraic cobordism of Levine-Morel Kazu satisfy analogous base change under condition 3). Derived algebraic cobordism have base change for all derived pullback squares.
Perfect complexes, vector bundles, Tor-amplitude

**Def.** $F \in \text{Qcoh}(X)$ is a perfect complex if on every open affine subscheme of $X$, $F$ is equivalent to an iterated cofiber of shifts of projective modules. $\text{Perf}(X) \subseteq \text{Qcoh}(X)$ is a full subcategory.

**Properties:**
- $\text{Perf}(X)$ is stable and has $\otimes$.
- If $X$ is a classical scheme, then perfect complexes are those that are locally $\mathbb{Q}$-is to a finite complex of vector bundles.

Moreover, if $X$ is defined over a field of char. 0, then $\text{Perf}(X)$ is equivalent to the category of perfect complexes on $X$.

$\text{Perf}(X)$ admits internal mapping objects $\mathcal{F}, \mathcal{G}$ such that $\text{Hom}_X(\mathcal{F}, \text{Hom}_X(\mathcal{E}, \mathcal{G})) = \text{Hom}_X(\mathcal{F} \otimes \mathcal{E}, \mathcal{G})$.

This is an equivalence of mapping spaces.

**Def.** $E \in \text{Qcoh}(X)$ is a vector bundle if locally $E \cong \mathcal{O}_X^n$ for some $n \in \mathbb{N}$.

- $E \to G$ is a surjective morphism of vector bundles.
- If $\pi_0(E) \to \pi_0(G)$ is a surjective morphism of vector bundles on $tX$, then $\text{fib}(\xi)$ is a vector bundle on $X$.

- A vector bundle $E$ is globally generated if $E \otimes \mathcal{O}_X^n \to E$ is a surjective morphism.

**Warning:** (If $X$ is not affine), it is not sufficient for $E$ to be globally generated that $\pi_0(E)$ is a globally generated vector bundle on $tX$.

Note that for $E \in \text{Qcoh}(X)$ we have $\pi_i E$ - classical quasi-coherent on $tX$.

- For discrete $E \in \text{Qcoh}(X)$, $\pi_k(F \otimes E) = 0$ except $k = |F|.$

**Facts:** every $F \in \text{Perf}(X)$ has finite Tor-amplitude.

Moreover, $F$ has amplitude if $0 \leq \text{deg} F$ is a vector bundle.
Projective bundles as functors of points, quasi-projective derived schemes

\[
\begin{align*}
X & \quad \text{scheme}, \quad \mathcal{E} \quad \text{loc. free sheaf on } X \\
\pi : \mathcal{P}_X(\mathcal{E}) & \rightarrow \mathcal{P}_X(\mathcal{E}) \\
\{g : Y \rightarrow \mathcal{P}_X(\mathcal{E})\} & \simeq \{s : Y \rightarrow \mathcal{P}_X(\mathcal{E}) \mid n_Y s = \text{id}_Y\} \\
Y & \rightarrow X \\
\text{So, what is a section of } \pi? \\
\end{align*}
\]

For every "point" of \( X \) one has to choose a line in \( \mathcal{E}_x \), or, equivalently, a linear quotient of \( \mathcal{E}_x \). 

\[
\begin{align*}
\{s : X \rightarrow \mathcal{P}_X(\mathcal{E}) \mid \pi \circ s = \text{id}_X\} & \simeq \{\mathcal{E}_x \rightarrow \mathcal{L} \mid \mathcal{L} \text{ is a line bundle on } X\} \\
\mathcal{E}_x & \rightarrow \mathcal{L} = s^*(\pi^*\mathcal{E} \rightarrow \mathcal{O}(1)) \\
\end{align*}
\]

\[\mathcal{O}(1) \]

Fact: \( \text{Qcoh, Perf, Vect : } \text{Aff}^{op} \rightarrow \text{Cat}_\infty \) (\( \text{Sp} \rightarrow \text{Spec} \) ) satisfy fpqc descent (are derived stacks).

Note that if \( F \) has descent, then \( \text{Mor}(F) : X \rightarrow \text{Fun}(A, F(x)) \) also satisfies descent, because \( \text{Fun}(A, -) \) is left exact.

\[\begin{array}{c}
\text{Mor}_{\text{Vect}}(\text{Vect}) \xrightarrow{\text{fib}} \text{Perf} \\
\text{Mor}_{\text{Vect}}(\text{Vect}) \rightarrow \text{Vect}_{\text{n+1}} \\
\text{Mor}_{\text{Vect}}(\text{Vect}) \rightarrow \text{Vect}_{\text{n+1}} \\
\text{Mor}_{\text{Vect}}(\text{Vect}) \rightarrow \text{Mor}_{\text{Vect}}(\text{Vect}) \\
\end{array}\]

This defines \( \mathcal{P}_X(\mathcal{E}) \) as a derived stack.

Properties:
1) there exist canonical \( \pi : \mathcal{P}_X(\mathcal{E}) \rightarrow X \)

- Line bundle \( \mathcal{O}(1) \) on \( \mathcal{P}_X(\mathcal{E}) \), \( Y : \pi^*\mathcal{E} \rightarrow \mathcal{O}(1) \)

2) \( \mathcal{P}_X(\mathcal{E}) \) is a derived pullback square

\[
\begin{array}{c}
\mathcal{P}_X(\mathcal{E}) \\
\downarrow \pi_X \quad \downarrow \pi_Y \\
Y \\
\rightarrow X
\end{array}
\]

3) \( \mathcal{P}_X(\mathcal{E}) \) is a derived scheme

Proof: Zariski locally \( \mathcal{P}_X(\mathcal{E}) \) is covered by \( \mathbb{A}^{n+1} \setminus \{0\} \)

Def.: \( f : X \rightarrow Y \) is a closed immersion/proper morphism

- \( tf : tX \rightarrow tY \) is a closed immersion/proper morphism.

