Quasi-smooth pullbacks in algebraic cobordism

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0. Glossary of derived algebraic geometry
I. Algebraic cobordism
II. Deformation to the normal cone (bundle)
III. Pullbacks along regular immersions
IV. Quasi-smooth pullbacks

Last week - derived alg. cobordism

char $k = 0$ $\quad X \in dQPr_k$

$$d\Omega_*(X) = \bigoplus_{\substack{f \text{ proper} \\ Y \in QSm_k}} \mathbb{Z}[Y \xrightarrow{f} X] \Big/ \begin{matrix}(HTP) \\ (FGL) \\ (SNC)\end{matrix}$$

- $d\Omega_*$ is the univ. OBMF w./ qSm pb.

Today $\underbrace{\Omega_*}_{\text{alg. cob}}$ is an OBMF on $dQPr_k$ w./ qSm pb.

Cor $\exists$ nat. tr. $\theta_{d\Omega} : \Omega_* \to d\Omega_*$ which has a left inverse ($\Rightarrow$ inj)

0. Glossary of DAG

Def 1 1) $f: Y \to X \in dSch$ is proper if $f_{cl}: Y_{cl} \to X_{cl}$ is proper

Rk valuative criterion holds:

R val. ring
$K \doteq \mathrm{Frac}\, R$

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}\, R \arrow[r] \arrow[ur, dashed, "\exists !"] & X \end{tikzcd}$$

2) - $i: \mathrm{Spec}\, B \to \mathrm{Spec}\, A \in dSch^{aff}$ is a closed immersion if $A \to B$ is surjective on π_0.

- $i: Z \to X \in dSch$ is a cl. imm. if $\forall R \in SCRings$ $\forall \mathrm{Spec}\, R \to X$

$$\begin{tikzcd} Z \times_X \mathrm{Spec}\, R \arrow[r, "i_R"] \arrow[d] & \mathrm{Spec}\, R \arrow[d] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

$Z \times_X \mathrm{Spec}\, R$ is affine & i_R is cl. imm.

Ex $X_{cl} \to X$

3) A cl. imm. $i: Z \hookrightarrow X \in dSch$ is regular if $\mathbb{L}_{Z/X}[-1]$ is a loc. free $\mathcal{O}_Z$-module of finite rank.

$\iff$ Zariski locally

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ 0 \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

$\Leftrightarrow$ Zar. loc, $\quad \mathrm{Spec}\, \underbrace{R/\!/(f_i)} \hookrightarrow \mathrm{Spec}\, R$

<u>Rk</u> If Z, X discrete, same as the usual reg. cl. imm.

(4) A morphism $Y \to X \in dSch$ is <u>quasi-proj</u> if it factors $Y \xrightarrow{\text{cl. im.}} U \xrightarrow{\text{o. i.}} \mathbb{P}^n_X \to X$

- any qproj mph is smoothable, $\quad Y \xrightarrow{\text{cl. im}} U \xrightarrow{\text{sm.}} X$

moreover, $Z \to Y \to X$ qproj

$$\begin{tikzcd} Z \arrow[r, "\text{c.i.}"] \arrow[dr] & V \arrow[r, "\text{c.i.}"] \arrow[d, "\text{sm}"] & W \arrow[d, "\text{sm}"] \\ & Y \arrow[r, "\text{c.i.}"] \arrow[dr] & U \arrow[d, "\text{sm}"] \\ & & X \end{tikzcd}$$

- a qproj mph is qsm $\Leftrightarrow$ it is lci, i.e.

$$Y \xrightarrow{\text{reg. imm.}} U \xrightarrow{\text{sm.}} X$$

<u>Virtual normal bundle</u>

<u>Def 2</u> $i: Z \to X$ reg imm. The <u>virtual normal bundle</u> $N_Z X \to Z$ is the v.b.

$$\mathrm{Spec}_{\mathcal{O}_Z} (\mathrm{Sym}^* \mathbb{L}_{Z/X}[-1])$$

<u>normal cone</u>: I = ideal sheaf of Z_{cl} in X_{cl}

$$C_Z X := \mathrm{Spec}_{\mathcal{O}_{Z_{cl}}} \left(\bigoplus_{n \geq 0} I^n / I^{n+1} \right) \to Z_{cl}.$$

We have $\mathcal{L}_{Z/X}|_{Z_{cl}}[-1] \to \mathcal{L}_{Z_{cl}/X_{cl}}[-1]$

$$\pi_1(\mathcal{L}_{Z_{cl}/X_{cl}}) \simeq I/I^2$$

$\Rightarrow \exists$ can morphism $\iota : C_Z X \to N_Z X$

<u>Lemma 3</u> ι is a closed imm.

D/ Suffices $\pi_1(\mathcal{L}_{Z/X}|_{Z_{cl}}) \twoheadrightarrow I/I^2$ surj.

may assume X, Z affine.

Let $f: A \to B \in$ SCRings be 0-connective.

