Algebraic Geometry 3-Homework 11

1. a. Let $O$ be a noetherian local unique factorization domain (UFD). Show that $O$ is a normal ring. Show that $O$ is a discrete valuation ring (DVR) if $O$ has Krull dimension one.

b. Let $X$ be an integral $k$-scheme. $X$ is called locally factorial if each local ring $O_{X,x}$ is a UFD (for example, a smooth $k$-scheme is locally factorial). Show that for a locally factorial $k$-scheme $X$ of dimension $n$, the divisor map

$$[-] : Div(X) \to Z_{n-1}(X)$$

is an isomorphism, inducing an isomorphism $Pic(X) \to CH_{n-1}(X)$.

2. a. Use the localization sequence to show that for $0 \leq m \leq n$, $CH_m(\mathbb{P}^n)$ is generated by the class of any linear subspace $\mathbb{P}^m \subset \mathbb{P}^n$.

b. Show that $CH_{n-1}(\mathbb{P}^n) \cong \mathbb{Z}$ and $CH_0(\mathbb{P}^n) \cong \mathbb{Z}$, the second isomorphism arising from the degree map $deg_k : CH_0(\mathbb{P}^n) \to CH_0(Spec \ k) = \mathbb{Z}$. For the first isomorphism, show that $O_{\mathbb{P}^n}(m) \cong O_{\mathbb{P}^n}$ if and only if $m = 0$.

3. Let $X$ be a $k$-scheme.

a. Let $\alpha \in Z_n(X)$ be a cycle, $D, D' \in Div(X)$ linear equivalent Cartier divisors. Show that $D \cdot \alpha = D' \cdot \alpha$ in $CH_{n-1}(|\alpha|)$.

b. Suppose $X$ is a proper $k$-scheme. Recall that for a proper $k$-scheme $p : Y \to Spec \ k$, a 0-cycle $z \in CH_0(Y)$ has degree $d$ over $k$ ($deg_k z = d$) if $p_*(z) = d \cdot [Spec \ k]$ in $CH_0(Spec \ k)$. Let $\alpha \in Z_1(X)$ be a cycle, $D, D' \in Div(X)$ linear equivalent Cartier divisors. Show that $deg_k D \cdot \alpha = deg_k D' \cdot \alpha$.

4. a. Let $\alpha \in Z_1(\mathbb{P}^2)$ be a cycle. Define $deg(\alpha)$ as $deg_k(\ell \cdot \alpha)$, where $\ell$ is a line in $\mathbb{P}^2$ considered as a Cartier divisor. Show that $deg(\alpha)$ is well-defined (i.e., independent of the choice of $\ell$).

b. Since $\mathbb{P}^2$ is smooth over $k$, we have $Div(\mathbb{P}^2) = Z_1(\mathbb{P}^2)$, so for each pair of 1-cycles, $\alpha, \beta \in Z_1(\mathbb{P}^2)$, the intersection produce $\alpha \cdot \beta \in CH_0(\mathbb{P}^2)$ is defined by considering $\alpha$ as in $Div(\mathbb{P}^2)$ and $\beta$ as in $Z_1(\mathbb{P}^2)$. Prove Bezout’s theorem:

$$deg_k(\alpha \cdot \beta) = deg(\alpha) \cdot deg(\beta).$$

Conclude that $\alpha \cdot \beta = \beta \cdot \alpha$ in $CH_0(\mathbb{P}^2)$.

5. Let $L$ be a line bundle on a smooth $k$-scheme $X$ of pure dimension $n$. We have the pseudo-divisor $(L, X, \emptyset)$ and the fundamental class $[X] \in Z_n(X)$. The first Chern class of $L$, $c_1(L)$, is defined as

$$c_1(L) := (L, X, \emptyset) \cdot [X] \in CH_{n-1}(X).$$

a. Let $f : Y \to X$ be a flat morphism of smooth $k$-schemes, $L$ a line bundle on $X$. Show that $f^*(c_1(L)) = c_1(f^*L)$.

b. Show that sending $L$ to $c_1(L)$ induces an isomorphism $Pic(X) \to CH_{n-1}(X)$, and that this is the same isomorphism as the one in 1(b).