- \( X \rightarrow U \times Y \rightarrow Y \) when \( \pi \) is proper & quasi-projective

Prop.: \( X \in \text{Sch}_k \), 1) \( X \) is quasi-proj. \( \iff \exists L - \text{ample line bundle on } X \)

2) if \( \mathcal{O}(1) \) is ample, \( E - \text{vector bundle, then } E(n) \) is globally generated

3) if \( X \) is quasi-proj., \( \text{Perf}_{\text{coh}}(\mathcal{E}) \approx \{F_d \rightarrow \cdots \rightarrow F_0 \mid F_i \in \text{Vect}_{\text{d}}\} \)
Total space of vector bundle, derived vanishing locus

$X$ - derived scheme, $F \in QCoh(X)$, $\pi(F) = 0$, i.e.

Fact: If derived scheme $W(F) \to X$, representing the functor $Y \mapsto X \to \text{Hom}_X(F, \mathcal{O}_Y)^\wedge = H^0(f^*(F), f_*\mathcal{O}_Y)$

locally, $W(F)$ can be defined as derived affine scheme given by $\text{Sym}^*_X(F)$

Def. If $E$ is a vector bundle on $X$, then $W(E^\vee) \to X$ is the geometric vector bundle associated to $E$

and usually also denoted by $E \to X$.

Prop. The space of sections of $E \to X$ is canonically identified with $\text{Hom}_X(O_X, E)$

"proof": $\text{Hom}_X(E^\vee, O_X) \cong \text{Hom}_X(O_X, E)$

Remark. If $X$ is a classical scheme, e.g. affine Spec $A$, $F$ is a bounded complex of projective $A$-modules of $f.t.$, then $W(F)$ is an algebraic (non-derived) $\infty$-stack

where $H^i(F) = 0$ for $i > n$.

Eg. if $F$ is $F^\vee \to F$, then $V(F)$ is an Artin stack. These were studied under the name of sheaves of Picard categories by Deligne and others.

Def. $X \in \text{Sch}$, $E \in \text{Veet}(X)$, $s \in \text{Hom}_X(O_X, E)$

the derived vanishing locus $V(s)$ is defined by

$V(s) \to X$

$\downarrow$ $
abla$

$X \to E$

Prop. Let $F$ be the cofibre of $s^\vee: E^\vee \to O_X$ in $\mathcal{D}_X$.

Then $V(s)$ is naturally identified as $W(F)$.

"Proof": morphisms from $Y$ to $V(s)$ correspond to $f: Y \to X$ & homotopy $Y \cong E$,

i.e. $f^* E^\vee \to O_Y$

morphisms from $Y$ to $W(F)$ correspond to $f: Y \to X$ & $f^* F \to O_Y$

Rk. $V(o)$ will give the top Chern class of $E$.

Note that $V(o) = \text{"Spec} \text{Sym}^*(E^\vee \wedge_n)$"
Inclusion of projective bundles

Let \( X \in \text{Sch}, \; E \to F \in \text{Vect}(X) \) s.t. \( F' \to E' \)

we get \( \pi^{*}_{IP(E)} \; F' \to \pi^{*}_{IP(E)} \; E' \to (U_{IP(E)})_{(1)} \)

which yields projection \( PI(E) \to IP(F) \)

| It is a closed immersion |

Projective bundles will come up when derived blow-ups are performed, then the following result will be needed.

Prop. Let \( X \in \text{Sch}, \; E, F \in \text{Vect}(X) \). \( \pi : IP(E) \oplus F \to X \)

and consider \( s \in \text{Hom}(U_{IP(E) \oplus F}, \pi^{*}(F) \otimes \mathcal{O}(1)) \)

\( = \text{Hom}(U_{X}, F \otimes \pi^{*}(\mathcal{O}(1))) \cong \text{Hom}(F', E' \oplus F') \)

Corresponding to the inclusion.

Then \( V(s) \hookrightarrow IP(E) \oplus F \) can be identified with \( IP(E)^{0} \to IP(E) \).

"Proof":

1. Find a morphism \( IP(E) \to V(s) \)
2. Prove that it is an equivalence
3. \( IP(E) \to V(s) \to IP(E) \oplus F \)

...}

To construct \( f \) we need to find homotopy between \( j^{*}(s) \) and zero in the space of \( \text{Hom}(U_{IP(E)}, j^{*}(\pi^{*}(F) \otimes \mathcal{O}(1))) \cong \text{Hom}(F', E') \)

The idea is to show that \( j \) we can identify with \( \text{Hom}_{X}(F', E' \oplus F') \to \text{Hom}_{X}(F', E') \)

with the map corresponding to the canonical projection.

After this identification \( j^{*}(s) \) is 0.

\( \begin{align*}
\text{(1) One has to show the following:} \\
\text{it is an isomorphism after truncation} \\
\text{IP(E) \to V(s) is quasi-smooth embedding of virtual codimension 0} \\
\text{for this suffices to show IP(E), V(s) are quasi-smooth over } X \\
\text{But virtual codimension 0 means locally cut out by 0 equations.}
\end{align*} \)