Surjective on π_0

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \pi_0 A \arrow[r] & \pi_0 B \end{tikzcd} \rightsquigarrow \begin{tikzcd} L_A \otimes_A \pi_0 B \arrow[r] \arrow[d] & L_B \otimes_B \pi_0 B \arrow[r] \arrow[d] & L_{B/A} \otimes_B \pi_0 A \arrow[d] \\ L_{\pi_0 A} \otimes_{\pi_0 A} \pi_0 B \arrow[r] \arrow[d] & L_{\pi_0 B} \arrow[r] \arrow[d] & L_{\pi_0 B/\pi_0 A} \arrow[d] \\ L_{\pi_0 A/A} \otimes_{\pi_0 A} \pi_0 B \arrow[r] & L_{\pi_0 B/B} \arrow[r] & M \end{tikzcd}$$

diagram of cofiber sequences

$A \to \pi_0 A$ 1-connective $\Rightarrow$ $L_{\pi_0 A/A}$ is 2-connective

$L_{\pi_0 B/B}$ is 2-connective

$\Rightarrow M$ is 2-connective. $\square$.

I Algebraic cobordism

Recall 4 $X \in dQPr_k$

$$\Omega_n(X) = \bigoplus \mathbb{Z}\left[f: Y \to X, \underbrace{L_1, \dots, L_r}_{L.B./Y}\right] \Big/ \begin{matrix}(\mathrm{Dim})\\(\mathrm{Sect})\\(\mathrm{FGL})\end{matrix}$$

f projective.

$Y \in Sm_k$ irre.

$n = \dim Y - r$

Lemma 5 $X_{cl} \to X$ induces $\Omega_*(X_{cl}) \simeq \Omega_*(X)$

P/ $Y \to X$ factors through X_{cl}. □.

$Y \in Sm_k$

Cor 6 1) Ω_* has | smooth pullbacks and
| proper pushforwards.

is the universal OBMT on $dQPr_k$.

2) $\exists$ can. nat. trans $\theta_{d\Omega}: \Omega_* \to d\Omega_*$

$$[Y \xrightarrow{f} X] \mapsto f_* \pi_Y^* \mathbb{1}$$

$\pi_Y: Y \to k$,

3) $Z \overset{c.i.}{\hookrightarrow} X \hookleftarrow U$ $\quad \Omega_*(Z) \to \Omega_*(X) \to \Omega_*(U) \to 0$

4) $p: E \to X$ rank n V.B. $\quad p^*: \Omega_*(X) \xrightarrow{\sim} \Omega_{*+n}(E)$.

II. Deformation to the normal cone (bundle)

Recall 7; $\exists!$ way to associate to every reg. imm.

$i : Z \hookrightarrow X \in dSch$ a morphism $Bl_Z X \to X \in dSch$

s.t. 1) Z, X discrete $\Rightarrow$ usual blow-up.

2)
$$\begin{tikzcd} Z' \arrow[r] \arrow[d, hook] & X' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd} \Rightarrow Bl_{Z'} X' \simeq Bl_Z X \times_X X'.$$

Def 8 $i : Z \hookrightarrow X \in dSch$ r. i.

deformation to the normal bundle

$$D_Z X := Bl_{Z \times \{0\}} (X \times \mathbb{A}^1) \setminus Bl_{Z \times \{0\}} (X \times \{0\})$$

$D_Z X$ is flat over $\mathbb{A}^1$.

$$\begin{tikzcd} N_Z X \arrow[r] \arrow[d] & D_Z X \arrow[d] & X \times \mathbb{G}_m \arrow[l] \arrow[d] \\ 0 \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l] \end{tikzcd}$$

III Pull back along reg. imm.

$N_Z X \to Z_{cl}$.

$i: Z \hookrightarrow X \in dSch$ reg. imm.

Localization $\Rightarrow$ $\Omega_*(N_Z X) \xrightarrow{f_*} \Omega_*(D_Z X) \to \Omega_*(X \times \mathbb{G}_m) \to 0$

$\downarrow f^*$ $\exists$

$\Omega_{*-1}(N_Z X)$

$f^* f_* = e(\underbrace{N_{N_Z X} D_Z X}_{\text{trivial}})$ intersection with a divisor

$= 0$

$\Rightarrow \exists\ \underline{s_{Z/X}} : \Omega_*(X \times \mathbb{G}_m) \to \Omega_{*-1}(N_Z X)$

Def 9 specialization to the normal cone (bundle)

$$\sigma_{Z/X} : \Omega_*(X) \to \Omega_{*+1}(X \times \mathbb{G}_m) \xrightarrow{s_{Z/X}} \Omega_*(N_Z X).$$

Rk 10 Alternatively, "motivic" specialization

$\Omega_n(X) \simeq MGL^{BM}_{2n,n}(X/k) = [\pi_{X!} \mathbb{1}_X(n)[2n], MGL_k]_{SH(k)}$

Let $\Omega_n(X,1) := MGL^{BM}_{2n-1,n}(X/k)$

- $\Omega_n(X \times \mathbb{G}_m, 1) \xrightarrow{\partial} \Omega_n(N_Z X)$.

- since $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$, consider $t \in \mathcal{O}^\times(X \times \mathbb{G}_m)$

$\rightsquigarrow \{t\} \in MGL^{1,1}(X \times \mathbb{G}_m) \simeq MGL^{BM}_{-1,0}(X \times \mathbb{G}_m / X)$

$\rightsquigarrow$ multiplication by $\{t\}$ $\gamma_t : \Omega_n(X) \longrightarrow \Omega_n(X \times \mathbb{G}_m, 1)$

Then $\Omega_n(X) \xrightarrow{\gamma_t} \Omega_n(X \times \mathbb{G}_m, 1) \xrightarrow{\partial} \Omega_n(N_Z X)$

agrees with $\sigma_{Z/X}$

Similarly, the same construction for $i_{cl} : Z_{cl} \to X_{cl}$ gives

$\sigma_{Z_{cl}/X_{cl}} : \Omega_*(X) \simeq \Omega_*(X_{cl}) \longrightarrow \Omega_*(C_Z X)$

By Lemma 3, $\iota : C_Z X \longrightarrow N_Z X$

Lemma 11 $\sigma_{Z/X} \simeq \iota_* \circ \sigma_{Z_{cl}/X_{cl}} : \Omega_*(X) \to \Omega_*(N_Z X)$.

Def 12 (Gysin morphism for reg imm.)

$i : Z \to X \in dSch_k$ reg imm. of virtual codim d

$\pi : N_Z X \to Z$ virtual normal bundle

$i^* : \Omega_*(X) \xrightarrow{\sigma_{Z/X}} \Omega_*(N_Z X) \xrightarrow{(\pi^*)^{-1}} \Omega_{*-d}(X)$

Lemma 13
$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$
$\in dQPr_k$ i reg. imm.

1) f proper $\Rightarrow$ $i^* f_* = f'_* i'^*$

2) f smooth $\Rightarrow$ $i'^* f^* = f'^* i^*$

Pf/ reduce to: - $\sigma_{Z/X}$ compat. with f^*, f_*

- proper pf / sm pb compat. w/ intersecting w/ divisor

(Levine-Morel, Ch. 6). $\square$.

Lemma 14 $\begin{tikzcd} Z \arrow[rr, hook, "i"] \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$ $\in dSch_k$ p, q smooth, i reg imm.

Then $q^* = i^* p^*$.

Prop 15 $Z \xrightarrow{k} Y \xrightarrow{i} X \in dSch_k$ i, k reg. imm.

Then $(i \circ k)^* = k^* \circ i^*$.

Pf/ Lemma 13 + 14, + key result:

$$\begin{tikzcd} \Omega_*(X) \arrow[r, "\sigma_{Y/X}"] \arrow[d, "\sigma_{Z/X}"'] & \Omega_*(N_Y X) \arrow[d, "\sigma_{N_Y X_{|Z} / N_Y X}"] \\ \Omega_*(N_Z X) \arrow[r, "\sigma_{N_Z Y / N_Z X}"] & \Omega_*(N) \end{tikzcd}$$

where $N = N(N_Z X, N_Z Y) \simeq N(N_Y X, N_Y X_{|Z})$

Idea: use double deformation space

$$D = D_{D_Z X_{|Y}} D_Z X$$ $\square$.

$X = \mathrm{Spec}\, R$ $Z = \mathrm{Spec}\, R // (0, \dots, 0)$

$$SL_r(X) \simeq SL_{r+n}(\underset{\underset{\mathbb{A}^n_X}{\|}}{N_Z X})$$

IV Quasi-smooth pullbacks

Def 16 $f: X \to Y \in dQPr_k$ quasi-smooth

Let $X \xrightarrow{i} P \xrightarrow{p} Y$ be a factorization n = rel. dim of p

reg im. sm. m = vir. codim of i.

$$f^*: \Omega_*(Y) \xrightarrow{p^*} \Omega_{*+n}(P) \xrightarrow{i^*} \Omega_{*+d}(X)$$

quasi-smooth pullback

Lemma 17 f^* does not depend on the factorization.

Lemma 18 $(g \circ f)^* = f^* \circ g^*$

Th 19 $\Omega_*: dQPr_k \to Gr^?Ab$

is an OBMF of geom type w/ qsm pb.

Cor 20 1) $\exists$ can map $\theta_{d\Omega}: d\Omega_* \to \Omega_*$

compat. w/ proper pf

sm pb.

2) $\theta_\Omega \circ \theta_{d\Omega} = id$

P/ 1) univ. prop. of $d\Omega_*$.

2) $\theta_\Omega \circ \theta_{d\Omega}([Y \xrightarrow{f} X]) = \theta_\Omega(f_*^{d\Omega} \pi_{Y,d\Omega}^* \mathbb{1})$

$= f_*^\Omega \pi_{Y,\Omega}^* \theta_\Omega(\mathbb{1}) = [Y \xrightarrow{f} X]$ $\square$.

In part, $\theta_{d\Omega}$ is inj.